

# Robust $H_\infty$ Control for Linear Switched Systems with Time Delay

Yan Li, Zhihuai Li and Xinmin Wang  
Northwestern Polytechnical University  
P. R. China

## 1. Introduction

Switched control has been applied widely in the intelligent robots, aerospace and aeronautics engineering and wireless communications. In this chapter, the robust  $H_\infty$  control for linear uncertain switched systems with time delay is studied.

Switched systems with time delay include the system with single time delay and the system with multiple time delays. Linear switched systems with time delay can be described as follows:

$$\begin{aligned}\dot{x}(t) &= A_{\sigma(t)}x(t) + \sum_{j=1}^N A_{dj\sigma(t)}x(t - \tau_j) + B_{\sigma(t)}u(t) + B_{1\sigma(t)}w(t) \\ z(t) &= C_{\sigma(t)}x(t) + \sum_{j=1}^N C_{dj\sigma(t)}x(t - \tau_j) + D_{\sigma(t)}u(t) + B_{2\sigma(t)}w(t) \\ x(t) &= \phi(t), t \in (-\max(\tau_j), 0)\end{aligned}\quad (1)$$

where  $x(t) \in R^n$  is the system state vector,  $u(t) \in R^{m_1}$  is the input vector,  $z(t) \in R^{m_2}$  is the output vector,  $w(t) \in l_2$  is the disturbance vector,  $\sigma(t): [0, \infty) \rightarrow M = \{1, 2, 3, \dots, m\}$  is the switching signal,  $A_{\sigma(t)}$ ,  $A_{dj\sigma(t)}$ ,  $B_{\sigma(t)}$ ,  $B_{1\sigma(t)}$ ,  $C_{\sigma(t)}$ ,  $C_{dj\sigma(t)}$ ,  $D_{\sigma(t)}$ ,  $B_{2\sigma(t)}$  are known constant matrices,  $\phi(t)$  represents the initial condition of the system,  $\tau_j$  represents the time delay. For the system (1), if  $N=1$ , it is a switched system with single time delay, otherwise it is a switched system with multiple time delays.

The state feedback control for switched systems can be designed with memory or without memory.

For the switched system (1), the state feedback control can be designed as follows:

$u(t) = K_j x(t)$  is the state feedback control without memory;

$u(t) = K_j x(t) + \sum_{j=1}^N K_j x(t - \tau_{ij})$  is the state feedback control with memory.

Compared with the results on the stability of switched systems, research on the  $H_\infty$  control for switched systems is not adequate yet. Attentions have been attracted to the  $H_\infty$  control

for switched systems since 1998, when Hespanha considered the problem firstly. Similar to the stability problem, the  $H_\infty$  control problem can be classified into:

Problem A. The  $H_\infty$  control under arbitrary switching signal;

Problem B. The  $H_\infty$  control under a certain switching signal.

Problem A means the internal stability and the  $L_2$  gain of the switched systems are independent of the switching signal. Problem A is usually solved through the common Lyapunov method which is conservative in that the common Lyapunov function is not easy to choose.

Wu and Meng (Wu & Meng, 2009) studied  $H_\infty$  model reduction for continuous-time linear switched systems with time-varying delay. By applying the average dwell time approach and the piecewise Lyapunov function technique, delay delay-dependent and delay-independent sufficient conditions are proposed in terms of linear matrix inequality (LMI) to guarantee the exponential stability and the weighted  $H_\infty$  performance for the error system. Zhang and Liu (Zhang & Liu 2008) studied the problem of delay-dependent robust  $H_\infty$  control for switched systems with disturbance and time-varying structured uncertainties. A sufficient condition ensuring the robust stabilization and  $H_\infty$  performance under arbitrary switching laws was obtained based on the Lyapunov function and Finslerpsilas lemma. Xie et al. (Xie *et al.*, 2004) proposed conditions for uniformly quadratic stability for uncertain switched systems based on common Lyapunov method and LMI formulation. Fu et al. (Fu et al., 2007) proposed a the sufficient condition for the design of dynamic output feedback control of switched systems based on the common Lyapunov function approach and convex combination technique. Song et al. (Song et al., 2007) present the switching law and robust  $H_\infty$  control design for a class of discrete switched systems with time-varying delay. Song (Song et al., 2006) also studied a class of uncertain discrete switched systems with time delay. The switching law and the  $H_\infty$  controller are given based on the Multi-Lyapunov Function method. Ma et al. (Ma et al., 2006) proposed an  $H_\infty$  controller with memory for discrete switched systems with time delay.

In this chapter, the robust  $H_\infty$  control based on multi-Lyapunov-Function approach and LMI formulation for general linear switched systems with time delay is first introduced. The results are then extended to robust  $H_\infty$  control without and with memory for uncertain linear switched systems with time-varying delay. Suppose all sub-systems are not robust stable, a sufficient condition for system stabilization with  $H_\infty$  bound is given, as well as the design algorithm for the robust  $H_\infty$  switched control and the switching law. The simulation results show the effectiveness of the methods.

## 2. Robust $H_\infty$ stability and stabilization for linear switched systems

Consider the following linear switched system:

$$\begin{aligned}\dot{x} &= A_i x + B_i w \\ z &= C_i x\end{aligned}\tag{2}$$

where,  $x(t) \in R^n$  is the state vector,  $z(t) \in R^{m_2}$  is the output vector,  $w(t) \in l_2$  is the disturbance vector,  $A_i, B_i, C_i$  are constant matrices with proper dimensions.  $i: [0, \infty) \rightarrow M = \{1, 2, 3, \dots, m\}$  is the switching signal.

**Lemma 1:**  $X, Y$  are matrices with proper dimensions. There exists a scalar  $\alpha > 0$  such that the following inequality holds :

$$X^T Y + Y^T X \leq \alpha X^T X + \alpha^{-1} Y^T Y \quad (3)$$

**Lemma 2:** For given symmetric matrix  $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ , the dimension of  $S_{11}$  is  $r \times r$ . The following three conditions are equivalent :

$$\begin{aligned} (1) \quad & S < 0 \\ (2) \quad & S_{11} < 0, S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0 \\ (3) \quad & S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0 \end{aligned} \quad (4)$$

**Lemma 3:** For a given scalar  $\gamma > 0$ ,  $\gamma = \max(\gamma_i), i \in M$ , if there is a switching law  $i = i(x(t), t)$  and a positive matrix  $P_i$  satisfying

$$\begin{bmatrix} A_i^T P_i + P_i A_i & P_i B_i & C_i^T \\ * & -\gamma_i^{-2} I & 0 \\ * & * & -I \end{bmatrix} < 0 \quad (5)$$

the system (2) is stable with  $H_\infty$  performance  $\gamma$ .

**Proof:** Choose the Lyapunov function of the sub-system of system (2) as  $V_i = x^T P_i x$ . The derivative of the Lyapunov function is

$$\dot{V}_i = \dot{x}^T P_i x + x^T P_i \dot{x} = x^T (A_i^T P_i + P_i A_i) x + w^T B_i^T P_i x + x^T B_i P_i w$$

When  $w=0$ , if the above equation satisfies

$$x^T (A_i^T P_i + P_i A_i) x < 0$$

the system (2) is asymptotic stable.

According Lemma 2, the inequality (5) is equivalent to

$$\begin{bmatrix} A_i^T P_i + P_i A_i & P_i B_i \\ * & -\gamma_i^2 I \end{bmatrix} + \begin{bmatrix} C_i^T \\ 0 \end{bmatrix} \begin{bmatrix} C_i & 0 \end{bmatrix} < 0$$

For any  $x(t)$  and  $w(t)$ , the following inequality holds

$$\begin{bmatrix} x^T(t) \\ w^T(t) \end{bmatrix} \left\{ \begin{bmatrix} A_i^T P_i + P_i A_i & P_i B_i \\ * & -\gamma_i^2 I \end{bmatrix} + \begin{bmatrix} C_i^T \\ 0 \end{bmatrix} \begin{bmatrix} C_i & 0 \end{bmatrix} \right\} \begin{bmatrix} x^T(t) \\ w^T(t) \end{bmatrix} < 0$$

which is equivalent to

$$\dot{V}_i(x(t), w(t), t) + z^T(t) z(t) - \gamma_i^2 w^T(t) w(t) < 0$$

Under the zero initial condition, we have

$$\begin{aligned} & \int_0^{+\infty} [z^T(t) z(t) - \gamma^2 w^T(t) w(t)] dt \\ & \leq \int_0^{+\infty} [\dot{V}_i(x(t), w(t), t) + z^T(t) z(t) - \gamma_i^2 w^T(t) w(t)] dt - V_M(x(+\infty), w(+\infty), +\infty) \\ & = \sum_{i=1}^M \int_{t_{i-1}}^{t_i} [\dot{V}_i(x(t), w(t), t) + z^T(t) z(t) - \gamma_i^2 w^T(t) w(t)] dt - V_M(x(+\infty), w(+\infty), +\infty) < 0 \end{aligned}$$

Thus

$$\int_0^{\infty} z^T(t)z(t)dt \leq \gamma^2 \int_0^{+\infty} w^T(t)w(t)dt$$

This completes the proof.

From the above proof we know that if inequality (5) holds,

- 1) When the disturbance  $w=0$ , the system is asymptotically stable;
- 2) There exist a scalar  $\gamma > 0$  satisfying the robust  $H_{\infty}$  performance

$$\int_0^{\infty} z^T(t)z(t)dt \leq \gamma^2 \int_0^{+\infty} w^T(t)w(t)dt$$

Therefore, we can conclude that the switched system (2) satisfies the condition of robust  $H_{\infty}$  control.

Consider the following linear switched system :

$$\begin{aligned} \dot{x} &= A_i x + B_i u + D_i w \\ z &= C_i x \end{aligned} \quad (6)$$

where,  $x(t) \in R^n$  is the state vector,  $u(t) \in R^{m1}$  is the control input,  $z(t) \in R^{m2}$  is the output,  $w(t) \in l_2$  is the disturbance.  $A_i, B_i, C_i, D_i$  are constant matrices with proper dimensions.  $i: [0, \infty) \rightarrow M = \{1, 2, 3, \dots, m\}$  is the switching signal.

**Definition 1.** For a given scalar  $\gamma > 0$ ,  $\gamma = \max(\gamma_i), i \in M$ , if there is a state feedback control without memory  $u = K_i x$ , such that the closed-loop subsystem of system (6) is stable with  $H_{\infty}$  performance  $\gamma$ , the system (6) is robust stabilizable with  $H_{\infty}$  performance  $\gamma$ .

With the above knowledge, we will study linear switched systems with time delay in the following sections. Firstly, the robust  $H_{\infty}$  control for general linear switched systems is analyzed. The results are then extended to uncertain switched systems with time-varying delay and uncertain switched systems with multiple time delays.

### 3. Robust $H_{\infty}$ stabilization for linear switched systems with time delay

Consider the following Linear switched systems with time delay

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)} x(t) + A_{d\sigma(t)} x(t - \tau) + B_{\sigma(t)} u(t) + B_{w\sigma(t)} w(t) \\ z(t) = C_{\sigma(t)} x(t) \\ x(t) = \phi(t) \end{cases} \quad (7)$$

Where  $x(t) \in R^n$  is the state vector,  $u(t) \in R^{m1}$  is the control input,  $z(t) \in R^{m2}$  is the output vector,  $w(t) \in l_2$  is the disturbance,  $z \in R^q$  is the controlled output,  $\sigma(t): [0, \infty) \rightarrow M = \{1, 2, 3, \dots, m\}$  is the switching signal,  $A_{\sigma(t)}, B_{\sigma(t)}, A_{d\sigma(t)}, B_{w\sigma(t)}, C_{\sigma(t)}$  are known constant matrices,  $\tau$  is the time delay,  $\phi(t)$  is a smooth function on  $R^n$  presenting the initial condition of the system.

**Theorem 1.** For system (7), and given scalar  $\gamma > 0$ ,  $\gamma = \max(\gamma_i)$ , If there exist a switching law  $\sigma(t) = i$  and positive matrices  $P_i, R_i \in R^{n \times n}$  such that the following inequality holds:

$$\begin{bmatrix} S_1 & P_i A_d & P_i B_w & C^T \\ * & -R_i & 0 & 0 \\ * & * & -\gamma_i^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{8}$$

where  $S_1 = (A + BK_i)^T P_i + P_i(A + BK_i) + R_i$ , system (7) is stabilizable with  $H_\infty$  performance  $\gamma$ .

The controller is  $u(t) = K_i x(t)$  and the switching law is  $\sigma(t) = \arg \min_{i \in M} \{x^T(t) P_i x(t)\}$ .

**Proof:** Suppose there are positive definite matrices  $P_i, R_i \in R^{n \times n}$  and matrix  $K_i \in R^{m \times n}$ , such that the linear matrix inequality (8) holds. The controller is  $u(t) = K_i x(t)$  and choose the Lyapunov function as

$$V(x(t), w(t), t) = x^T(t) P_i x(t) + \int_{t-\tau}^t x^T(s) R_i x(s) ds$$

Then,

$$\begin{aligned} \dot{V}(x(t), w(t), t) &= \dot{x}^T(t) P_i x(t) + x^T(t) P_i \dot{x}(t) + x^T(t) R_i x(t) - x^T(t - \tau) R_i x(t - \tau) \\ &= x^T(t) [(A + BK_i)^T P_i + P_i(A + BK_i) + R_i] x(t) + \\ &\quad x^T(t) P_i B_w w(t) + w^T(t) B_w^T P_i x(t) + x^T(t - \tau) A_d^T P_i x(t) \\ &\quad + x^T(t) P_i A_d x(t - \tau) - x^T(t - \tau) R_i x(t - \tau) \end{aligned} \tag{9}$$

When  $w(t) = 0$ , if (10) holds, the closed-loop system of system (7) is asymptotically stable.

Thus,

$$\begin{aligned} &x^T(t) [(A + BK_i)^T P_i + P_i(A + BK_i) + R_i] x(t) + x^T(t - \tau) A_d^T P_i x(t) \\ &+ x^T(t) P_i A_d x(t - \tau) - x^T(t - \tau) R_i x(t - \tau) < 0 \end{aligned} \tag{10}$$

Rewrite the inequality (10) as

$$\begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix}^T \begin{bmatrix} S_1 & P_i A_d \\ * & -R_i \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix} < 0 \tag{11}$$

By Lemma 2 and inequality (8), we have

$$\begin{bmatrix} S_1 & P_i A_d & P_i B_w \\ * & -R_i & 0 \\ * & * & -\gamma_i^2 I \end{bmatrix} + \begin{bmatrix} C^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} C^T \\ 0 \\ 0 \end{bmatrix}^T < 0 \tag{12}$$

For any  $x(t), x(t - \tau), w(t)$ , the following inequality holds.

$$\begin{bmatrix} x(t) \\ x(t - \tau) \\ w(t) \end{bmatrix}^T \begin{bmatrix} S_1 & P_i A_d & P_i B_w \\ * & -R_i & 0 \\ * & * & -\gamma_i^2 I \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - \tau) \\ w(t) \end{bmatrix} + \begin{bmatrix} x(t) \\ x(t - \tau) \\ w(t) \end{bmatrix}^T \begin{bmatrix} C^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} C^T \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} x(t) \\ x(t - \tau) \\ w(t) \end{bmatrix} < 0 \tag{13}$$

Thus, we have

$$\dot{V}(x(t), w(t), t) + z^T(t) z(t) - \gamma_i^2 w^T(t) w(t) < 0$$

With  $\gamma = \max(\gamma_i), i \in M$  and under zero initial condition, we have

$$\begin{aligned}
& \int_0^{+\infty} [z^T(t)z(t) - \gamma^2 w^T(t)w(t)] dt \\
& \leq \int_0^{+\infty} [\dot{V}(x(t), w(t), t) + z^T(t)z(t) - \gamma_i^2 w^T(t)w(t)] dt - V(x(+\infty), w(+\infty), +\infty) \\
& = \sum_{i=1}^M \int_{t_{i-1}}^{t_i} [\dot{V}(x(t), w(t), t) + z^T(t)z(t) - \gamma_i^2 w^T(t)w(t)] dt - V(x(+\infty), w(+\infty), +\infty) < 0
\end{aligned}$$

Thus

$$\int_0^{\infty} z^T(t)z(t) dt \leq \gamma^2 \int_0^{\infty} w^T(t)w(t) dt$$

that is

$$z^T(t)z(t) dt < \gamma^2 w^T(t)w(t) dt.$$

This complete the proof.

**Remark 1.** To convert the inequality (8) into an LMI, right and left multiplying the following matrix

$$\begin{bmatrix}
X_i & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}$$

to inequality (8) lead to

$$\begin{bmatrix}
S & A_d & B_w & X_i C^T \\
* & -R_i & 0 & 0 \\
* & * & -\gamma_i^2 I & 0 \\
* & * & * & -I
\end{bmatrix} < 0 \quad (14)$$

where,  $S = AX_i + X_i A^T + BY_i + Y_i^T B^T + Q_i$ ,  $X_i = P_i^{-1}$ ,  $Y_i = K_i X_i$ ,  $Q_i = X_i R_i X_i$ . The state feedback gain  $K_i = Y_i X_i^{-1}$ ,  $i \in M$  can be obtained by solving inequality (14).

**Example 1** Consider the linear switched system (7) with

$$\begin{aligned}
A^1 &= \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}, A_1^1 = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}, B^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, B_1^1 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}, C^1 = [1 \quad 1], \\
\gamma^1 &= 3, w^1(t) = \cos(t), \tau^1 = 0.2. \\
A^2 &= \begin{bmatrix} -1.5 & -2 \\ 1 & 3 \end{bmatrix}, A_1^2 = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}, B^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, B_1^2 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}, C^2 = [1 \quad 1], \\
\gamma^2 &= 3, w^2(t) = \cos(t), \tau^2 = 0.2.
\end{aligned} \quad (15)$$

By solving the linear matrix inequality (14), we have:

$$\begin{aligned}
P^1 &= \begin{bmatrix} 0.1967 & -1.2115 \\ -1.2115 & 7.4606 \end{bmatrix}, K^1 = [1.1140 \quad -6.8599] \\
P^2 &= \begin{bmatrix} 0.0281 & -0.1732 \\ -0.1732 & 1.0670 \end{bmatrix}, K^2 = [0.7320 \quad -4.5086]
\end{aligned}$$

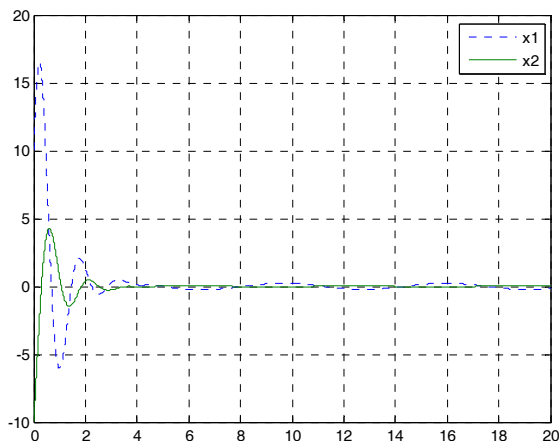


Fig. 1. State response of Example 1

By Theorem 1, the switching law is designed as:

$$\sigma(t) = i = \begin{cases} 1, & x^T(P^1 - P^2)x \leq 0 \\ 2, & x^T(P^1 - P^2)x > 0 \end{cases}$$

The state response of the closed-loop system is shown in Figure 1.  $x_1$  and  $x_2$  are system states, the initial condition is  $[x_1, x_2] = [10, -10]$ . The result shows the system is stable under the switching law when it is switched among the closed-loop subsystems.

#### 4. Robust $H_\infty$ stabilization for uncertain linear switched systems with time-varying delay

Consider the following linear uncertain switched system with time-varying delay

$$\begin{aligned} \dot{x}(t) &= (A_{\sigma(t)} + \Delta A_{\sigma(t)})x(t) + (A_{d\sigma(t)} + \Delta A_{d\sigma(t)})x(t - d(t)) + (B_{\sigma(t)} + \Delta B_{\sigma(t)})u(t) \\ &\quad + (B_{h\sigma(t)} + \Delta B_{h\sigma(t)})u(t - h(t)) + (B_{1w\sigma(t)} + \Delta B_{1w\sigma(t)})w(t) \\ z(t) &= (C_{\sigma(t)} + \Delta C_{\sigma(t)})x(t) + (C_{d\sigma(t)} + \Delta C_{d\sigma(t)})x(t - d(t)) + (D_{\sigma(t)} + \Delta D_{\sigma(t)})u(t) \\ &\quad + (D_{h\sigma(t)} + \Delta D_{h\sigma(t)})u(t - h(t)) + (B_{2w\sigma(t)} + \Delta B_{2w\sigma(t)})w(t) \\ x(t) &= \phi(t), t \in [-\max(d(t), h(t)), 0] \end{aligned} \tag{16}$$

where,  $x(t) \in R^n$  is state vector,  $u(t) \in R^{m1}$  is the control input,  $z(t) \in R^{m2}$  is the output,  $w(t) \in l_2$  is the disturbance,  $\sigma(t) : [0, \infty) \rightarrow M = \{1, 2, 3, \dots, m\}$  is the switching signal,  $A_{\sigma(t)}$ ,  $A_{d\sigma(t)}$ ,  $B_{\sigma(t)}$ ,  $B_{h\sigma(t)}$ ,  $B_{1w\sigma(t)}$ ,  $C_{\sigma(t)}$ ,  $C_{d\sigma(t)}$ ,  $D_{\sigma(t)}$ ,  $D_{h\sigma(t)}$ ,  $B_{2w\sigma(t)}$  are known constant matrices,  $\Delta A_{d\sigma(t)}$ ,  $\Delta B_{\sigma(t)}$ ,  $\Delta B_{h\sigma(t)}$ ,  $\Delta B_{1w\sigma(t)}$ ,  $\Delta C_{\sigma(t)}$ ,  $\Delta C_{d\sigma(t)}$ ,  $\Delta D_{\sigma(t)}$ ,  $\Delta D_{h\sigma(t)}$ ,  $\Delta B_{2w\sigma(t)}$  are bounded real functional matrices with proper dimensions, representing the uncertainties,  $\phi(t)$  is the initial condition.  $d(t)$  and  $h(t)$  are the state delay and the control delay respectively. There are positive scalars  $d, h, \rho_d, \rho_h$ , such that

$$\begin{aligned} 0 \leq d(t) \leq d < \infty, \dot{d}(t) \leq \rho_d < 1 \\ 0 \leq h(t) \leq h < \infty, \dot{h}(t) \leq \rho_h < 1 \end{aligned} \tag{17}$$

Denote

$$\begin{aligned} \bar{A} &= A_{\sigma(t)} + \Delta A_{\sigma(t)}, \bar{A}_d = A_{d\sigma(t)} + \Delta A_{d\sigma(t)}, \bar{B} = B_{\sigma(t)} + \Delta B_{\sigma(t)} \\ \bar{B}_h &= B_{h\sigma(t)} + \Delta B_{h\sigma(t)}, \bar{B}_1 = B_{1w\sigma(t)} + \Delta B_{1w\sigma(t)} \\ \bar{C} &= C_{\sigma(t)} + \Delta C_{\sigma(t)}, \bar{C}_d = C_{d\sigma(t)} + \Delta C_{d\sigma(t)}, \bar{D} = D_{\sigma(t)} + \Delta D_{\sigma(t)} \\ \bar{D}_h &= D_{h\sigma(t)} + \Delta D_{h\sigma(t)}, \bar{B}_2 = B_{2w\sigma(t)} + \Delta B_{2w\sigma(t)} \end{aligned} \tag{18}$$

and suppose:

$$\begin{bmatrix} \Delta A_{\sigma(t)} & \Delta A_{d\sigma(t)} & \Delta B_{\sigma(t)} & \Delta B_{h\sigma(t)} & \Delta B_{1w\sigma(t)} \\ \Delta C_{\sigma(t)} & \Delta C_{d\sigma(t)} & \Delta D_{\sigma(t)} & \Delta D_{h\sigma(t)} & \Delta B_{2w\sigma(t)} \end{bmatrix} = \begin{bmatrix} H_{1\sigma} \\ H_{2\sigma} \end{bmatrix} F_{\sigma(t)} \begin{bmatrix} E_{1\sigma} & E_{2\sigma} & E_{3\sigma} & E_{4\sigma} & E_{5\sigma} \end{bmatrix} \tag{19}$$

where  $H_{1\sigma}, H_{2\sigma}, E_{1\sigma}, E_{2\sigma}, E_{3\sigma}, E_{4\sigma}, E_{5\sigma}$  are constant matrices with proper dimensions and  $F_{\sigma(t)}$  satisfying

$$F_{\sigma(t)}^T F_{\sigma(t)} \leq I \tag{20}$$

**Theorem 2** For a given scalar  $\gamma_i > 0$ , if there are positive definite matrices  $P_i, R_{1i}, R_{2i} \in R^{n \times n}$ , such that :

$$\begin{bmatrix} S_1 & P_i \bar{A}_d & P_i \bar{B}_h K_i & P_i \bar{B}_1 & \bar{C}^T + K_i^T \bar{D}^T \\ * & -(1 - \rho_d) R_{1i} & 0 & 0 & \bar{C}_d^T \\ * & * & -(1 - \rho_h) R_{2i} & 0 & K_i^T \bar{D}_h^T \\ * & * & * & -\gamma_i^2 I & \bar{B}_2^T \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{21}$$

where

$$S_1 = \bar{A}^T P_i + P_i \bar{A} + K_i^T \bar{B}^T P_i + P_i \bar{B} K_i + R_{1i} + R_{2i}$$

the system (16) is robust stabilizable with  $H_\infty$  performance  $\gamma$ ,  $\gamma = \max(\gamma_i)$ .  $u(t) = K_i x(t)$  is the switched robust  $H_\infty$  controller. The switching law is  $\sigma(t) = i = \arg \min_{i \in M} \{x^T(t) P_i x(t)\}$ .

**Proof:** If there are positive definite matrices  $P_i, R_{1i}, R_{2i} \in R^{n \times n}$  and matrix  $K_i \in R^{m \times n}$  satisfying the inequality (21) with the controller  $u(t) = K_i x(t)$  and Lyapunov function:

$$V(x(t), w(t), t) = x^T(t) P_i x(t) + \int_{t-d(t)}^t x^T(s) R_{1i} x(s) ds + \int_{t-h(t)}^t x^T(s) R_{2i} x(s) ds$$

Then

$$\begin{aligned} \dot{V}(x(t), w(t), t) &= \dot{x}^T(t) P_i x(t) + x^T(t) P_i \dot{x}(t) + x^T(t) (R_{1i} + R_{2i}) x(t) \\ &\quad - (1 - \dot{d}(t)) x^T(t - d(t)) R_{1i} x(t - d(t)) - (1 - \dot{h}(t)) x^T(t - h(t)) R_{2i} x(t - h(t)) \\ &= x^T(t) \bar{A}^T P_i x(t) + x^T(t) P_i \bar{A} x(t) + x^T(t) K_i^T \bar{B}^T P_i x(t) + x^T(t) P_i \bar{B} K_i x(t) \\ &\quad + x^T(t) (R_{1i} + R_{2i}) x(t) + x^T(t) P_i \bar{B}_1 w(t) + w^T(t) \bar{B}_1^T P_i x(t) \\ &\quad + x^T(t - d(t)) \bar{A}_d^T P_i x(t) + x^T(t) P_i \bar{A}_d x(t - d(t)) + x^T(t - h(t)) K_i^T \bar{B}_h^T P_i x(t) \\ &\quad + x^T(t) P_i \bar{B}_h K_i x(t - h(t)) - (1 - \dot{d}(t)) x^T(t - d(t)) R_{1i} x(t - d(t)) \\ &\quad - (1 - \dot{h}(t)) x^T(t - h(t)) R_{2i} x(t - h(t)) \end{aligned} \tag{22}$$

When  $w(t) = 0$ , considering condition (17), if inequality (23) holds, the closed-loop system is robust asymptotically stable



$$\begin{aligned}
 &x^T(t)\bar{A}^T P_i x(t) + x^T(t)P_i \bar{A} x(t) + x^T(t)K_i^T \bar{B}^T P_i x(t) + x^T(t)P_i \bar{B} K_i x(t) \\
 &+ x^T(t)(R_{1i} + R_{2i})x(t) + x^T(t-d(t))\bar{A}_d^T P_i x(t) + x^T(t)P_i \bar{A}_d x(t-d(t)) \\
 &+ x^T(t-h(t))K_i^T \bar{B}_h^T P_i x(t) + x^T(t)P_i \bar{B}_h K_i x(t-h(t)) \\
 &-(1-\rho_d)x^T(t-d(t))R_{1i}x(t-d(t)) - (1-\rho_h)x^T(t-h(t))R_{2i}x(t-h(t)) < 0
 \end{aligned} \tag{23}$$

Rewrite inequality (23) as

$$\begin{bmatrix} x^T & x^T(t-d(t)) & x^T(t-h(t)) \end{bmatrix} W \begin{bmatrix} x^T & x^T(t-d(t)) & x^T(t-h(t)) \end{bmatrix}^T < 0 \tag{24}$$

where

$$W = \begin{bmatrix} \bar{A}^T P_i + P_i \bar{A} + K_i^T \bar{B} K_i + R_{1i} + R_{2i} & P_i \bar{A}_d & P_i \bar{B}_h K_i \\ * & -(1-\rho_d)R_{1i} & 0 \\ * & * & -(1-\rho_h)R_{2i} \end{bmatrix} < 0$$

By inequality (21) and Lemma 2, the following inequality follows

$$\begin{bmatrix} \bar{A}^T P_i + P_i \bar{A} + K_i^T \bar{B} K_i + R_{1i} + R_{2i} & P_i \bar{A}_d & P_i \bar{B}_h K_i & P_i \bar{B}_1 \\ * & -(1-\rho_d)R_{1i} & 0 & 0 \\ * & * & -(1-\rho_h)R_{2i} & 0 \\ * & * & * & -\gamma_i^2 I \end{bmatrix} + \begin{bmatrix} \bar{C}^T + K_i^T \bar{D}^T \\ \bar{C}_d^T \\ K_i^T \bar{D}_h^T \\ \bar{B}_3^T \end{bmatrix} \begin{bmatrix} \bar{C} + \bar{D} K_i & \bar{C}_d & \bar{D}_h K_i & \bar{B}_2 \end{bmatrix} < 0 \tag{25}$$

For any  $x(t), x(t-d(t)), x(t-h(t)), w(t)$ , the following inequality holds

$$\begin{bmatrix} x(t) \\ x(t-d(t)) \\ x(t-h(t)) \\ w(t) \end{bmatrix}^T \begin{bmatrix} \bar{A}^T P_i + P_i \bar{A} + K_i^T \bar{B} K_i + R_{1i} + R_{2i} & P_i \bar{A}_d & P_i \bar{B}_h K_i & P_i \bar{B}_1 \\ * & -(1-\rho_d)R_{1i} & 0 & 0 \\ * & * & -(1-\rho_h)R_{2i} & 0 \\ * & * & * & -\gamma_i^2 I \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d(t)) \\ x(t-h(t)) \\ w(t) \end{bmatrix} + \begin{bmatrix} x(t) \\ x(t-d(t)) \\ x(t-h(t)) \\ w(t) \end{bmatrix}^T \begin{bmatrix} \bar{C}^T + K_i^T \bar{D}^T \\ \bar{C}_d^T \\ K_i^T \bar{D}_h^T \\ \bar{B}_3^T \end{bmatrix} \begin{bmatrix} \bar{C} + \bar{D} K_i & \bar{C}_d & \bar{D}_h K_i & \bar{B}_2 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d(t)) \\ x(t-h(t)) \\ w(t) \end{bmatrix} < 0 \tag{26}$$

Thus

$$\dot{V} + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0 \tag{27}$$

Under zero initial condition with  $\gamma = \max(\gamma_i), i \in M$ , we have

$$\begin{aligned}
 &\int_0^{+\infty} [z^T(t)z(t) - \gamma^2 w^T(t)w(t)] dt \\
 &\leq \int_0^{+\infty} [\dot{V}(x(t), w(t), t) + z^T(t)z(t) - \gamma_i^2 w^T(t)w(t)] dt - V(x(+\infty), w(+\infty), +\infty) \\
 &= \sum_{i=1}^M \int_{t_{i-1}}^{t_i} [\dot{V}(x(t), w(t), t) + z^T(t)z(t) - \gamma_i^2 w^T(t)w(t)] dt - V(x(+\infty), w(+\infty), +\infty) < 0
 \end{aligned} \tag{28}$$

Therefore

$$\int_0^\infty z^T(t)z(t)dt \leq \gamma^2 \int_0^{+\infty} w^T(t)w(t)dt$$

This completes the proof.

**Remark 2:** Although theorem 2 presents a sufficient condition for the robust stabilization with  $H_\infty$  performance  $\gamma$ , there are still uncertainties in the inequality (21).

**Theorem 3** For the switched system (16) and a given positive scalar  $\gamma_i$ , if there are matrix  $Y_i$  with proper dimension, positive definite matrices  $X_i, Q_{1i}, Q_{2i}$  and scalar  $\alpha > 0$  such that

$$\begin{bmatrix} S & A_d X_i & B_h Y_i & B_{1w} & \varphi & X_i E_1^T + Y_i^T E_3^T \\ * & -(1-\rho_d)Q_{1i} & 0 & 0 & X_i C_d^T & X_i E_2^T \\ * & * & -(1-\rho_h)Q_{2i} & 0 & Y_i^T D_h^T & Y_i^T E_4^T \\ * & * & * & -\gamma_i^2 I & B_{2w}^T & E_5^T \\ * & * & * & * & -I + \alpha H_2 H_2^T & 0 \\ * & * & * & * & * & -\alpha I \end{bmatrix} < 0 \tag{29}$$

where

$$S = X_i A^T + A X_i + Y_i^T B^T + B Y_i + Q_{1i} + Q_{2i} + \alpha H_1 H_1^T,$$

and

$$\varphi = X_i C^T + Y_i^T D^T + \alpha H_1 H_2^T$$

holds, the system is robust stabilizable with  $H_\infty$  performance  $\gamma$ ,  $\gamma = \max(\gamma_i)$ . The robust  $H_\infty$  switched controller is given by:

$$K_i = Y_i X_i^{-1}, i \in M \tag{30}$$

And the switching law is

$$\sigma(t) = i = \arg \min_{i \in M} \{x^T(t) X_i^{-1} x(t)\} \tag{31}$$

**Proof:** For any non-zero vector  $\xi$ , by inequality (21), we have

$$\xi^T \begin{bmatrix} S_1 & P_i \bar{A}_d & P_i \bar{B}_h K_i & P_i \bar{B}_1 & \bar{C}^T + K_i^T \bar{D}^T \\ * & -(1-\rho_d)R_{1i} & 0 & 0 & \bar{C}_d^T \\ * & * & -(1-\rho_h)R_{2i} & 0 & K_i^T \bar{D}_h^T \\ * & * & * & -\gamma_i^2 I & \bar{B}_2^T \\ * & * & * & * & -I \end{bmatrix} \xi = \xi^T L \xi + \xi^T \Delta L \xi < 0 \tag{32}$$

where

$$L = \begin{bmatrix} S_2 & P_i A_d & P_i B_h K_i & P_i B_{1w} & C^T + K_i^T D^T \\ * & -(1-\rho_d)R_{1i} & 0 & 0 & C_d^T \\ * & * & -(1-\rho_h)R_{2i} & 0 & K_i^T D_h^T \\ * & * & * & -\gamma_i^2 I & B_{2w}^T \\ * & * & * & * & -I \end{bmatrix} \tag{33}$$

$$S_2 = A^T P_i + P_i A + K_i B^T P_i + P_i B K_i + R_{1i} + R_{2i}$$

and

$$\Delta L = \begin{bmatrix} P_i H_1 \\ 0 \\ 0 \\ 0 \\ H_2 \end{bmatrix} F(t) \begin{bmatrix} E_1 + E_3 K_i & E_2 & E_4 K_i & E_5 & 0 \end{bmatrix} + \begin{bmatrix} E_1 + E_3 K_i & E_2 & E_4 K_i & E_5 & 0 \end{bmatrix}^T F^T(t) \begin{bmatrix} P_i \\ 0 \\ 0 \\ 0 \\ H_2 \end{bmatrix} \quad (34)$$

By Lemma 1 and inequality (20), we have

$$\begin{aligned} \Delta L &= \begin{bmatrix} P_i H_1 \\ 0 \\ 0 \\ 0 \\ H_2 \end{bmatrix} F(t) F^T(t) \begin{bmatrix} P_i H_1 \\ 0 \\ 0 \\ 0 \\ H_2 \end{bmatrix}^T \\ &\quad + \begin{bmatrix} E_1 + E_3 K_i & E_2 & E_4 K_i & E_5 & 0 \end{bmatrix}^T F(t) F^T(t) \begin{bmatrix} E_1 + E_3 K_i & E_2 & E_4 K_i & E_5 & 0 \end{bmatrix} \\ &\leq \alpha L_1 + \alpha^{-1} L_2 \end{aligned} \quad (35)$$

where  $\alpha > 0$ ,

$$L_1 = \begin{bmatrix} P_i H_1 \\ 0 \\ 0 \\ 0 \\ H_2 \end{bmatrix} \begin{bmatrix} P_i H_1 \\ 0 \\ 0 \\ 0 \\ H_2 \end{bmatrix}^T$$

$$L_2 = \begin{bmatrix} E_1 + E_3 K_i & E_2 & E_4 K_i & E_5 & 0 \end{bmatrix}^T \begin{bmatrix} E_1 + E_3 K_i & E_2 & E_4 K_i & E_5 & 0 \end{bmatrix}$$

If

$$L + \alpha L_1 + \alpha^{-1} L_2 < 0 \quad (36)$$

the inequality (21) follows

By Lemma 2, inequality (36) is equivalent to:

$$\begin{bmatrix} S_3 & P_i A_d & P_i B_h K_i & P_i B_{1w} & C^T + K_i^T D^T + \alpha P_i H_1 H_2^T & E_1^T + K_i^T E_3^T \\ * & -(1 - \rho_d) R_{1i} & 0 & 0 & C_d^T & E_2^T \\ * & * & -(1 - \rho_h) R_{2i} & 0 & K_i^T D_h^T & K_i^T E_4^T \\ * & * & * & -\gamma_i^2 I & B_{2w}^T & E_5^T \\ * & * & * & * & -I + \alpha H_2 H_2^T & 0 \\ * & * & * & * & * & -\alpha I \end{bmatrix} < 0 \quad (37)$$

Denote

$$X_i = P_i^{-1}, Q_{1i} = X_i R_{1i} X_i, Q_{2i} = X_i R_{2i} X_i,$$

and

$$K_i = Y_i X_i^{-1}$$

where  $S_3 = A^T P_i + P_i A + K_i^T B^T P_i + P_i B K_i + R_{1i} + R_{2i} + \alpha P_i H_1 H_1^T P_i$ ,  $Y_i$  is an arbitrary matrix with proper dimension.  $Q_{1i}, Q_{2i}$  are positive definite matrices with proper dimensions. By right and left multiplying the following matrix to the inequality (37)

$$\begin{bmatrix} X_i & 0 & 0 & 0 & 0 & 0 \\ 0 & X_i & 0 & 0 & 0 & 0 \\ 0 & 0 & X_i & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

the inequality (29) follows.

This completes the proof.

**Example 2** Consider uncertain switched system (16) with

$$\begin{aligned} A^1 &= \begin{bmatrix} -2 & 3 \\ 3 & 4 \end{bmatrix}, A^{1^1} = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, B^1 = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, B^{1^1} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \\ B^{2^1} &= \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}, B^{3^1} = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}, C^1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, C^{1^1} = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}, D^1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}, \\ D^{1^1} &= \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix}, H^{1^1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, H^{2^1} = \begin{bmatrix} 0.5 & 0.4 \\ 0 & 0 \end{bmatrix}, E^{1^1} = \begin{bmatrix} 0.5 & 0.3 \\ 0 & 0.6 \end{bmatrix}, \\ E^{2^1} &= \begin{bmatrix} 0.5 & 0.4 \\ 0 & 0 \end{bmatrix}, E^{3^1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, E^{4^1} = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}, E^{5^1} = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \\ A^2 &= \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}, A^{1^2} = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}, B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B^{1^2} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix}, \\ B^{2^2} &= \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}, B^{3^2} = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}, C^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, C^{1^2} = \begin{bmatrix} 0.3 & 0.5 \\ 0.2 & 0.1 \end{bmatrix}, D^2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.1 \end{bmatrix}, \\ D^{1^2} &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2 \end{bmatrix}, H^{1^2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, H^{2^2} = \begin{bmatrix} 0.5 & 0.4 \\ 0 & 0 \end{bmatrix}, E^{1^2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ E^{2^2} &= \begin{bmatrix} 0.5 & 0.4 \\ 0 & 0 \end{bmatrix}, E^{3^2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E^{4^2} = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}, E^{5^2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned} \quad (38)$$

Choose

$$\begin{aligned} F^1(t) &= \sin(t), F^2(t) = \sin(t), w^1(t) = c \cos(t), w^2(t) = \cos(t), \alpha^1 = \alpha^2 = 0.5, \\ d^1(t) &= 0.3 \sin(t/3.85), d^2(t) = 0.3 \sin(t/3.85), h^1(t) = 0.2 \sin(t/3.85), \\ h^2(t) &= 0.2 \sin(t/3.85), \rho_d^1 = 0.3, \rho_h^1 = 0.2, \gamma_1 = 3, \rho_d^2 = 0.3, \rho_h^2 = 0.2, \gamma_2 = 3 \end{aligned}$$

By Theorem 3, we have:

$$\begin{aligned} P^1 &= \begin{bmatrix} 4.0697 & 0 \\ 0 & 4.0697 \end{bmatrix}, K^1 = \begin{bmatrix} -8.6086 & -0.7780 \\ -2.3540 & -6.1658 \end{bmatrix} \\ P^2 &= \begin{bmatrix} 1.9297 & 3.1929 \\ 3.1929 & 5.2830 \end{bmatrix}, K^2 = \begin{bmatrix} -0.6417 & -1.0618 \\ -3.6147 & -5.9809 \end{bmatrix} \end{aligned}$$

and the switching law is designed as

$$\sigma(t) = i = \begin{cases} 1, & x^T (P^1 - P^2)x \leq 0 \\ 2, & x^T (P^1 - P^2)x > 0 \end{cases}$$

The state response is shown in Figure 2.

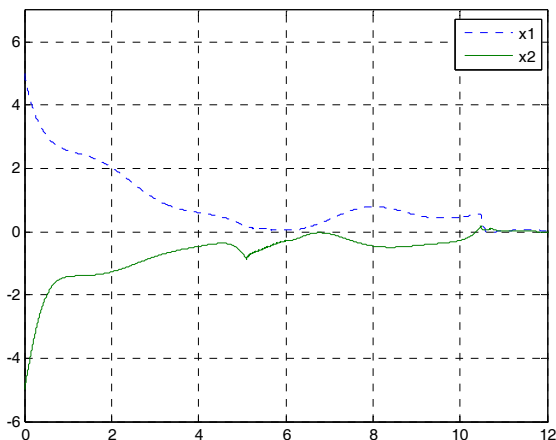


Fig. 2. State response of Example 2

$x_1$  and  $x_2$  are system states. The initial condition is  $[x_1, x_2] = [5, -5]$ . The result shows the system is stable under the switching law when it is switched among the closed-loop sub-systems.

### 5. State feedback robust $H_\infty$ stabilization for linear uncertain switched systems with multiple time delays

Consider the following linear switched system with multiple time delays

$$\begin{aligned}
 \dot{x}(t) &= (A_{\sigma(t)} + \Delta A_{\sigma(t)})x(t) + \sum_{j=1}^N (A_{dj\sigma(t)} + \Delta A_{dj\sigma(t)})x(t - \tau_j) \\
 &+ (B_{\sigma(t)} + \Delta B_{\sigma(t)})u(t) + B_{1\sigma(t)}w(t) \\
 z(t) &= (C_{\sigma(t)} + \Delta C_{\sigma(t)})x(t) + \sum_{j=1}^N (C_{dj\sigma(t)} + \Delta C_{dj\sigma(t)})x(t - \tau_j) \\
 &+ (D_{\sigma(t)} + \Delta D_{\sigma(t)})u(t) + B_{2\sigma(t)}w(t) \\
 x(t) &= \phi(t), t \in (-\max(\tau_j), 0)
 \end{aligned} \tag{39}$$

where  $x(t) \in R^n$  is the system state vector,  $u(t) \in R^{m1}$  is the control input of the system,  $z(t) \in R^{m2}$  is the output vector,  $w(t) \in l_2$  is the disturbance,  $\sigma(t) : [0, \infty) \rightarrow M = \{1, 2, 3, \dots, m\}$  is the switching signal,  $A_{\sigma(t)}$ ,  $A_{dj\sigma(t)}$ ,  $B_{\sigma(t)}$ ,  $B_{1\sigma(t)}$ ,  $C_{\sigma(t)}$ ,  $C_{dj\sigma(t)}$ ,  $D_{\sigma(t)}$ ,  $B_{2\sigma(t)}$  are known constant matrices,  $\Delta A_{d\sigma(t)}$ ,  $\Delta A_{dj\sigma(t)}$ ,  $\Delta B_{\sigma(t)}$ ,  $\Delta C_{\sigma(t)}$ ,  $\Delta C_{dj\sigma(t)}$ ,  $\Delta D_{\sigma(t)}$  are bounded time-varying real functional matrices with proper dimensions, denoting the uncertainties of the switched systems.  $\phi(t)$  is the initial condition of the system.  $\tau_j$  is the delay of the system state.

Denote

$$\begin{aligned} \bar{A} &= A_{\sigma(t)} + \Delta A_{\sigma(t)}, \bar{A}_{dj} = A_{dj\sigma(t)} + \Delta A_{dj\sigma(t)}, \bar{B} = B_{\sigma(t)} + \Delta B_{\sigma(t)} \\ \bar{C} &= C_{\sigma(t)} + \Delta C_{\sigma(t)}, \bar{C}_{dj} = C_{dj\sigma(t)} + \Delta C_{dj\sigma(t)}, \bar{D} = D_{\sigma(t)} + \Delta D_{\sigma(t)} \end{aligned} \tag{40}$$

and suppose :

$$\begin{bmatrix} \Delta A_{\sigma(t)} & \Delta A_{dj\sigma(t)} & \Delta B_{\sigma(t)} \\ \Delta C_{\sigma(t)} & \Delta C_{dj\sigma(t)} & \Delta D_{\sigma(t)} \end{bmatrix} = \begin{bmatrix} H_{1\sigma} \\ H_{2\sigma} \end{bmatrix} F_{\sigma(t)} \begin{bmatrix} E_{1\sigma} & E_{2j\sigma} & E_{3\sigma} \end{bmatrix} \tag{41}$$

where  $H_{1\sigma}, H_{2\sigma}, E_{1\sigma}, E_{2j\sigma}, E_{3\sigma}$  are real constant matrices with proper dimensions, and  $F_{\sigma(t)}$  satisfies:

$$F_{\sigma(t)}^T F_{\sigma(t)} \leq I \tag{42}$$

**Theorem 4.** For a given scalar  $\gamma_i > 0$ , if there are positive definite matrices  $P_i, Q_{li}, \dots, Q_{Ni} \in R^{n \times n}$ , such that:

$$\begin{bmatrix} S_1 & P_i(\bar{A}_{d1} + \bar{B}K_{li}) & \dots & P_i(\bar{A}_{dN} + \bar{B}K_{Ni}) & P_i B_1 & \bar{C}^T + K_i^T \bar{D}^T \\ * & -Q_{li} & 0 & 0 & 0 & \bar{C}_{d1}^T + K_{li}^T \bar{D}^T \\ * & 0 & \dots & 0 & 0 & \dots \\ * & 0 & 0 & -Q_{Ni} & 0 & \bar{C}_{dN}^T + K_{Ni}^T \bar{D}^T \\ * & * & * & * & -\gamma_i^2 I & B_2^T \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{43}$$

where  $S_1 = (\bar{A}^T + K_i^T \bar{B}^T)P_i + P_i(\bar{A} + \bar{B}K_i) + \sum_{j=1}^N Q_{ji}$ , the system (39) is robust stabilizable with  $H_\infty$  performance  $\gamma$ ,  $\gamma = \max(\gamma_i)$ . The state feedback switched  $H_\infty$  control with memory is

$$u(t) = K_i x(t) + \sum_{j=1}^N K_{ji} x(t - \tau_{ji})$$

**Proof:** Suppose there are positive definite matrices  $P_i, Q_{li}, \dots, Q_{Ni} \in R^{n \times n}$  and matrices  $K_i, K_{ji} \in R^{m \times n}$ , such that the inequality (43) holds. The state feedback control is

$$u(t) = K_i x(t) + \sum_{j=1}^N K_{ji} x(t - \tau_{ji}).$$

Choose the Lyapunov function as

$$V(x(t), w(t), t) = x^T(t)P_i x(t) + \sum_{j=1}^N \int_{t-\tau_j}^t x^T(s)Q_{ji} x(s)ds$$

The derivative of the Lyapunov function is:

$$\begin{aligned} \dot{V}(x(t), w(t), t) &= \dot{x}^T(t)P_i x(t) + x^T(t)P_i \dot{x}(t) + \sum_{j=1}^N \dot{x}^T(t)Q_{ji} x(t) - \sum_{j=1}^N \dot{x}^T(t-\tau_j)Q_{ji} x(t-\tau_j) \\ &= x^T(t)[P_i(\bar{A} + \bar{B}K_i) + (\bar{A} + \bar{B}K_i)^T P_i + \sum_{j=1}^N Q_{ji}]x(t) + \sum_{j=1}^N x^T(t-\tau_j)(\bar{A}_{dj} + \bar{B}K_{ji})^T P_i x(t) \\ &\quad + \sum_{j=1}^N x^T(t)P_i(\bar{A}_{dj} + \bar{B}K_{ji})x(t-\tau_j) + w^T(t)B_1^T P_i x(t) + x^T(t)P_i B_1 w(t) - \sum_{j=1}^N x^T(t-\tau_j)Q_{ji} x(t-\tau_j) \end{aligned} \tag{44}$$

When  $w(t) = 0$ , if the following inequality holds, the closed-loop is asymptotically stable

$$\begin{aligned}
 & x^T(t)[P_i(\bar{A} + \bar{B}K_i) + (\bar{A} + \bar{B}K_i)^T P_i + \sum_{j=1}^N Q_{ji}]x(t) + \sum_{j=1}^N x^T(t - \tau_{ji})(\bar{A}_{dj} + \bar{B}K_{ji})^T P_i x(t) \\
 & + \sum_{j=1}^N x^T(t) P_i (\bar{A}_{dj} + \bar{B}K_{ji}) x(t - \tau_{ji}) - \sum_{j=1}^N x^T(t - \tau_{ji}) Q_{ji} x(t - \tau_{ji}) < 0
 \end{aligned} \tag{45}$$

Rewrite the above inequality as

$$\begin{bmatrix} x(t) \\ x(t - \tau_{i1}) \\ \dots \\ x(t - \tau_{Ni}) \end{bmatrix}^T \begin{bmatrix} S_i & P_i(\bar{A}_{d1} + \bar{B}K_{i1}) & \dots & P_i(\bar{A}_{dN} + \bar{B}K_{iN}) \\ * & -Q_{i1} & 0 & 0 \\ * & 0 & \dots & 0 \\ * & 0 & 0 & -Q_{iN} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - \tau_{i1}) \\ \dots \\ x(t - \tau_{Ni}) \end{bmatrix} < 0$$

By Lemma 2 and inequality (43), we have

$$\begin{bmatrix} S_i & P_i(\bar{A}_{d1} + \bar{B}K_{i1}) & \dots & P_i(\bar{A}_{dN} + \bar{B}K_{iN}) & P_i B_i \\ * & -Q_{i1} & 0 & 0 & 0 \\ * & * & \dots & 0 & 0 \\ * & * & * & -Q_{iN} & 0 \\ * & * & * & * & -\gamma_i^2 I \end{bmatrix} + \begin{bmatrix} \bar{C}^T + K_i^T \bar{D}^T \\ \bar{C}_{d1}^T + K_{i1}^T \bar{D}^T \\ \dots \\ \bar{C}_{dN}^T + K_{iN}^T \bar{D}^T \\ B_i^T \end{bmatrix} \begin{bmatrix} \bar{C}^T + K_i^T \bar{D}^T \\ \bar{C}_{d1}^T + K_{i1}^T \bar{D}^T \\ \dots \\ \bar{C}_{dN}^T + K_{iN}^T \bar{D}^T \\ B_i^T \end{bmatrix}^T < 0 \tag{46}$$

For any  $x(t), x(t - \tau_{i1}), \dots, x(t - \tau_{Ni}), w(t)$ , the following inequality holds.

$$\begin{aligned}
 & \begin{bmatrix} x(t) \\ x(t - \tau_{i1}) \\ \dots \\ x(t - \tau_{Ni}) \\ w(t) \end{bmatrix}^T \begin{bmatrix} S_i & P_i(\bar{A}_{d1} + \bar{B}K_{i1}) & \dots & P_i(\bar{A}_{dN} + \bar{B}K_{iN}) & P_i B_i \\ * & -Q_{i1} & 0 & 0 & 0 \\ * & * & \dots & 0 & 0 \\ * & * & * & -Q_{iN} & 0 \\ * & * & * & * & -\gamma_i^2 I \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - \tau_{i1}) \\ \dots \\ x(t - \tau_{Ni}) \\ w(t) \end{bmatrix} + \\
 & \begin{bmatrix} x(t) \\ x(t - \tau_{i1}) \\ \dots \\ x(t - \tau_{Ni}) \\ w(t) \end{bmatrix}^T \begin{bmatrix} \bar{C}^T + K_i^T \bar{D}^T \\ \bar{C}_{d1}^T + K_{i1}^T \bar{D}^T \\ \dots \\ \bar{C}_{dN}^T + K_{iN}^T \bar{D}^T \\ B_i^T \end{bmatrix} \begin{bmatrix} \bar{C}^T + K_i^T \bar{D}^T \\ \bar{C}_{d1}^T + K_{i1}^T \bar{D}^T \\ \dots \\ \bar{C}_{dN}^T + K_{iN}^T \bar{D}^T \\ B_i^T \end{bmatrix}^T \begin{bmatrix} x(t) \\ x(t - \tau_{i1}) \\ \dots \\ x(t - \tau_{Ni}) \\ w(t) \end{bmatrix} < 0
 \end{aligned} \tag{47}$$

Thus

$$\dot{V}(x(t), w(t), t) + z^T(t)z(t) - \gamma_i^2 w^T(t)w(t) < 0$$

Under the zero initial condition, by setting  $\gamma = \max(\gamma_i), i \in M$ , we have

$$\begin{aligned}
 & \int_0^{+\infty} [z^T(t)z(t) - \gamma^2 w^T(t)w(t)] dt \\
 & \leq \int_0^{+\infty} [\dot{V}(x(t), w(t), t) + z^T(t)z(t) - \gamma_i^2 w^T(t)w(t)] dt - V(x(+\infty), w(+\infty), +\infty) \\
 & = \sum_{i=1}^M \int_{t_{i-1}}^{t_i} [\dot{V}(x(t), w(t), t) + z^T(t)z(t) - \gamma_i^2 w^T(t)w(t)] dt - V(x(+\infty), w(+\infty), +\infty) < 0
 \end{aligned}$$

Therefore

$$\int_0^{+\infty} z^T(t)z(t) dt \leq \gamma^2 \int_0^{+\infty} w^T(t)w(t) dt$$

This completes the proof.

**Remark 3.** Theorem 4 presents a sufficient condition for robust stabilization with  $H_\infty$  performance  $\gamma$ . Since there are uncertainties in inequality (43), it can not be solved directly.

**Lemma 5 :** For system (39), and given positive scalar  $\gamma_i$ , if there are matrices  $Y_i, Y_{1i}, \dots, Y_{Ni}$ , positive definite matrices  $X_i, R_{1i}, \dots, R_{Ni}$  with proper dimensions, and scalar  $\alpha > 0$ , such that the following inequality holds

$$\begin{bmatrix} S & A_{d1}X_i + BY_{1i} & \dots & A_{dN}X_i + BY_{Ni} & B_1 & \varphi & X_i E_1^T + Y_i^T E_3^T \\ * & -R_{1i} & 0 & 0 & 0 & X_i C_{d1}^T + Y_{1i}^T D^T & X_i E_{21}^T + Y_{1i}^T E_3^T \\ * & * & \dots & 0 & 0 & \dots & \dots \\ * & * & * & -R_{Ni} & 0 & X_i C_{dN}^T + Y_{Ni}^T D^T & X_i E_{2N}^T + Y_{Ni}^T E_3^T \\ * & * & * & * & -\gamma_i^2 I & B_2^T & 0 \\ * & * & * & * & * & -I + \alpha H_2 H_2^T & 0 \\ * & * & * & * & * & * & -\alpha I \end{bmatrix} < 0 \tag{48}$$

where  $S = X_i A^T + A X_i + Y_i^T B^T + B Y_i + \sum_{j=1}^N R_{ji} + \alpha H_1 H_1^T$ ,  $\varphi = X_i C^T + Y_i^T D^T + \alpha H_1 H_2^T$ , the system (39)

is robust stabilizable with  $H_\infty$  performance  $\gamma = \max(\gamma_i)$ . The robust  $H_\infty$  control is given by :

$$u(t) = K_i x(t) + \sum_{j=1}^N K_{ji} x(t - \tau_{ji}), K_i = Y_i X_i^{-1}, K_{ji} = Y_{ji} X_i^{-1}, i \in M, j \in N$$

The switching law is  $\sigma(t) = \arg \min_{i \in M} \{x^T(t) X_i^{-1} x(t)\}$ .

**Proof.** The proof is similar to Theorem 3, and is omitted.

**Example 3** The linear uncertain switched system (39) with multiple time delays is given below.

$$\begin{aligned} A^1 &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, A_1^1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, A_2^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_1^1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \\ B_2^1 &= \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, C^1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, C_1^1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}, C_2^1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.2 \end{bmatrix}, D^1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix} \\ H_1^1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, H_2^1 = \begin{bmatrix} 0.5 & 0.4 \\ 0 & 0 \end{bmatrix}, E_1^1 = \begin{bmatrix} 0.5 & 0.2 \\ 0 & 0 \end{bmatrix}, E_{21}^1 = \begin{bmatrix} 0.5 & 0.4 \\ 0 & 0 \end{bmatrix}, E_{22}^1 = \begin{bmatrix} 0.2 & 0.5 \\ 0 & 0 \end{bmatrix} \\ E_3^1 &= \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{49}$$

and  $\gamma^1 = 3, w^1(t) = \cos(t), F^1(t) = \sin(t), \alpha^1 = 0.5, \tau_1^1 = \tau_2^1 = 0.02$

$$\begin{aligned} A^2 &= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, A_1^2 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, A_2^2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_1^2 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} \\ B_2^2 &= \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}, C^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, C_1^2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}, C_2^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0.2 \end{bmatrix}, D^2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix} \\ H_1^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, H_2^2 = \begin{bmatrix} 0.3 & 0.4 \\ 0 & 0 \end{bmatrix}, E_1^2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2 \end{bmatrix}, E_{21}^2 = \begin{bmatrix} 0.2 & 0.4 \\ 0 & 0 \end{bmatrix}, E_{22}^2 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix} \\ E_3^2 &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$



and  $\gamma^2 = 1, w^2(t) = \cos(t), F^2(t) = \sin(t), \alpha^2 = 0.5, \tau_1^2 = \tau_2^2 = 0.02$

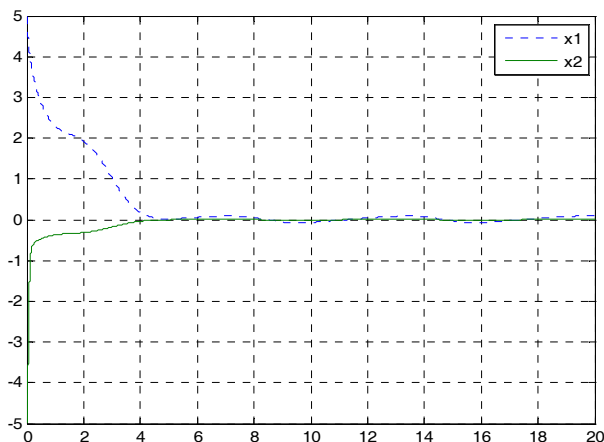


Fig. 3. State response of Example 3

By Lemma 5, the state feedback gain for system stabilization with  $H_\infty$  performance are

$$P^1 = X_1^- = \begin{bmatrix} 0.0352 & 0.2345 \\ 0.2345 & 1.5610 \end{bmatrix}, K^1 = \begin{bmatrix} -0.0171 & -0.1135 \\ -0.3927 & -2.6146 \end{bmatrix}, K_1^1 = \begin{bmatrix} 0.0559 & 0.3723 \\ -1.0425 & -6.9413 \end{bmatrix}, K_2^1 = \begin{bmatrix} 0.0559 & 0.3723 \\ -1.0425 & -6.9413 \end{bmatrix}$$

$$P^2 = X_2^- = \begin{bmatrix} 0.0025 & -0.0842 \\ -0.0842 & 2.6723 \end{bmatrix}, K^2 = \begin{bmatrix} -0.0075 & 0.2345 \\ 0.1130 & -3.6644 \end{bmatrix}, K_1^2 = \begin{bmatrix} -0.0714 & 2.3169 \\ 0.2707 & -8.7784 \end{bmatrix}, K_2^2 = \begin{bmatrix} -0.0721 & 2.3395 \\ 0.2729 & -8.8480 \end{bmatrix}$$

The switching law is:

$$\sigma(t) = i = \begin{cases} 1, & x^T (P^1 - P^2)x \leq 0 \\ 2, & x^T (P^1 - P^2)x > 0 \end{cases}$$

The state responses are shown in Figure 3.

$x_1$  and  $x_2$  are system states, the initial condition is  $[x_1, x_2] = [5, -5]$ . The result shows the system is stable under the switching law when it is switched among the closed-loop subsystems.

## 6. Conclusion

This chapter studies the robust  $H_\infty$  control for linear switched systems with time delay. After introducing robust  $H_\infty$  stability and stabilization of linear switched systems, we firstly analyzed robust  $H_\infty$  control for general linear switched systems with time delay. Based on the multi-Lyapunov-Function method, a sufficient condition is derived in terms of LMI. The robust  $H_\infty$  control and the switching law design are also given.

By involving uncertainties and time-varying delay, the robust  $H_\infty$  control for uncertain linear switched systems with time varying delay is studied. Through the multi-Lyapunov-Function approach, a sufficient condition is given in LMI formulation. The robust switched  $H_\infty$  control and the the switching law design are presented as well.

The state feedback robust  $H_\infty$  control with memory is also studied for uncertain linear switched systems with multiple time delays. A sufficient condition is given as well as the robust switched  $H_\infty$  control and the switching law.

Illustrative examples are given to show the effectiveness of the proposed methods.

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This book presents selected issues related to switched systems, including practical examples of such systems. This book is intended for people interested in switched systems, especially researchers and engineers. Graduate and undergraduate students in the area of switched systems can find this book useful to broaden their knowledge concerning control and switching systems.

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### **InTech Europe**

University Campus STeP Ri  
Slavka Krautzeka 83/A  
51000 Rijeka, Croatia  
Phone: +385 (51) 770 447  
Fax: +385 (51) 686 166  
[www.intechopen.com](http://www.intechopen.com)

### **InTech China**

Unit 405, Office Block, Hotel Equatorial Shanghai  
No.65, Yan An Road (West), Shanghai, 200040, China  
中国上海市延安西路65号上海国际贵都大饭店办公楼405单元  
Phone: +86-21-62489820  
Fax: +86-21-62489821

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