

# Control Design of Fuzzy Systems with Immeasurable Premise Variables

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## 1. Introduction

The Takagi-Sugeno fuzzy model is described by fuzzy if-then rules which represent local linear systems of the underlining nonlinear systems (Takagi & Sugeno, 1985; Tanaka et al., 1996; Tanaka & Sugeno, 1992) and thus it can describe a wide class of nonlinear systems. In the last decade, nonlinear control design methods based on Takagi-Sugeno fuzzy system have been explored. Since the stability analysis and state feedback stabilization first made in (Tanaka & Sugeno, 1992), system theory and various control schemes for fuzzy systems have been developed. Parallel to state feedback control design, observer problems were also considered in (Tanaka & Sano, 1994; Tanaka & Wang, 1997; Yoneyama et al., 2000; Yoneyama et al., 2001a). When the observer is available, it is known in (Yoneyama et al., 2000) that for fuzzy systems, the separation property of designing state feedback controller and observer is established. Thus, an output feedback controller for fuzzy systems was proposed in (Yoneyama et al., 2001a). Theory was extended to  $H_\infty$  control (Cao et al., 1996; Chen et al., 2005; Feng et al., 1996; Hong & Langari, 1998; Katayama & Ichikawa, 2002; Yoneyama et al., 2001b). In spite of these developments in fuzzy system control theory, the separation property holds only for a limited class of fuzzy systems where the premise variable is measurable.

When we consider a fuzzy system, the selection of the premise variables plays an important role in system representation. The premise variable is usually given, and hence the output is naturally selected. In this case, however, a class of fuzzy systems is limited. If the premise variable is the state of the system, a fuzzy system can represent the widest class of nonlinear systems. In this case, output feedback controller design is difficult because the state variable is immeasurable and it is not available for the premise variable of an output feedback controller. For this class of fuzzy systems, output feedback control design schemes based on parallel distributed compensator (PDC) have been considered in (Ma et al., 1998; Tanaka & Wang, 2001; Yoneyama et al., 2001a) where the premise variable of the controller was replaced by its estimate. Furthermore, Linear Matrix Inequality (Boyd et al, 1994) approach was introduced in (Guerra et al., 2006). Uncertain system approach was taken for stabilization and  $H_\infty$  control of both continuous-time and discrete-time fuzzy systems in (Assawinchaichote, 2006; Tanaka et al., 1998; Yoneyama, 2008a; Yoneyama, 2008b). However, controller design conditions given in these approaches are still conservative.

This chapter is concerned with robust  $H_\infty$  output feedback controller design for a class of uncertain continuous-time Takagi-Sugeno fuzzy systems where the premise variable is the immeasurable state variable. This class of fuzzy systems covers a general nonlinear system

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and its output feedback control problem is of practical importance. First, it is shown that Takagi-Sugeno fuzzy system with immeasurable premise variables can be written as an uncertain linear system, and robust stability with  $H_\infty$  disturbance attenuation via output feedback control for a fuzzy system is converted into the same problem for an uncertain linear system. Then, an original robust control problem is shown to be equivalent to stabilization with  $H_\infty$  disturbance attenuation problem for nominal system with no uncertainty. The discrete-time counterpart of the same robust control problems is also considered. Numerical examples of both continuous-time and discrete-time fuzzy systems are shown to illustrate our design methods. Finally, an extension to Takagi-Sugeno fuzzy time-delay systems is given. The same technique is used to write a fuzzy time-delay system as an uncertain linear time-delay system.

## 2. Robust output feedback control for uncertain continuous-time fuzzy systems with immeasurable premise variables

We consider robust stability with  $H_\infty$  disturbance attenuation problems via output feedback control for continuous-time Takagi-Sugeno fuzzy systems with immeasurable premise variables. We first show that such a fuzzy system can be written as an uncertain linear system. Then, robust stability with  $H_\infty$  disturbance attenuation problem for an uncertain system is converted into stability with  $H_\infty$  disturbance attenuation problem for a nominal system. Based on such a relationship, a solution to a robust stability with  $H_\infty$  disturbance attenuation via output feedback control for a fuzzy system with immeasurable premise variables is given. Finally, a numerical example illustrates our theory.

### 2.1 Fuzzy systems and problem formulation

In this section, we introduce continuous-time Takagi-Sugeno fuzzy systems with immeasurable premise variables. Consider the Takagi-Sugeno fuzzy model, described by the following IF-THEN rule:

$$\begin{aligned} \text{IF} \quad & \xi_1 \text{ is } M_{i1} \text{ and } \dots \text{ and } \xi_p \text{ is } M_{ip}, \\ \text{THEN} \quad & \dot{x}(t) = (A_i + \Delta A_i)x(t) + (B_{1i} + \Delta B_{1i})w(t) + (B_{2i} + \Delta B_{2i})u(t), \\ & z(t) = (C_{1i} + \Delta C_{1i})x(t) + (D_{11i} + \Delta D_{11i})w(t) + (D_{12i} + \Delta D_{12i})u(t), \\ & y(t) = (C_{2i} + \Delta C_{2i})x(t) + (D_{21i} + \Delta D_{21i})w(t) + (D_{22i} + \Delta D_{22i})u(t), \quad i = 1, \dots, r \end{aligned}$$

where  $x(t) \in \mathfrak{R}^n$  is the state,  $w(t) \in \mathfrak{R}^{m_1}$  is the disturbance,  $u(t) \in \mathfrak{R}^{m_2}$  is the control input,  $z(t) \in \mathfrak{R}^{q_1}$  is the controlled output,  $y(t) \in \mathfrak{R}^{q_2}$  is the measurement output.  $r$  is the number of IF-THEN rules.  $M_{ij}$  is a fuzzy set and  $\xi_1, \dots, \xi_p$  are premise variables. We set  $\xi = [\xi_1 \ \dots \ \xi_p]^T$ . We assume that the premise variables do not depend on  $u(t)$ .  $A_i, B_{1i}, B_{2i}, C_{1i}, C_{2i}, D_{11i}, D_{12i}, D_{21i}$  and  $D_{22i}$  are constant matrices of appropriate dimensions. The uncertain matrices are of the form

$$\begin{bmatrix} \Delta A_i & \Delta B_{1i} & \Delta B_{2i} \\ \Delta C_{1i} & \Delta D_{11i} & \Delta D_{12i} \\ \Delta C_{2i} & \Delta D_{21i} & \Delta D_{22i} \end{bmatrix} = \begin{bmatrix} H_{1i} \\ H_{2i} \\ H_{3i} \end{bmatrix} F_i(t) [E_{1i} \ E_{2i} \ E_{3i}], \quad i = 1, \dots, r \quad (1)$$

where each  $F_i(t) \in \mathfrak{R}^{l \times j}$  is an uncertain matrix satisfying  $F_i^T(t)F_i(t) \leq I$ , and  $H_{1i}, H_{2i}, H_{3i}, E_{1i}, E_{2i}$  and  $E_{3i}$  are known constant matrices of appropriate dimensions.

**Assumption 2.1** The system  $(A_r, B_{1r}, B_{2r}, C_{1r}, C_{2r}, D_{11r}, D_{12r}, D_{21r}, D_{22r})$  represents a nominal system that can be chosen as a subsystem including the equilibrium point of the original system.

Throughout the chapter, we assume  $\xi(t)$  is a function of the immeasurable state  $x(t)$ . Then, the state equation, the controlled output and the output equation are defined as follows:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^r \lambda_i(x(t)) \{ (A_i + \Delta A_i)x(t) + (B_{1i} + \Delta B_{1i})w(t) + (B_{2i} + \Delta B_{2i})u(t) \}, \\ z(t) &= \sum_{i=1}^r \lambda_i(x(t)) \{ (C_{1i} + \Delta C_{1i})x(t) + (D_{11i} + \Delta D_{11i})w(t) + (D_{12i} + \Delta D_{12i})u(t) \}, \\ y(t) &= \sum_{i=1}^r \lambda_i(x(t)) \{ (C_{2i} + \Delta C_{2i})x(t) + (D_{21i} + \Delta D_{21i})w(t) + (D_{22i} + \Delta D_{22i})u(t) \} \end{aligned} \quad (2)$$

where

$$\lambda_i(x) = \frac{\beta_i(x)}{\sum_{i=1}^r \beta_i(x)}, \quad \beta_i(x) = \prod_{j=1}^p M_{ij}(x_j) \quad (3)$$

and  $M_{ij}(\cdot)$  is the grade of the membership function of  $M_{ij}$ . We assume

$$\beta_i(x(t)) \geq 0, \quad i = 1, \dots, r, \quad \sum_{i=1}^r \beta_i(x(t)) > 0$$

for any  $x(t)$ . Hence  $\lambda_i(x(t))$  satisfies

$$\lambda_i(x(t)) \geq 0, \quad i = 1, \dots, r, \quad \sum_{i=1}^r \lambda_i(x(t)) = 1 \quad (4)$$

for any  $x(t)$ . Our problem is to find a control  $u(\cdot)$  for the system (2) given the output measurements  $y(\cdot)$  such that the controlled output  $z(\cdot)$  satisfies

$$\int_0^\infty z^T(t)z(t)dt < \gamma^2 \int_0^\infty w^T(t)w(t)dt \quad (5)$$

for a prescribed scalar  $\gamma > 0$ . If such a controller exists, we call it a robust  $H_\infty$  output feedback controller. Here, we consider the robust stabilization and the robust  $H_\infty$  disturbance attenuation of uncertain fuzzy system (2) with the immeasurable premise variable.

## 2.2 Robust stability of equivalent uncertain systems

When we consider the  $H_\infty$  output feedback control problem for fuzzy systems, the selection of the premise variables plays an important role. The premise variable is usually given, and so the output is selected as the premise variable. In this case, however, the system covers a very limited class of nonlinear systems. When the premise variable is the state of the system, a fuzzy system describes a widest class of nonlinear systems. However, the controller design

based on PDC concept is infeasible due to the immeasurable premise variable imposed on the control law (Tanaka & Wang, 2001; Yoneyama et al., 2001a; Yoneyama et al., 2001b). To avoid a difficulty due to the parallel distributed compensation (PDC) concept, we rewrite a fuzzy system, and consider a controller design for an equivalent uncertain system. We rewrite (2) as an uncertain system. Since  $\lambda_i(x(t))$  satisfies (4), we have

$$\lambda_r(x(t)) = 1 - [\lambda_1(x(t)) + \lambda_2(x(t)) + \dots + \lambda_{r-1}(x(t))].$$

It follows from this relationship that

$$\begin{aligned} \sum_{i=1}^r \lambda_i(x(t))(A_i + \Delta A_i) &= A_r + H_{1r}F(t)E_{1r} + \lambda_1(x(t))[A_{1r} + \bar{H}_{1r}\bar{F}_{1r}(t)\bar{E}_{1r}] \\ &\quad + \dots + \lambda_{r-1}(x(t))[A_{r-1,r} + \bar{H}_{r-1,r}\bar{F}_{r-1,r}(t)\bar{E}_{r-1,r}] \\ &= A_r + H_1\tilde{F}(t)\tilde{A} \end{aligned}$$

where

$$\begin{aligned} A_{ir} &= A_i - A_r, \quad i = 1, \dots, r-1 \\ H_1 &= [I \quad I \quad \dots \quad I \quad H_{11} \quad H_{12} \quad \dots \quad H_{1r}], \\ \tilde{F}(t) &= \text{diag}[\lambda_1(x(t))I \quad \dots \quad \lambda_{r-1}(x(t))I \quad \lambda_1(x(t))F_1(t) \quad \dots \quad \lambda_r(x(t))F_r(t)], \\ \tilde{A} &= [A_{1r}^T \quad A_{2r}^T \quad \dots \quad A_{r-1,r}^T \quad E_{11}^T \quad E_{12}^T \quad \dots \quad E_{1r}^T]^T. \end{aligned}$$

Similarly, we rewrite other matrices and have an equivalent description for (1):

$$\begin{aligned} \dot{x}(t) &= (A_r + \Delta A)x(t) + (B_{1r} + \Delta B_1)w(t) + (B_{2r} + \Delta B_2)u(t) \\ &\triangleq A_\Delta x(t) + B_{1\Delta}w(t) + B_{2\Delta}u(t), \\ z(t) &= (C_{1r} + \Delta C_1)x(t) + (D_{11r} + \Delta D_{11})w(t) + (D_{12r} + \Delta D_{12})u(t) \\ &\triangleq C_{1\Delta}x(t) + D_{11\Delta}w(t) + D_{12\Delta}u(t), \\ y(t) &= (C_{2r} + \Delta C_2)x(t) + (D_{21r} + \Delta D_{21})w(t) + (D_{22r} + \Delta D_{22})u(t) \\ &\triangleq C_{2\Delta}x(t) + D_{21\Delta}w(t) + D_{22\Delta}u(t) \end{aligned} \tag{6}$$

where

$$\begin{aligned} \begin{bmatrix} \Delta A & \Delta B_1 & \Delta B_2 \\ \Delta C_1 & \Delta D_{11} & \Delta D_{12} \\ \Delta C_2 & \Delta D_{21} & \Delta D_{22} \end{bmatrix} &= \begin{bmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{bmatrix} \begin{bmatrix} \tilde{F}(t) & 0 & 0 \\ 0 & \tilde{F}(t) & 0 \\ 0 & 0 & \tilde{F}(t) \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix}, \\ H_2 &= [I \quad I \quad \dots \quad I \quad H_{21} \quad H_{22} \quad \dots \quad H_{2r}], \\ H_3 &= [I \quad I \quad \dots \quad I \quad H_{31} \quad H_{32} \quad \dots \quad H_{3r}], \\ B_{1ir} &= B_{1i} - B_{1r}, \quad B_{2ir} = B_{2i} - B_{2r}, \quad C_{1ir} = C_{1i} - C_{1r}, \\ C_{2ir} &= C_{2i} - C_{2r}, \quad D_{11ir} = D_{11i} - D_{11r}, \quad D_{12ir} = D_{12i} - D_{12r}, \\ D_{21ir} &= D_{21i} - D_{21r}, \quad D_{22ir} = D_{22i} - D_{22r}, \quad i = 1, \dots, r-1, \\ \tilde{B}_1 &= [B_{11r}^T \quad B_{12r}^T \quad \dots \quad B_{1r-1,r}^T \quad E_{21}^T \quad E_{22}^T \quad \dots \quad E_{2r}^T]^T, \end{aligned} \tag{7}$$

$$\begin{aligned}
\tilde{B}_2 &= [B_{21r}^T \quad B_{22r}^T \quad \cdots \quad B_{2r-1,r}^T \quad E_{31}^T \quad E_{32}^T \quad \cdots \quad E_{3r}^T]^T, \\
\tilde{C}_1 &= [C_{11r}^T \quad C_{12r}^T \quad \cdots \quad C_{1r-1,r}^T \quad E_{11}^T \quad E_{12}^T \quad \cdots \quad E_{1r}^T]^T, \\
\tilde{C}_2 &= [C_{21r}^T \quad C_{22r}^T \quad \cdots \quad C_{2r-1,r}^T \quad E_{11}^T \quad E_{12}^T \quad \cdots \quad E_{1r}^T]^T, \\
\tilde{D}_{11} &= [D_{111r}^T \quad D_{112r}^T \quad \cdots \quad D_{11r-1,r}^T \quad E_{21}^T \quad E_{22}^T \quad \cdots \quad E_{2r}^T]^T, \\
\tilde{D}_{12} &= [D_{121r}^T \quad D_{122r}^T \quad \cdots \quad D_{12r-1,r}^T \quad E_{31}^T \quad E_{32}^T \quad \cdots \quad E_{3r}^T]^T, \\
\tilde{D}_{21} &= [D_{211r}^T \quad D_{212r}^T \quad \cdots \quad D_{21r-1,r}^T \quad E_{21}^T \quad E_{22}^T \quad \cdots \quad E_{2r}^T]^T, \\
\tilde{D}_{22} &= [D_{221r}^T \quad D_{222r}^T \quad \cdots \quad D_{22r-1,r}^T \quad E_{31}^T \quad E_{32}^T \quad \cdots \quad E_{3r}^T]^T.
\end{aligned}$$

We note that since the state  $x(t)$  is not measurable,  $\lambda_i(x(t))$  is unknown. This implies that  $\tilde{F}(t)$  is a time varying unknown function. However, it is easy to see that  $\tilde{F}(t)$  satisfies  $\tilde{F}^T(t)\tilde{F}(t) \leq I$ , because  $0 \leq \lambda_i(x(t)) \leq 1$ ,  $E_i^T(t)F_i(t) \leq I$ ,  $i = 1, \dots, r$ . Hence, we can see (6) as a linear system with time varying uncertainties.

**Remark 2.1** Representation (6) has less uncertain matrices than that of (Yoneyama, 2008a; Yoneyama, 2008b), which leads to less conservative results.  $H_1$  and  $\tilde{A}$  in (7) are not unique and can be chosen such that  $H_1\tilde{F}(t)\tilde{A} = H_1F(t)E_{1r} + \lambda_1(x(t))[A_{1r} + \bar{H}_{1r}\bar{F}_{1r}(t)\bar{E}_{1r}] + \cdots + \lambda_{r-1}(x(t))[A_{r-1,r} + \bar{H}_{r-1,r}\bar{F}_{r-1,r}(t)\bar{E}_{r-1,r}]$ . This is true for other matrices  $H_i$ ,  $i = 2, 3$  and  $\tilde{B}_1, \tilde{B}_2, \tilde{C}_1, \tilde{C}_2, \tilde{D}_{11}, \tilde{D}_{12}, \tilde{D}_{21}, \tilde{D}_{22}$ .

Now, our problem of finding a robust stabilizing output feedback controller with  $H_\infty$  disturbance attenuation for the system (2) is to find a controller of the form (8) for the uncertain system (6).

$$\begin{aligned}
\dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}y(t), \\
u(t) &= \hat{C}\hat{x}(t).
\end{aligned} \tag{8}$$

**Definition 2.1** (i) Consider the unforced system (6) with  $w(t) = 0$ ,  $u(t) = 0$ . The uncertain system (6) is said to be robustly stable if there exists a matrix  $X > 0$  such that

$$A_\Delta^T X + X A_\Delta < 0$$

for all admissible uncertainties.

(ii) The uncertain system (6) is said to be robustly stabilizable via output feedback controller if there exists an output feedback controller of the form (4) such that the resulting closed-loop system (6) with (4) is robustly stable.

**Definition 2.2** (i) Given a scalar  $\gamma > 0$ , the system (6) is said to be robustly stable with  $H_\infty$  disturbance attenuation  $\gamma$  if there exists a matrix  $X > 0$  such that

$$\begin{bmatrix}
A_\Delta^T X + X A_\Delta & X B_{1\Delta} & C_{1\Delta}^T \\
B_{1\Delta}^T X & -\gamma^2 I & D_{11\Delta}^T \\
C_{1\Delta} & D_{11\Delta} & -I
\end{bmatrix} < 0.$$

(ii) Given a scalar  $\gamma > 0$ , the uncertain system (6) is said to be robustly stabilizable with  $H_\infty$  disturbance attenuation  $\gamma$  via output feedback controller if there exists an output feedback controller of the form (8) such that the resulting closed-loop system (6) with (8) is robustly stable with  $H_\infty$  disturbance attenuation  $\gamma$ .

**Definition 2.3** Given a scalar  $\gamma > 0$ , the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \\ z(t) &= Cx(t) + Dw(t) \end{aligned}$$

is said to be stable with  $H_\infty$  disturbance attenuation  $\gamma$  if it is exponentially stable and input-output stable with (5).

The following lemma is well known, and we need it to prove our main results.

**Lemma 2.1** (Xie, 1996) Given constant matrices  $Q = Q^T$ ,  $H$ ,  $E$  of appropriate dimensions. Suppose a time varying matrix  $F(t)$  satisfies  $F^T(t)F(t) \leq I$ . Then, the following holds

$$Q + E^T F^T(t) H^T + H F(t) E \leq Q + \frac{1}{\varepsilon} H H^T + \varepsilon E^T E$$

for some  $\varepsilon > 0$ .

Now, we state our key results that show the relationship between robust stability with  $H_\infty$  disturbance attenuation and stability with  $H_\infty$  disturbance attenuation.

**Theorem 2.1** The system (6) with  $w(t) = 0$  is robustly stable with  $H_\infty$  disturbance attenuation  $\gamma$  if and only if for  $\varepsilon > 0$  that

$$\begin{aligned} \dot{x}(t) &= A_r x(t) + [\gamma^{-1} B_{1r} \quad \varepsilon^{-1} H_1 \quad 0] \tilde{w}(t), \\ \tilde{z}(t) &= \begin{bmatrix} C_{1r} \\ \varepsilon A \\ \varepsilon \tilde{C}_1 \end{bmatrix} x(t) + \begin{bmatrix} \gamma^{-1} D_{11r} & 0 & \varepsilon^{-1} H_2 \\ \varepsilon \gamma^{-1} \tilde{B}_1 & 0 & 0 \\ \varepsilon \gamma^{-1} \tilde{D}_{11r} & 0 & 0 \end{bmatrix} \tilde{w}(t) \end{aligned}$$

is stable with unitary  $H_\infty$  disturbance attenuation  $\gamma = 1$ .

**Proof:** The system (6) is robustly stable with  $H_\infty$  disturbance attenuation  $\gamma$  if and only if there exists a matrix  $X > 0$  such that

$$\begin{bmatrix} (A_r + H_1 \tilde{F}_1(t) \tilde{A})^T X + X(A_r + H_1 \tilde{F}_1(t) \tilde{A}) & X(B_{1r} + H_1 \tilde{F}_1(t) \tilde{B}_1) & (C_{1r} + H_2 \tilde{F}_2(t) \tilde{C}_1)^T \\ (B_{1r} + H_1 \tilde{F}_1(t) \tilde{B}_1)^T X & -\gamma^2 I & (D_{11r} + H_2 \tilde{F}_2(t) \tilde{D}_{11})^T \\ (C_{1r} + H_2 \tilde{F}_2(t) \tilde{C}_1) & D_{11r} + H_2 \tilde{F}_2(t) \tilde{D}_{11} & -I \end{bmatrix} < 0,$$

which can be written as

$$\hat{A} + \hat{H} \hat{F}(t) \hat{E} + \hat{E}^T \hat{F}^T(t) \hat{H}^T < 0 \tag{9}$$

where

$$\hat{A} = \begin{bmatrix} A_r^T X + X A_r & X B_{1r} & C_{1r}^T \\ B_{1r}^T X & -\gamma^2 I & D_{11r}^T \\ C_{1r} & D_{11r} & -I \end{bmatrix}, \quad \hat{H} = \begin{bmatrix} X H_1 & 0 \\ 0 & 0 \\ 0 & H_2 \end{bmatrix}, \quad \hat{F}(t) = \begin{bmatrix} \tilde{F}_1(t) & 0 \\ 0 & \tilde{F}_2(t) \end{bmatrix}, \quad \hat{E} = \begin{bmatrix} \tilde{A} & \tilde{B}_1 & 0 \\ \tilde{C}_1 & \tilde{D}_{11} & 0 \end{bmatrix}.$$

It can be shown by Lemma 2.1 that there exists  $X > 0$  such that (9) holds if and only if there exist  $X > 0$  and a scalar  $\varepsilon > 0$  such that

$$\hat{A} + \frac{1}{\varepsilon^2} \hat{H} \hat{H}^T + \varepsilon^2 \hat{E}^T \hat{E} < 0,$$

which can be written as

$$\begin{bmatrix} \hat{A} & \varepsilon^{-1} \hat{H} & \varepsilon \hat{E}^T \\ \varepsilon^{-1} \hat{H}^T & -I & 0 \\ \varepsilon \hat{E} & 0 & -I \end{bmatrix} < 0.$$

Pre-multiplying and post-multiplying the above LMI by

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma^{-1} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

and its transpose, respectively, we have

$$\begin{bmatrix} A_r^T X + X A_r & X \tilde{B} & \tilde{C}^T \\ \tilde{B}^T X & -I & \tilde{D}^T \\ \tilde{C} & \tilde{D} & -I \end{bmatrix} < 0.$$

The result follows from Definition 2.1.

**Theorem 2.2** The system (6) with  $w(t) = 0$ ,  $u(t) = 0$  is robustly stable if and only if for  $\varepsilon > 0$  the system

$$\begin{aligned} \dot{x}(t) &= A_r x(t) + \varepsilon^{-1} H_1 \tilde{w}(t), \\ \tilde{z}(t) &= \varepsilon \tilde{A} x(t) \end{aligned}$$

is stable with unitary  $H_\infty$  disturbance attenuation  $\gamma = 1$ .

**Proof:** The system (6) is robustly stable if and only if there exists a matrix  $X > 0$  such that

$$(A_r + H_1 \tilde{F}(t) \tilde{A})^T X + X (A_r + H_1 \tilde{F}(t) \tilde{A}) < 0.$$

It can be shown by Lemma 2.1 that there exists  $X > 0$  such that the above inequality holds if and only if there exist  $X > 0$  and a scalar  $\varepsilon > 0$  such that

$$A_r^T X + X A_r + \frac{1}{\varepsilon^2} X H_1 H_1^T X + \varepsilon^2 \tilde{A}^T \tilde{A} < 0,$$

which can be written as

$$\begin{bmatrix} A_r^T X + X A_r & \varepsilon^{-1} X H_1 & \varepsilon \tilde{A}^T \\ \varepsilon^{-1} H_1^T X & -I & 0 \\ \varepsilon \tilde{A} & 0 & -I \end{bmatrix} < 0.$$

The result follows from Definition 2.1.

### 2.3 Robust controller design

Based on the results in the previous section, we consider the design of robust  $H_\infty$  output feedback controller for the system (2). Such a controller can be designed for a nominal linear system with no uncertainty. For the following two auxiliary systems:

$$\begin{aligned} \dot{x}(t) &= A_r x(t) + [\gamma^{-1} B_{1r} \quad \varepsilon^{-1} H_1 \quad 0 \quad 0] \tilde{w}(t) + B_{2r} u(t), \\ \tilde{z}(t) &= \begin{bmatrix} C_{1r} \\ \varepsilon \tilde{A} \\ \varepsilon \tilde{C}_2 \\ \varepsilon \tilde{C}_1 \end{bmatrix} x(t) + \begin{bmatrix} \gamma^{-1} D_{11r} & 0 & 0 & \varepsilon^{-1} H_2 \\ \varepsilon \gamma^{-1} \tilde{B}_1 & 0 & 0 & 0 \\ \varepsilon \gamma^{-1} \tilde{D}_{21} & 0 & 0 & 0 \\ \varepsilon \gamma^{-1} \tilde{D}_{11} & 0 & 0 & 0 \end{bmatrix} \tilde{w}(t) + \begin{bmatrix} D_{12r} \\ \varepsilon \tilde{B}_2 \\ \varepsilon \tilde{D}_{22} \\ \varepsilon \tilde{D}_{12} \end{bmatrix} u(t), \\ y(t) &= C_{2r} x(t) + [\gamma^{-1} D_{21r} \quad 0 \quad \varepsilon^{-1} H_3 \quad 0] \tilde{w}(t) + D_{22r} u(t), \end{aligned} \tag{10}$$

and

$$\begin{aligned} \dot{x}(t) &= A_r x(t) + [\varepsilon^{-1} H_1 \quad 0] \bar{w}(t) + B_{2r} u(t), \\ \bar{z}(t) &= \begin{bmatrix} \varepsilon \tilde{A} \\ \varepsilon \tilde{C}_2 \end{bmatrix} x(t) + \begin{bmatrix} \varepsilon \tilde{B}_2 \\ \varepsilon \tilde{D}_{22} \end{bmatrix} u(t), \\ y(t) &= C_{2r} x(t) + [0 \quad \varepsilon^{-1} H_3] \bar{w}(t) + D_{22r} u(t) \end{aligned} \tag{11}$$

where  $\varepsilon > 0$  is a scaling parameter, we can show that the following theorems hold.

**Theorem 2.3** The system (6) is robustly stabilizable with  $H_\infty$  disturbance attenuation with  $\gamma$  via the output feedback controller (8) if the closed-loop system corresponding to (10) and (8) is stable with unitary  $H_\infty$  disturbance attenuation.

**Proof:** The closed-loop system (6) with (8) is given by

$$\begin{aligned} \dot{x}_c(t) &= (A_c + H_{1c} \tilde{F}_{1c}(t) E_{1c}) x_c(t) + (B_c + H_{1c} \tilde{F}_{1c}(t) E_{2c}) w(t), \\ z(t) &= (C_c + H_2 \tilde{F}_{2c}(t) E_{3c}) x_c(t) + (D_{11r} + H_2 \tilde{F}_{2c}(t) \tilde{D}_{11}) w(t) \end{aligned}$$

where  $x_c = [x^T \quad \hat{x}^T]^T$  and



$$A_c = \begin{bmatrix} A_r & B_{2r}\hat{C} \\ \hat{B}C_{2r} & \hat{A} + \hat{B}D_{22r}\hat{C} \end{bmatrix}, \quad B_c = \begin{bmatrix} B_{1r} \\ \hat{B}D_{21r} \end{bmatrix}, \quad C_c = [C_{1r} \quad D_{12r}\hat{C}], \quad H_{1c} = \begin{bmatrix} H_1 & 0 \\ 0 & \hat{B}H_3 \end{bmatrix},$$

$$\tilde{F}_{1c}(t) = \begin{bmatrix} \tilde{F}_1(t) & 0 \\ 0 & \tilde{F}_3(t) \end{bmatrix}, \quad E_{1c} = \begin{bmatrix} \tilde{A} & \tilde{B}_2\hat{C} \\ \tilde{C}_2 & \tilde{D}_{22}\hat{C} \end{bmatrix}, \quad E_{2c} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{D}_{21} \end{bmatrix}, \quad E_{3c} = [\tilde{C}_1 \quad \tilde{D}_{12}\hat{C}].$$

On the other hand, the closed-loop system (10) with (8) is given by

$$\dot{x}_c(t) = A_c x_c(t) + [\gamma^{-1}B_c \quad \varepsilon^{-1}H_{1c} \quad 0] \tilde{w}(t),$$

$$\tilde{z}(t) = \begin{bmatrix} C_c \\ \varepsilon E_{1c} \\ \varepsilon E_{3c} \end{bmatrix} x_c(t) + \begin{bmatrix} \gamma^{-1}D_{11r} & 0 & \varepsilon^{-1}H_2 \\ \varepsilon\gamma^{-1}E_{2c} & 0 & 0 \\ \varepsilon\gamma^{-1}\tilde{D}_{11r} & 0 & 0 \end{bmatrix} \tilde{w}(t).$$

The result follows from Theorem 2.1.

Similar to Theorem 2.3, a robust stabilization is obtained from Theorem 2.2 as follows:

**Theorem 2.4** The system (6) with  $w(t) = 0$  is robustly stabilizable via the output feedback controller (8) if the closed-loop system corresponding to (11) and (8) is stable with unitary  $H_\infty$  disturbance attenuation.

**Remark 2.2** Theorem 2.3 shows that a controller that achieves a unitary  $H_\infty$  disturbance attenuation for the nominal system (10) can robustly stabilize the uncertain fuzzy system (6). This implies that the same controller can be used to robustly stabilize the fuzzy system (6). Similarly, Theorem 2.4 indicates that a controller that achieves a unitary  $H_\infty$  disturbance attenuation  $\gamma$ . This leads to the fact that the same controller can be used to achieve the stability and the prescribed  $H_\infty$  disturbance attenuation level of the fuzzy system (6). We also note that research on an  $H_\infty$  output feedback controller design has been extensively investigated, and a design method of  $H_\infty$  controllers has already been given. Therefore, the existing results on stability with  $H_\infty$  disturbance attenuation can be applied to solve the robust  $H_\infty$  output feedback stabilization problem for fuzzy systems with the immeasurable premise variables. In addition, if there exist solutions to Theorems 2.3 and 2.4, then, controllers keep certain robustness.

## 2.4 Numerical examples

We give an illustrative example of designing robust  $H_\infty$  output feedback controller for a continuous-time Takagi-Sugeno fuzzy system with immeasurable premise variables. Let us consider the following continuous-time nonlinear system with uncertain parameters.

$$\begin{aligned} \dot{x}_1(t) &= (-3.2 - a)x_1(t) - 0.6x_2(t) + (0.8 + \beta)x_1(t)x_2(t) + 0.3w_1(t), \\ \dot{x}_2(t) &= 0.4x_1(t) - 1.1x_1(t)x_2(t) + 0.2w_1(t) + 0.7u(t), \\ z(t) &= \begin{bmatrix} 0.3x_1(t) + 0.2x_2(t) \\ 0.1u(t) \end{bmatrix}, \\ y(t) &= 0.3x_1(t) - 0.2x_2(t) + w_2(t) \end{aligned}$$

where  $a$  and  $\beta$  are uncertain scalars which satisfy  $|a| \leq 0.1$  and  $|\beta| \leq 0.3$ , respectively. Defining  $x(t) = [x_1(t) \ x_2(t)]$ ,  $w(t) = [w_1(t) \ w_2(t)]$  and assuming  $x_2(t) \in [-1, 1]$ , we have an equivalent fuzzy system description

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^2 \lambda_i(x_2(t)) \left\{ (A_i + H_{1i}F_i(t)E_{1i})x(t) + \begin{bmatrix} 0.3 & 0 \\ 0.2 & 0 \end{bmatrix} w(t) + \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} u(t) \right\}, \\ z(t) &= \begin{bmatrix} 0.3 & 0.2 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t), \\ y(t) &= [0.3 \ -0.2]x(t) + [0 \ 1]w(t)\end{aligned}$$

where  $\lambda_1(x_2(t)) = (-x_2(t) + 1) / 2$ ,  $\lambda_2(x_2(t)) = (x_2(t) + 1) / 2$  and

$$\begin{aligned}A_1 &= \begin{bmatrix} -2.4 & -0.6 \\ -1.1 & 0.4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & -0.6 \\ 1.1 & 0.4 \end{bmatrix}, \quad H_{11} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad H_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ F_1(t) &= \frac{a}{|a|}, \quad F_2(t) = \frac{\beta}{|\beta|}, \quad E_{11} = [0.1 \ 0], \quad E_{12} = [0.1 \ 0]\end{aligned}$$

which can be written as

$$\begin{aligned}\dot{x}(t) &= (A_2 + H_1 \tilde{F}_1(t) \tilde{A})x(t) + \begin{bmatrix} 0.3 & 0 \\ 0.2 & 0 \end{bmatrix} w(t) + \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} u(t), \\ z(t) &= \begin{bmatrix} 0.3 & 0.2 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t), \\ y(t) &= [0.3 \ -0.2]x(t) + [0 \ 1]w(t)\end{aligned}$$

where

$$H_1 = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \tilde{F}_1(t) = \text{diag}[\lambda_1(t) \ \lambda_1(t) \ F_1(t) \ F_2(t)], \quad \tilde{A} = \begin{bmatrix} 1.6 & 0 \\ -2.2 & 0 \\ 0.1 & 0 \\ 0.1 & 0 \end{bmatrix}$$

It is easy to see that the original system with  $u(t) = 0$  is unstable. Theorem 2.3 allows us to design a robust stabilizing controller with  $H_\infty$  disturbance attenuation  $\gamma = 10$ :

$$\begin{aligned}\dot{\hat{x}}(t) &= \begin{bmatrix} -24.6351 & -4.8920 \\ 8.5631 & -13.3744 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 48.7026 \\ -32.8950 \end{bmatrix} y(t), \\ u(t) &= [-37.4520 \ -90.1612] \hat{x}(t)\end{aligned}$$

This controller is applied to the system. A simulation result with the initial conditions  $x(0) = [-0.5 \ 0.6]^T$ ,  $\hat{x}(0) = [0 \ 0]^T$ , the noises  $w_1(t) = w_2(t) = 0.1e^{-0.2t} \sin(0.1t)$  and the assumption  $F_1(t) = F_2(t) = \sin(10t)$  is depicted in Figures 1 and 2, which show the trajectories of the state and control, respectively. It is easy to see that the obtained controller stabilizes the nonlinear system.

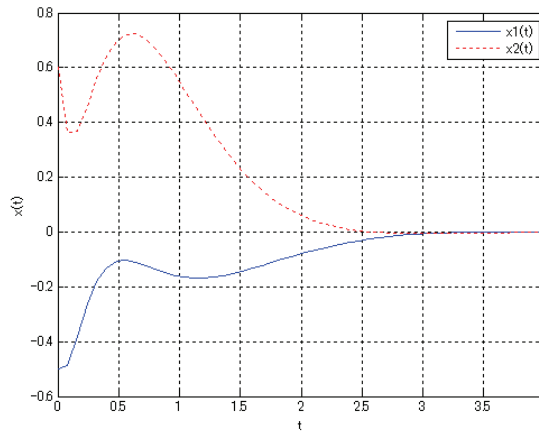


Fig. 1. The state trajectories

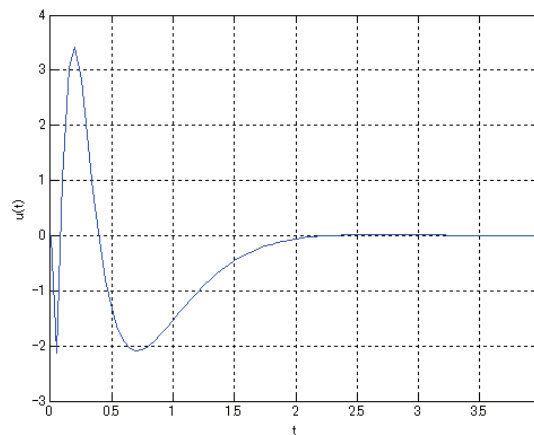


Fig. 2. The control trajectory

### 3. Robust output feedback control for uncertain discrete-time fuzzy systems with immeasurable premise variables

We now consider the discrete-time counterpart of the previous section. We first show that a discrete-time Takagi-Sugeno fuzzy system with immeasurable premise variables can be written as an uncertain discrete-time linear system. Then, robust stability with  $H_\infty$  disturbance attenuation problem for such an uncertain system is converted into stability with  $H_\infty$  disturbance attenuation for a nominal system. Based on such a relationship, a solution to a robust stability with  $H_\infty$  disturbance attenuation problem via output feedback controller for a discrete-time fuzzy system with immeasurable premise variables is given. Finally, a numerical example illustrates our theory.

### 3.1 Fuzzy system and problem formulation

In this section, we consider discrete-time Takagi-Sugeno fuzzy systems with immeasurable premise variables. The Takagi-Sugeno fuzzy model is described by the following IF-THEN rules:

$$\begin{aligned}
 \text{IF} \quad & \xi_1 \text{ is } M_{i1} \text{ and } \dots \text{ and } \xi_p \text{ is } M_{ip}, \\
 \text{THEN} \quad & x(k+1) = (A_i + \Delta A_i)x(k) + (B_{1i} + \Delta B_{1i})w(k) + (B_{2i} + \Delta B_{2i})u(k), \\
 & z(k) = (C_{1i} + \Delta C_{1i})x(k) + (D_{11i} + \Delta D_{11i})w(k) + (D_{12i} + \Delta D_{12i})u(k), \\
 & y(k) = (C_{2i} + \Delta C_{2i})x(k) + (D_{21i} + \Delta D_{21i})w(k) + (D_{22i} + \Delta D_{22i})u(k), \quad i = 1, \dots, r
 \end{aligned}$$

where  $x(k) \in \mathfrak{R}^n$  is the state,  $w(k) \in \mathfrak{R}^{m_1}$  is the disturbance,  $u(k) \in \mathfrak{R}^{m_2}$  is the control input,  $z(k) \in \mathfrak{R}^{q_1}$  is the controlled output,  $y(k) \in \mathfrak{R}^{q_2}$  is the measurement output.  $r$  is the number of IF-THEN rules.  $M_{ij}$  are fuzzy sets and  $\xi_1, \dots, \xi_p$  are premise variables. We set  $\xi = [\xi_1 \dots \xi_p]^T$  and  $\xi(k)$  is assumed to be a function of  $x(k)$ , which is not measurable. We assume that the premise variables do not depend on  $u(k)$ , and that  $A_i, B_{1i}, B_{2i}, C_{1i}, C_{2i}, D_{11i}, D_{12i}, D_{21i}$  and  $D_{22i}$  are constant matrices of appropriate dimensions. The uncertain matrices are of the form

$$\begin{bmatrix} \Delta A_i & \Delta B_{1i} & \Delta B_{2i} \\ \Delta C_{1i} & \Delta D_{11i} & \Delta D_{12i} \\ \Delta C_{2i} & \Delta D_{21i} & \Delta D_{22i} \end{bmatrix} = \begin{bmatrix} H_{1i} \\ H_{2i} \\ H_{3i} \end{bmatrix} F_i(k) \begin{bmatrix} E_{1i} & E_{2i} & E_{3i} \end{bmatrix}, \quad i = 1, \dots, r$$

where  $F_i(k) \in \mathfrak{R}^{l \times j}$  is an uncertain matrix satisfying  $F_i^T(k)F_i(k) \leq I$ , and  $H_{1i}, H_{2i}, H_{3i}, E_{1i}, E_{2i}$  and  $E_{3i}$  are known constant matrices of appropriate dimensions.

**Assumption 3.1** The system  $(A_r, B_{1r}, B_{2r}, C_{1r}, C_{2r}, D_{11r}, D_{12r}, D_{21r}, D_{22r})$  represents a nominal system that can be chosen as a subsystem including the equilibrium point of the original system.

$\xi(k)$  is assumed to be a function of the immeasurable state  $x(k)$ . The state, the controlled output and the measurement output equations are defined as follows:

$$\begin{aligned}
 x(k+1) &= \sum_{i=1}^r \lambda_i(\xi(k)) \{ (A_i + \Delta A_i)x(k) + (B_{1i} + \Delta B_{1i})w(k) + (B_{2i} + \Delta B_{2i})u(k) \}, \\
 z(k) &= \sum_{i=1}^r \lambda_i(\xi(k)) \{ (C_{1i} + \Delta C_{1i})x(k) + (D_{11i} + \Delta D_{11i})w(k) + (D_{12i} + \Delta D_{12i})u(k) \}, \quad (12) \\
 y(k) &= \sum_{i=1}^r \lambda_i(\xi(k)) \{ (C_{2i} + \Delta C_{2i})x(k) + (D_{21i} + \Delta D_{21i})w(k) + (D_{22i} + \Delta D_{22i})u(k) \}
 \end{aligned}$$

where

$$\lambda_i(\xi) = \frac{\beta_i(\xi)}{\sum_{i=1}^r \beta_i(\xi)}, \quad \beta_i(\xi) = \prod_{j=1}^p M_{ij}(\xi_j)$$

and  $M_{ij}(\cdot)$  is the grade of the membership function of  $M_{ij}$ . We assume

$$\beta_i(\xi(k)) \geq 0, \quad i = 1, \dots, r, \quad \sum_{i=1}^r \beta_i(\xi(k)) > 0$$

for any  $\xi(k)$ . Hence  $\lambda_i(\xi(k))$  satisfies

$$\lambda_i(\xi(k)) \geq 0, \quad i = 1, \dots, r, \quad \sum_{i=1}^r \lambda_i(\xi(k)) = 1 \quad (13)$$

for any  $\xi(k)$ . Our problem is to find a control  $u(\cdot)$  for the system (1) given the output measurements  $y(\cdot)$  such that the controlled output  $z(\cdot)$  satisfies

$$\sum_{i=0}^{\infty} z^T(i)z(i) < \gamma^2 \sum_{i=1}^{\infty} w^T(i)w(i) \quad (14)$$

for a prescribed scalar  $\gamma > 0$ . If such a controller exists, we call it an  $H_{\infty}$  output feedback controller. Here, we consider the robust stabilization and the robust  $H_{\infty}$  disturbance attenuation of uncertain fuzzy system (9) with the immeasurable premise variable.

### 3.2 Robust stability of equivalent uncertain systems

Similar to continuous-time case, we rewrite (12) as an uncertain system. Then, the robust stability with  $H_{\infty}$  disturbance attenuation problem of fuzzy system is converted as the robust stability and robust stability with  $H_{\infty}$  disturbance attenuation problems of equivalent uncertain system. Since  $\lambda_i(x(k))$  satisfies (13), we have

$$\lambda_r(x(k)) = 1 - [\lambda_1(x(k)) + \lambda_2(x(k)) + \dots + \lambda_{r-1}(x(k))].$$

It follows from this relationship that

$$\begin{aligned} \sum_{i=1}^r \lambda_i(x(k))(A_i + \Delta A_i) &= A_r + H_{1r}F(k)E_{1r} + \lambda_1(x(k))[A_{1r} + \bar{H}_{1r}\bar{F}_{1r}(k)\bar{E}_{1r}] \\ &\quad + \dots + \lambda_{r-1}(x(k))[A_{r-1,r} + \bar{H}_{r-1,r}\bar{F}_{r-1,r}(k)\bar{E}_{r-1,r}] \\ &= A_r + H_1\tilde{F}(k)\tilde{A} \end{aligned}$$

where

$$\begin{aligned} A_{ir} &= A_i - A_r, \quad i = 1, \dots, r-1 \\ H_1 &= [I \quad I \quad \dots \quad I \quad H_{11} \quad H_{12} \quad \dots \quad H_{1r}], \\ \tilde{F}(k) &= \text{diag} [\lambda_1(k)I \quad \dots \quad \lambda_{r-1}(k)I \quad \lambda_1(t)E_1(k) \quad \dots \quad \lambda_r(t)E_r(k)], \\ \tilde{A} &= [A_{1r}^T \quad A_{2r}^T \quad \dots \quad A_{r-1,r}^T \quad E_{11}^T \quad E_{12}^T \quad \dots \quad E_{1r}^T]^T. \end{aligned}$$

Similarly, we rewrite  $\sum_{i=1}^r \lambda_i(x(k))(B_{1i} + \Delta B_{1i})$ ,  $\sum_{i=1}^r \lambda_i(x(k))(B_{2i} + \Delta B_{2i})$ ,  $\sum_{i=1}^r \lambda_i(x(k))(C_{1i} + \Delta C_{1i})$ ,  $\sum_{i=1}^r \lambda_i(x(k))(C_{2i} + \Delta C_{2i})$ ,  $\sum_{i=1}^r \lambda_i(x(k))(D_{11i} + \Delta D_{11i})$ ,  $\sum_{i=1}^r \lambda_i(x(k))(D_{12i} + \Delta D_{12i})$ ,  $\sum_{i=1}^r \lambda_i(x(k))(D_{21i} + \Delta D_{21i})$  and  $\sum_{i=1}^r \lambda_i(x(k))(D_{22i} + \Delta D_{22i})$ , and we have an equivalent description for (12):

$$\begin{aligned}
x(k+1) &= (A_r + \Delta A)x(k) + (B_{1r} + \Delta B_1)w(k) + (B_{2r} + \Delta B_2)u(k) \\
&\triangleq A_\Delta x(k) + B_{1\Delta}w(k) + B_{2\Delta}u(k), \\
z(k) &= (C_{1r} + \Delta C_1)x(k) + (D_{11r} + \Delta D_{11})w(k) + (D_{12r} + \Delta D_{12})u(k) \\
&\triangleq C_{1\Delta}x(k) + D_{11\Delta}w(k) + D_{12\Delta}u(k), \\
y(k) &= (C_{2r} + \Delta C_2)x(k) + (D_{21r} + \Delta D_{21})w(k) + (D_{22r} + \Delta D_{22})u(k) \\
&\triangleq C_{2\Delta}x(k) + D_{21\Delta}w(k) + D_{22\Delta}u(k)
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
\begin{bmatrix} \Delta A & \Delta B_1 & \Delta B_2 \\ \Delta C_1 & \Delta D_{11} & \Delta D_{12} \\ \Delta C_2 & \Delta D_{21} & \Delta D_{22} \end{bmatrix} &= \begin{bmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{bmatrix} \begin{bmatrix} \tilde{F}(k) & 0 & 0 \\ 0 & \tilde{F}(k) & 0 \\ 0 & 0 & \tilde{F}(k) \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix}, \\
H_2 &= [I \quad I \quad \cdots \quad I \quad H_{21} \quad H_{22} \quad \cdots \quad H_{2r}], \\
H_3 &= [I \quad I \quad \cdots \quad I \quad H_{31} \quad H_{32} \quad \cdots \quad H_{3r}], \\
B_{1ir} &= B_{1i} - B_{1r}, \quad B_{2ir} = B_{2i} - B_{2r}, \quad C_{1ir} = C_{1i} - C_{1r}, \\
C_{2ir} &= C_{2i} - C_{2r}, \quad D_{11ir} = D_{11i} - D_{11r}, \quad D_{12ir} = D_{12i} - D_{12r}, \\
D_{21ir} &= D_{21i} - D_{21r}, \quad D_{22ir} = D_{22i} - D_{22r}, \quad i = 1, \dots, r-1, \\
\tilde{B}_1 &= [B_{11r}^T \quad B_{12r}^T \quad \cdots \quad B_{1r-1,r}^T \quad E_{21}^T \quad E_{22}^T \quad \cdots \quad E_{2r}^T]^T, \\
\tilde{B}_2 &= [B_{21r}^T \quad B_{22r}^T \quad \cdots \quad B_{2r-1,r}^T \quad E_{31}^T \quad E_{32}^T \quad \cdots \quad E_{3r}^T]^T, \\
\tilde{C}_1 &= [C_{11r}^T \quad C_{12r}^T \quad \cdots \quad C_{1r-1,r}^T \quad E_{11}^T \quad E_{12}^T \quad \cdots \quad E_{1r}^T]^T, \\
\tilde{C}_2 &= [C_{21r}^T \quad C_{22r}^T \quad \cdots \quad C_{2r-1,r}^T \quad E_{11}^T \quad E_{12}^T \quad \cdots \quad E_{1r}^T]^T, \\
\tilde{D}_{11} &= [D_{111r}^T \quad D_{112r}^T \quad \cdots \quad D_{11r-1,r}^T \quad E_{21}^T \quad E_{22}^T \quad \cdots \quad E_{2r}^T]^T, \\
\tilde{D}_{12} &= [D_{121r}^T \quad D_{122r}^T \quad \cdots \quad D_{12r-1,r}^T \quad E_{31}^T \quad E_{32}^T \quad \cdots \quad E_{3r}^T]^T, \\
\tilde{D}_{21} &= [D_{211r}^T \quad D_{212r}^T \quad \cdots \quad D_{21r-1,r}^T \quad E_{21}^T \quad E_{22}^T \quad \cdots \quad E_{2r}^T]^T, \\
\tilde{D}_{22} &= [D_{221r}^T \quad D_{222r}^T \quad \cdots \quad D_{22r-1,r}^T \quad E_{31}^T \quad E_{32}^T \quad \cdots \quad E_{3r}^T]^T.
\end{aligned}$$

We note that since the state  $x(k)$  is not measurable,  $\lambda_i(x(k))$  are unknown. This implies that  $\tilde{F}(k)$  is a time varying unknown function. However, it is easy to see that  $\tilde{F}(k)$  satisfies  $\tilde{F}^T(k)\tilde{F}(k) \leq I$ , because  $0 \leq \lambda_i(x(k)) \leq 1$ ,  $F_i^T(k)F_i(k) \leq I$ ,  $i = 1, 2, 3$ . Hence, with Assumption 3.1, we can see (12) as a linear nominal system with time varying uncertainties. The importance on this description is that the system (15) is exactly the same as the system (12). Now, our problem of finding an  $H_\infty$  controller for the system (12) is to find a robustly stabilizing output feedback controller with  $H_\infty$  disturbance attenuation of the form (16) for the uncertain system (15).

$$\begin{aligned}
\hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}y(k), \\
u(k) &= \hat{C}\hat{x}(k).
\end{aligned} \tag{16}$$

**Definition 3.1** (i) Consider the unforced system (12) with  $w(k) = 0$ ,  $u(k) = 0$ . The uncertain system (12) is said to be robustly stable if there exists a matrix  $X > 0$  such that

$$A_{\Delta}^T X A_{\Delta} - X < 0$$

for all admissible uncertainties.

(ii) The uncertain system (12) is said to be robustly stabilizable via output feedback controller if there exists an output feedback controller of the form (16) such that the resulting closed-loop system (12) with (16) is robustly stable.

**Definition 3.2** (i) Given a scalar  $\gamma > 0$ , the system (12) is said to be robustly stable with  $H_{\infty}$  disturbance attenuation  $\gamma$  if there exists a matrix  $X > 0$  such that

$$\begin{bmatrix} -X & 0 & A_{\Delta}^T & C_{1\Delta}^T \\ 0 & -\gamma^2 I & B_{1\Delta}^T & D_{11\Delta}^T \\ A_{\Delta} & B_{1\Delta} & -X^{-1} & 0 \\ C_{1\Delta} & D_{11\Delta} & 0 & -I \end{bmatrix} < 0.$$

(ii) Given a scalar  $\gamma > 0$ , the uncertain system (12) is said to be robustly stabilizable with  $H_{\infty}$  disturbance attenuation  $\gamma$  via output feedback controller if there exists an output feedback controller of the form (16) such that the resulting closed-loop system (12) with (16) is robustly stable with  $H_{\infty}$  disturbance attenuation  $\gamma$ .

The robust stability and the robust stability with  $H_{\infty}$  disturbance attenuation are converted into the stability with  $H_{\infty}$  disturbance attenuation.

**Definition 3.3** Given a scalar  $\gamma > 0$ , the system

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k), \\ z(k) &= Cx(k) + Dw(k) \end{aligned} \quad (17)$$

is said to be stable with  $H_{\infty}$  disturbance attenuation  $\gamma$  if it is exponentially stable and input-output stable with (14).

Now, we state our key results that show the relationship between the robust stability and the robust stability with  $H_{\infty}$  disturbance attenuation of an uncertain system, and stability with  $H_{\infty}$  disturbance attenuation of a nominal system.

**Theorem 3.1** The system (12) with  $w(k) = 0$  is robustly stable if and only if for  $\varepsilon > 0$  the system

$$\begin{aligned} x(k+1) &= A_r x(k) + \varepsilon^{-1} H_1 \bar{w}(k), \\ \bar{z}(k) &= \varepsilon \tilde{A} x(k) \end{aligned}$$

where  $\bar{w}$  and  $\bar{z}$  are of appropriate dimensions, is stable with unitary  $H_{\infty}$  disturbance attenuation  $\gamma = 1$ .

**Proof:** The system (12) is robustly stable if and only if there exists a matrix  $X > 0$  such that

$$(A_r + H_1 F_1(k) \tilde{A})^T X (A_r + H_1 F_1(k) \tilde{A}) - X < 0,$$

which can be written as

$$Q + HF_1(k)E + E^T F_1^T(k)H^T < 0 \quad (18)$$

where

$$Q = \begin{bmatrix} -X & A_r^T \\ A_r & -X^{-1} \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ H_1 \end{bmatrix}, \quad E = [\tilde{A} \quad 0].$$

It follows from Lemma 2.1 that there exists  $X > 0$  such that (18) holds if and only if there exist a matrix  $X > 0$  and a scalar  $\varepsilon > 0$  such that

$$Q + \frac{1}{\varepsilon^2} HH^T + \varepsilon^2 E^T E < 0,$$

which can be written as

$$Y_1 = \begin{bmatrix} Q & \varepsilon^{-1}H & \varepsilon E^T \\ \varepsilon^{-1}H^T & -I & 0 \\ \varepsilon E & 0 & -I \end{bmatrix} < 0.$$

Pre-multiplying and post-multiplying

$$S_1 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

we have

$$S_1 Y_1 S_1 = \begin{bmatrix} -X & 0 & A_r^T & \varepsilon \tilde{A}^T \\ 0 & -I & \varepsilon^{-1} H_1^T & 0 \\ A_r & \varepsilon^{-1} H_1 & -X^{-1} & 0 \\ \varepsilon A & 0 & 0 & -I \end{bmatrix} < 0.$$

The desired result follows from Definition 3.1.

**Theorem 3.2** The system (12) with  $u(k)=0$  is robustly stable with  $H_\infty$  disturbance attenuation  $\gamma$  if and only if for  $\varepsilon > 0$  the system

$$\begin{aligned} x(k+1) &= A_r x(k) + [\gamma^{-1} B_{1r} \quad \varepsilon^{-1} H_1 \quad 0] \tilde{w}(k), \\ \tilde{z}(k) &= \begin{bmatrix} C_{1r} \\ \varepsilon \tilde{A} \\ \varepsilon \tilde{C}_1 \end{bmatrix} x(k) + \begin{bmatrix} \gamma^{-1} D_{11r} & 0 & \varepsilon^{-1} H_2 \\ \varepsilon \gamma^{-1} \tilde{B}_1 & 0 & 0 \\ \varepsilon \gamma^{-1} \tilde{D}_{11r} & 0 & 0 \end{bmatrix} \tilde{w}(k) \end{aligned}$$

where  $\tilde{w}$  and  $\tilde{z}$  are of appropriate dimensions, is stable with unitary  $H_\infty$  disturbance attenuation  $\gamma = 1$ .



**Proof:** The system (12) is robustly stable with  $H_\infty$  disturbance attenuation  $\gamma$  if and only if there exists a matrix  $X > 0$  such that

$$\begin{bmatrix} -X & 0 & (A_r + H_1 F_1(k) \tilde{A})^T & (C_{1r} + H_2 F_2(k) \tilde{C}_1)^T \\ 0 & -\gamma^2 I & (B_{1r} + H_1 F_1(k) \tilde{B}_1)^T & (D_{11r} + H_2 F_2(k) \tilde{D}_{11})^T \\ A_r + H_1 F_1(k) \tilde{A} & B_{1r} + H_1 F_1(k) \tilde{B}_1 & -X^{-1} & 0 \\ C_{1r} + H_2 F_2(k) \tilde{C}_1 & D_{11r} + H_2 F_2(k) \tilde{D}_{11} & 0 & -I \end{bmatrix} < 0,$$

which can be written as

$$\hat{Q} + \hat{H} \hat{F}(k) \hat{E} + \hat{E}^T \hat{F}^T(k) \hat{H}^T < 0 \quad (19)$$

where

$$\hat{Q} = \begin{bmatrix} -X & 0 & A_r^T & C_{1r}^T \\ 0 & -\gamma^2 I & B_{1r}^T & D_{11r}^T \\ A_r & B_{1r} & -X^{-1} & 0 \\ C_{1r} & D_{11r} & 0 & -I \end{bmatrix}, \quad \hat{H} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ H_1 & 0 \\ 0 & H_2 \end{bmatrix}, \quad \hat{F}(k) = \begin{bmatrix} F_1(k) & 0 \\ 0 & F_2(k) \end{bmatrix}, \quad \hat{E} = \begin{bmatrix} \tilde{A} & \tilde{B}_1 & 0 & 0 \\ \tilde{C}_1 & \tilde{D}_{11} & 0 & 0 \end{bmatrix}.$$

It can be shown from Lemma 2.1 that there exists  $X > 0$  such that (19) holds if and only if there exist  $X > 0$  and a scalar  $\varepsilon > 0$  such that

$$\hat{Q} + \frac{1}{\varepsilon^2} \hat{H} \hat{H}^T + \varepsilon^2 \hat{E}^T \hat{E} < 0,$$

which can be written as

$$Y_2 = \begin{bmatrix} \hat{Q} & \varepsilon^{-1} \hat{H} & \varepsilon \hat{E}^T \\ \varepsilon^{-1} \hat{H}^T & -I & 0 \\ \varepsilon \hat{E} & 0 & -I \end{bmatrix} < 0.$$

Pre-multiplying and post-multiplying the above LMI by

$$S_2 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma^{-1} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

we have

$$S_2 Y_2 S_2 = \begin{bmatrix} -X & 0 & A_r^T & \tilde{C}^T \\ 0 & -I & \tilde{B}^T & \tilde{D}^T \\ A_r & \tilde{B} & -X^{-1} & 0 \\ \tilde{C} & \tilde{D} & 0 & -I \end{bmatrix} < 0.$$

The result follows from Definition 3.1.

### 3.3 Robust controller design

We are now at the position where we propose the control design of an  $H_\infty$  output feedback controller for the system (12). The controller design is based on the equivalent system (15). The design of a robustly stabilizing output feedback controller with  $H_\infty$  disturbance attenuation for the system (15) can be converted into that of a stabilizing controller with  $H_\infty$  disturbance attenuation controllers for a nominal system. For the following auxiliary systems, we can show that the following theorems hold. Consider the following systems:

$$\begin{aligned} x(k+1) &= A_r x(k) + [\gamma^{-1} B_{1r} \quad \varepsilon^{-1} H_1 \quad 0 \quad 0] \tilde{w}(k) + B_{2r} u(k), \\ \tilde{z}(k) &= \begin{bmatrix} C_{1r} \\ \varepsilon \tilde{A} \\ \varepsilon \tilde{C}_2 \\ \varepsilon \tilde{C}_1 \end{bmatrix} x(k) + \begin{bmatrix} \gamma^{-1} D_{11r} & 0 & 0 & \varepsilon^{-1} H_2 \\ \varepsilon \gamma^{-1} \tilde{B}_1 & 0 & 0 & 0 \\ \varepsilon \gamma^{-1} \tilde{D}_{21} & 0 & 0 & 0 \\ \varepsilon \gamma^{-1} \tilde{D}_{11} & 0 & 0 & 0 \end{bmatrix} \tilde{w}(k) + \begin{bmatrix} D_{12r} \\ \varepsilon \tilde{B}_2 \\ \varepsilon \tilde{D}_{22} \\ \varepsilon \tilde{D}_{12} \end{bmatrix} u(k), \\ y(k) &= C_{2r} x(k) + [\gamma^{-1} D_{21r} \quad 0 \quad \varepsilon^{-1} H_3 \quad 0] \tilde{w}(k) + D_{22r} u(k) \end{aligned} \tag{20}$$

and

$$\begin{aligned} x(k+1) &= A_r x(k) + [\varepsilon^{-1} H_1 \quad 0] \bar{w}(k) + B_{2r} u(k), \\ \bar{z}(k) &= \begin{bmatrix} \varepsilon \tilde{A} \\ \varepsilon \tilde{C}_2 \end{bmatrix} x(k) + \begin{bmatrix} \varepsilon \tilde{B}_2 \\ \varepsilon \tilde{D}_{22} \end{bmatrix} u(k), \\ y(k) &= C_{2r} x(k) + [0 \quad \varepsilon^{-1} H_3] \bar{w}(k) + D_{22r} u(k), \end{aligned} \tag{21}$$

where  $\varepsilon > 0$  is a scaling parameter.

**Theorem 3.3** The system (12) is robustly stabilizable with  $H_\infty$  disturbance attenuation with  $\gamma$  via the output feedback controller (16) if the closed-loop system corresponding to (20) and (16) is stable with unitary  $H_\infty$  disturbance attenuation.

**Proof:** The closed-loop system (12) with (16) is given by

$$\begin{aligned} x_c(k+1) &= (A_c + H_{1c} F_{1c}(k) E_{1c}) x_c(k) + (B_c + H_{1c} F_{1c}(k) E_{2c}) w(k), \\ z(k) &= (C_c + H_2 F_2(k) E_{3c}) x(k) + (D_{11r} + H_2 F_2(k) \tilde{D}_{11}) w(k). \end{aligned}$$

where  $x_c = [x^T \quad \hat{x}^T]^T$  and

$$A_c = \begin{bmatrix} A_r & B_{2r} \hat{C} \\ \hat{B} C_{2r} & \hat{A} + \hat{B} D_{22r} \hat{C} \end{bmatrix}, \quad B_c = \begin{bmatrix} B_{1r} \\ \hat{B} D_{21r} \end{bmatrix}, \quad C_c = [C_{1r} \quad D_{12r} \hat{C}], \quad H_{1c} = \begin{bmatrix} H_1 & 0 \\ 0 & \hat{B} H_3 \end{bmatrix},$$

$$F_{1c}(k) = \begin{bmatrix} F_1(k) & 0 \\ 0 & F_3(k) \end{bmatrix}, \quad E_{1c} = \begin{bmatrix} \tilde{A} & \tilde{B}_2 \hat{C} \\ \tilde{C}_2 & \tilde{D}_{22} \hat{C} \end{bmatrix}, \quad E_{2c} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{D}_{21} \end{bmatrix}, \quad E_{3c} = [\tilde{C}_1 \quad \tilde{D}_{12} \hat{C}].$$

On the other hand, the closed-loop system (20) with (17) is given by

$$x_c(k+1) = A_c x_c(k) + [\gamma^{-1} B_c \quad \varepsilon^{-1} H_{1c} \quad 0] \tilde{w}(k),$$

$$\tilde{z}(k) = \begin{bmatrix} C_c \\ \varepsilon E_{1c} \\ \varepsilon E_{3c} \end{bmatrix} x_c(k) + \begin{bmatrix} \gamma^{-1} D_{11r} & 0 & \varepsilon^{-1} H_2 \\ \varepsilon \gamma^{-1} E_{2c} & 0 & 0 \\ \varepsilon \gamma^{-1} \tilde{D}_{11r} & 0 & 0 \end{bmatrix} \tilde{w}(k).$$

The result follows from Theorem 3.2.

Similar to Theorem 3.3, a robust stabilization is obtained from Theorem 3.2 as follows:

**Theorem 3.4** The system (12) with  $w(k) = 0$  is robustly stabilizable via the output feedback controller (16) if the closed-loop system corresponding to (21) and (16) is stable with unitary  $H_\infty$  disturbance attenuation.

**Remark 3.1** Theorem 3.3 indicates that a controller that achieves a unitary  $H_\infty$  disturbance attenuation for the nominal system (20) can robustly stabilize the fuzzy system (12) with  $H_\infty$  disturbance attenuation  $\gamma$ . Similar argument can be made on robust stabilization of Theorem 3.4. Therefore, the existing results on stability with  $H_\infty$  disturbance attenuation can be applied to solve our main problems.

### 3.4 Numerical examples

Now, we illustrate a control design of a simple discrete-time Takagi-Sugeno fuzzy system with immeasurable premise variables. We consider the following nonlinear system with uncertain parameters.

$$x_1(k+1) = (0.9 + a)x_1(k) - 0.2x_2(k) + 0.2x_1(k)x_2^2(k) + 0.3w_1(k),$$

$$x_2(k+1) = 0.2x_1(k) - (0.4 + \beta)x_2^3(k) + 0.5w_1(k) + 0.7u(k),$$

$$z(k) = \begin{bmatrix} 1.5x_1(k) + 0.5x_2(k) \\ u(k) \end{bmatrix},$$

$$y(k) = 0.3x_1(k) - 0.1x_2(k) + w_2(k)$$

where  $a$  and  $\beta$  are uncertain scalars which satisfy  $|a| \leq 0.1$  and  $|\beta| \leq 0.02$ , respectively. Defining  $x(k) = [x_1(k) \quad x_2(k)]$ ,  $w(k) = [w_1(k) \quad w_2(k)]$  and assuming  $x_2(k) \in [-1, 1]$ , we have an equivalent fuzzy system description

$$x_1(k+1) = \sum_{i=1}^2 \lambda_i(x_2(k)) \left\{ (A_i + H_{1i} F_i(k) E_{1i}) x(k) + \begin{bmatrix} 0.3 & 0 \\ 0.5 & 0 \end{bmatrix} w(k) + \begin{bmatrix} 0 \\ 0.7 \end{bmatrix} u(k) \right\},$$

$$z(k) = \begin{bmatrix} 1.5 & 0.5 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k),$$

$$y(k) = [0.3 \quad -0.1] x(k) + [0 \quad 1] w(k)$$

where  $\lambda_1(x_2(k)) = 1 - x_2^2(k)$ ,  $\lambda_2(x_2(k)) = x_2^2(k)$  and

$$A_1 = \begin{bmatrix} 0.9 & -0.2 \\ 0.2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.1 & -0.2 \\ 0.2 & -0.3 \end{bmatrix}, \quad H_{11} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \quad H_{12} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix},$$

$$F_1(k) = \frac{a}{|a|}, \quad F_2(k) = \frac{\beta}{|\beta|}, \quad E_{11} = [0.2 \ 0], \quad E_{12} = [0 \ 0.1],$$

which can be written as

$$x_1(k+1) = (A_2 + H_1 \tilde{F}(k) \tilde{A})x(k) + \begin{bmatrix} 0.3 & 0 \\ 0.5 & 0 \end{bmatrix}w(k) + \begin{bmatrix} 0 \\ 0.7 \end{bmatrix}u(k),$$

$$z(k) = \begin{bmatrix} 1.5 & 0.5 \\ 0 & 0 \end{bmatrix}x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k),$$

$$y(k) = [0.3 \ -0.1]x(k) + [0 \ 1]w(k)$$

where

$$H_1 = \begin{bmatrix} 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0.2 \end{bmatrix}, \quad \tilde{F}(k) = \text{diag}[\lambda_1(k) \ \lambda_1(k) \ F_1(k) \ F_2(k)], \quad \tilde{A} = \begin{bmatrix} -0.2 & 0 \\ 0 & 0.3 \\ 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

The open-loop system is originally unstable. Theorem 3.3 allows us to design a robust stabilizing controller with  $H_\infty$  disturbance attenuation  $\gamma = 20$  :

$$\hat{x}(k+1) = \begin{bmatrix} 0.0971 & -0.0063 \\ 2.3181 & -0.1348 \end{bmatrix}\hat{x}(k) + \begin{bmatrix} 3.8463 \\ 7.9861 \end{bmatrix}y(k),$$

$$u(k) = [3.0029 \ -0.1726]\hat{x}(k)$$

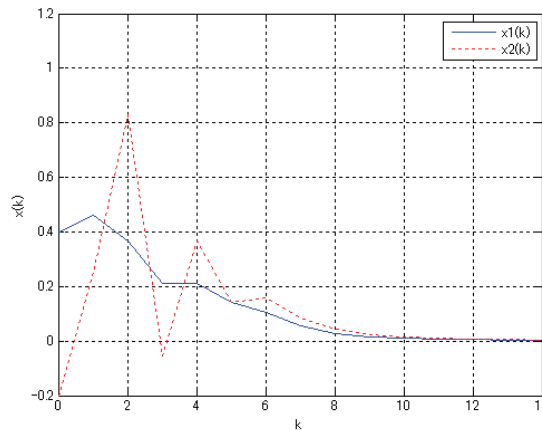


Fig. 3. The state trajectories

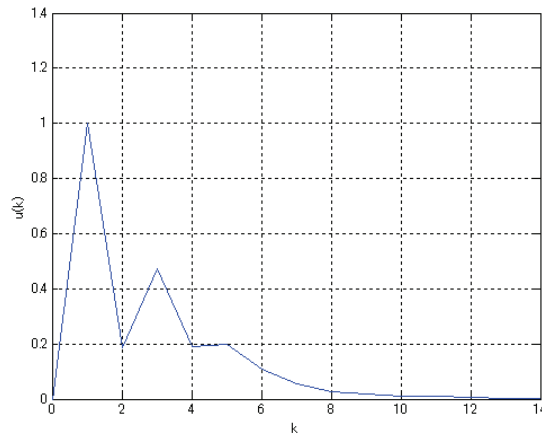


Fig. 4. The control trajectory

This controller is applied to the system. A simulation result with the initial conditions  $x(0) = [0.4 \ -0.3]^T$ ,  $\hat{x}(0) = [0 \ 0]^T$ , the noises  $w_1(k) = e^{-k} \cos(k)$ ,  $w_2(k) = e^{-k} \sin(k)$  and the assumption  $F_1(k) = F_2(k) = \sin(k)$  is depicted in Figures 3 and 4, which show the trajectories of the state and control, respectively. We easily see that the obtained controller stabilizes the system.

#### 4. Extension to fuzzy time-delay systems

In this section, we consider an extension to robust control problems for Takagi-Sugeno fuzzy time-delay systems. Consider the Takagi-Sugeno fuzzy model, described by the following IF-THEN rule:

**IF**  $\xi_1$  is  $M_{i1}$  and ... and  $\xi_p$  is  $M_{ip}$ ,

**THEN**

$$\begin{aligned} \dot{x}(t) &= (A_i + \Delta A_i)x(t) + (A_{di} + \Delta A_{di})x(t-h) + (B_{1i} + \Delta B_{1i})w(t) + (B_{2i} + \Delta B_{2i})u(t), \\ z(t) &= (C_{1i} + \Delta C_{1i})x(t) + (C_{1di} + \Delta C_{1di})x(t-h) + (D_{11i} + \Delta D_{11i})w(t) \\ &\quad + (D_{12i} + \Delta D_{12i})u(t), \\ y(t) &= (C_{2i} + \Delta C_{2i})x(t) + (C_{2di} + \Delta C_{2di})x(t-h) + (D_{21i} + \Delta D_{21i})w(t) \\ &\quad + (D_{22i} + \Delta D_{22i})u(t), \end{aligned} \quad i = 1, \dots, r$$

where  $x(t) \in \mathfrak{R}^n$  is the state,  $w(t) \in \mathfrak{R}^{m_1}$  is the disturbance,  $u(t) \in \mathfrak{R}^{m_2}$  is the control input,  $z(t) \in \mathfrak{R}^{q_1}$  is the controlled output,  $y(t) \in \mathfrak{R}^{q_2}$  is the measurement output.  $r$  is the number of IF-THEN rules.  $M_{ij}$  is a fuzzy set and  $\xi_1, \dots, \xi_p$  are premise variables. We set  $\xi = [\xi_1 \ \dots \ \xi_p]^T$ . We assume that the premise variables do not depend on  $u(t)$ .  $A_i, A_{di}, B_{1i}, B_{2i}, C_{1i}, C_{2i}, C_{1di}, C_{2di}, D_{11i}, D_{12i}, D_{21i}$  and  $D_{22i}$  are constant matrices of appropriate dimensions. The uncertain matrices are of the form (1) with  $\Delta A_{di} = H_{1i}F_i(t)E_{di}$ ,  $\Delta C_{1di} = H_{2i}F_i(t)E_{di}$  and  $\Delta C_{2di} = H_{3i}F_i(t)E_{di}$  where  $H_{1i}, H_{2i}, H_{3i}$  and  $E_{di}$  are known constant matrices of appropriate dimensions.

**Assumption 4.1** The system  $(A_r, A_{dr}, B_{1r}, B_{2r}, C_{1r}, C_{2r}, C_{1dr}, C_{2dr}, D_{11r}, D_{12r}, D_{21r}, D_{22r})$  represents a nominal system that can be chosen as a subsystem including the equilibrium point of the original system.

The state equation, the controlled output and the output equation are defined as follows:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^r \lambda_i(x(t)) \{ (A_i + \Delta A_i)x(t) + (A_{di} + \Delta A_{di})x(t-h) + (B_{1i} + \Delta B_{1i})w(t) \\ &\quad + (B_{2i} + \Delta B_{2i})u(t) \}, \\ z(t) &= \sum_{i=1}^r \lambda_i(x(t)) \{ (C_{1i} + \Delta C_{1i})x(t) + (C_{1di} + \Delta C_{1di})x(t-h) + (D_{11i} + \Delta D_{11i})w(t) \\ &\quad + (D_{12i} + \Delta D_{12i})u(t) \}, \\ y(t) &= \sum_{i=1}^r \lambda_i(x(t)) \{ (C_{2i} + \Delta C_{2i})x(t) + (C_{2di} + \Delta C_{2di})x(t-h) + (D_{21i} + \Delta D_{21i})w(t) \\ &\quad + (D_{22i} + \Delta D_{22i})u(t) \} \end{aligned} \quad (22)$$

where  $\lambda_i(x(t))$  is defined in (3) and satisfies (4). Our problem is to find a control  $u(\cdot)$  for the system (22) given the output measurements  $y(\cdot)$  such that the controlled output  $z(\cdot)$  satisfies (5) for a prescribed scalar  $\gamma > 0$ . Using the same technique as in the previous sections, we have an equivalent description for (22):

$$\begin{aligned} \dot{x}(t) &= (A_r + \Delta A)x(t) + (A_{dr} + \Delta A_d)x(t-h) + (B_{1r} + \Delta B_1)w(t) + (B_{2r} + \Delta B_2)u(t) \\ &\triangleq A_\Delta x(t) + A_{d\Delta}x(t-h) + B_{1\Delta}w(t) + B_{2\Delta}u(t), \\ z(t) &= (C_{1r} + \Delta C_1)x(t) + (C_{1dr} + \Delta C_{1d})x(t-h) + (D_{11r} + \Delta D_{11})w(t) + (D_{12r} + \Delta D_{12})u(t) \\ &\triangleq C_{1\Delta}x(t) + C_{1d\Delta}x(t-h) + D_{11\Delta}w(t) + D_{12\Delta}u(t), \\ y(t) &= (C_{2r} + \Delta C_2)x(t) + (C_{2dr} + \Delta C_{2d})x(t-h) + (D_{21r} + \Delta D_{21})w(t) + (D_{22r} + \Delta D_{22})u(t) \\ &\triangleq C_{2\Delta}x(t) + C_{2d\Delta}x(t-h) + D_{21\Delta}w(t) + D_{22\Delta}u(t) \end{aligned} \quad (23)$$

where  $\Delta A_d = H_1 \tilde{F}(t) \tilde{A}_d$ ,  $\Delta C_{1d} = H_1 \tilde{F}(t) \tilde{C}_{1d}$ ,  $\Delta C_{2d} = H_1 \tilde{F}(t) \tilde{C}_{2d}$ , and other uncertain matrices are given in (7). As we can see from (23) that uncertain Takagi-Sugeno fuzzy time-delay system (22) can be written as an uncertain linear time-delay system. Thus, robust control problems for uncertain fuzzy time-delay system (22) can be converted into those for an uncertain linear time-delay system (23). Solutions to various control problems for an uncertain linear time-delay system have been given (for example, see Gu et al., 2003; Mahmoud, 2000) and hence the existing results can be applied to solve robust control problems for fuzzy time-delay systems.

## 5. Conclusion

This chapter has considered robust  $H_\infty$  control problems for uncertain Takagi-Sugeno fuzzy systems with immeasurable premise variables. A continuous-time Takagi-Sugeno fuzzy system was first considered. Takagi-Sugeno fuzzy system with immeasurable premise variables can be written as an uncertain linear system. Based on such an uncertain system representation, robust stabilization and robust  $H_\infty$  output feedback controller design method was proposed. The same control problems for discrete-time counterpart were also

considered. For both continuous-time and discrete-time control problems, numerical examples were shown to illustrate our design methods. Finally, an extension to fuzzy time-delay systems was given and a way to robust control problems for them was shown. Uncertain system approach taken in this chapter is applicable to filtering problems where the state variable is assumed to be immeasurable.

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While several books are available today that address the mathematical and philosophical foundations of fuzzy logic, none, unfortunately, provides the practicing knowledge engineer, system analyst, and project manager with specific, practical information about fuzzy system modeling. Those few books that include applications and case studies concentrate almost exclusively on engineering problems: pendulum balancing, truck backeruppers, cement kilns, antilock braking systems, image pattern recognition, and digital signal processing. Yet the application of fuzzy logic to engineering problems represents only a fraction of its real potential. As a method of encoding and using human knowledge in a form that is very close to the way experts think about difficult, complex problems, fuzzy systems provide the facilities necessary to break through the computational bottlenecks associated with traditional decision support and expert systems. Additionally, fuzzy systems provide a rich and robust method of building systems that include multiple conflicting, cooperating, and collaborating experts (a capability that generally eludes not only symbolic expert system users but analysts who have turned to such related technologies as neural networks and genetic algorithms). Yet the application of fuzzy logic in the areas of decision support, medical systems, database analysis and mining has been largely ignored by both the commercial vendors of decision support products and the knowledge engineers who use them.

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