Differential Electromagnetic Forms in Rotating Frames

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1. Introduction

Differential forms are completely antisymmetric homogeneous r-tensors on a differentiable n-manifold $0 \leq r \leq n$ belonging to the Grasssman algebra [1] and endowed by Cartan [2] with an exterior calculus. These differential forms found an immediate application in geometry and mechanics; introduced by Deschamps [3,4] in electromagnetism, they have known in parallel with the expansion of computers, an increasing interest [5-9] because Maxwell's equations and the constitutive relations are put in a manifestly independent coordinate form.

In the Newton (3+1) space-time, with the euclidean metric $ds^2 = dx^2 + dy^2 + dz^2$ the conventional Maxwell equations in which the **E**, **B**, **D**, **H** fields are 3-vectors have, in absence of charge and current, the Gibbs representation

$$\nabla.\mathbf{B} = 0, \qquad \nabla\wedge\mathbf{E} + 1/c\partial_t\mathbf{B} = 0 \tag{1a}$$

$$\nabla.\mathbf{D} = 0, \qquad \nabla\wedge\mathbf{H} - 1/c\partial_t\mathbf{D} = 0 \tag{1b}$$

and, they also have the differential form representation ($\partial_\tau = 1/c\partial_t$) [5,7]

$$d \wedge E + \partial_\tau B = 0, \qquad d \wedge B = 0 \tag{2a}$$

$$d \wedge H - \partial_\tau D = 0, \qquad d \wedge D = 0 \tag{2b}$$

$d = dx\partial_x + dy\partial_y + dz\partial_z$ is the exterior derivative, E , H the differential 1-forms

$$E = E_x\,dx + E_y\,dy + E_z\,dz, \qquad H = H_x\,dx + H_y\,dy + H_z\,dz \tag{3a}$$

and B , D the differential 2-forms

$$B = B_x(dy\wedge dz) + B_y(dz\wedge dx) + B_z(dx\wedge dy) , \; D = -[\,D_x(dy\wedge dz) + D_y(dz\wedge dx) + D_z(dx\wedge dy)] \tag{3b}$$

We are interested here, for reasons to be discussed in Sec.(6) in a Frenet-Serret frame rotating around oz with a constant angular velocity requiring a relativistic processing, We shall prove that this situation leads to an Einstein space-time with a riemannian metric. As an introduction to this problem, we give a succcinct presentation of differential electromagnetic forms in a Minkowski space-time with the metric $ds^2 = dx^2 + dy^2 + dz^2 - c^{-2}\partial_t^2$.

2. Differential forms in Minkowski space-time [7]

In absence of charge and current, the Maxwell equations have the tensor representaation [10, 11]

$$\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \text{ a)} \qquad \partial_\nu F^{\mu\nu} = 0 \text{ b)} \qquad (4)$$

the greek (resp.latin) indices take the values 1,2,3,4 (resp.1,2,3) with $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = ct$, $\partial_j = \partial/\partial x_j$, $\partial_4 = 1/c\partial/\partial_t$ and the summation convention is used. The components of the tensors $F_{\mu\nu}$ and $F^{\mu\nu}$ are with the 3D Levi-Civita tensor ε_{ijk}

$$B_i = \tfrac{1}{2}\,\varepsilon_{ijk}\, F_{jk}, \quad E_i = -\,F_{i4}\,, \qquad H_i = \tfrac{1}{2}\,\varepsilon_{ijk}\, F^{jk}\,, \quad D_i = -\,F^{i4} \qquad (5)$$

and in vacuum

$$\mathbf{D} = \varepsilon_0\, \mathbf{E}, \qquad \mathbf{H} = \mu_0^{-1}\mathbf{B}\,, \qquad (\varepsilon_0\mu_0)^{1/2} = 1/c \qquad (5a)$$

Let d be the exterior derivative operator

$$d = (\partial_x\, dx + \partial_y\, dy + \partial_z\, dz + \partial_t\, dt\,)\wedge \qquad (6)$$

and $F = E + B$ be the two-form in which :

$$E = E_x\,(dx\wedge cdt) + E_y\,(dy\wedge cdt) + E_z\,(dz\wedge cdt)$$

$$B = B_x\,(dy\wedge dz) + B_y\,(dz\wedge dx) + B_z\,(dx\wedge dy) \qquad (7)$$

Then the Maxwell equations (4a) have the differential 3-form representation d F = 0. Similarly for $G = D + H$ with :

$$H = H_x\,(dx\wedge cdt) + H_y\,(dy\wedge cdt) + H_z\,(dz\wedge cdt)$$

$$D = -[D_x\,(dy\wedge dz) + D_y\,(dz\wedge dx) + D_z\,(dx\wedge dy)] \qquad (8)$$

the differential 3-form representation of Maxwell's equations (4b) is d G = 0.
To manage the constituive relations (5a) the Hodge star operator [6,9] is introduced

$$* (dx\wedge cdt) = c^{-1}\,(dy\wedge dz)\,, \qquad * (dy\wedge dz) = c\,(dx\wedge cdt)$$

$$* (dy\wedge cdt) = c^{-1}\,(dz\wedge dx)\,, \qquad * (dz\wedge dx) = c\,(dy\wedge cdt)$$

$$* (dz\wedge cdt) = c^{-1}\,(dx\wedge dy)\,, \qquad * (dx\wedge dy) = c\,(dz\wedge cdt) \qquad (9)$$

Applying the Hodge star operator to F gives $*F$ = $*E$ + $*B$ and one checks easily the relation G = $\lambda_0\, *F$ with $\lambda_0 = (\varepsilon_0/\mu_0)^{1/2}$ so that the Maxwell equations in the Minkowski vacuum, have the diffrerential 3-form representation

$$dF = 0, \qquad d *F = 0 \qquad (10)$$

3. Electromagnetidsm in a Frenet-Serret rotating frame

We consider a frame rotating with a constant angular velocity Ω around oz. Then, using the Trocheris-Takeno relativistic description of rotation [12, 13], the relations   between the

cylindrical coordinates R,Φ,Z,T and r,ϕ,z,t in the natural (fixed) and rotating frames are with ß = ΩR /c

$$R = r, \qquad \Phi = \phi \cosh\beta - ct/r \sinh\beta$$

$$Z = z, \qquad cT = ct \cosh\beta - r\phi \sinh\beta \tag{11}$$

and a simple calculation gives the metric ds^2 in the rotating frame

$$ds^2 = c^2 dt^2 - dz^2 - r^2 d\phi^2 - (1 + B^2 - A^2) dr^2 - 2(A \sinh\beta + B \cosh\beta) cdt\, dr -$$

$$2(A \cosh\beta + B \sinh\beta)\, r\, dr\, d\phi \tag{12}$$

$$A = \beta \sinh\beta\, ct/r + \beta \cosh\beta\, \phi + \sinh\beta\, \phi, \quad B = \beta \sinh\beta\, \phi + \beta \cosh\beta\, ct/r - \sinh\beta\, ct/r \tag{12a}$$

Using the notations $x_4 = ct$, $x_3 = z$, $x_2 = \phi$, $x_1 = r$, we get from (12) $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ with

$$g_{44} = 1, \quad g_{33} = -1, \quad g_{22} = -r^2, \quad g_{11} = -(1 + B^2 - A^2)$$

$$g_{14} = g_{41} = 2(A \sinh\beta + B \cosh\beta), \quad g_{12} = g_{21} = 2(A \cosh\beta + B \sinh\beta) \tag{13}$$

The determinant g of $g_{\mu\nu}$ is

$$g = g_{33}[\, g_{11}g_{22}g_{44} - g_{12}^2 g_{44} - g_{14}^2 g_{22}\,]$$

$$= R\,[g_{11} - g_{12}^2 r^{-2} - g_{14}^2] \tag{14}$$

but

$$g_{12}^2 r^{-2} + g_{14}^2 = 4(A^2 - B^2) \tag{14a}$$

and, taking into account the expression (13) of g_{11}, we get finally

$$g = r^2[5(A^2 - B^2) - 1], \quad A^2 - B^2 = (\phi^2 - c^2 t^2/r^2)(\beta^2 + \sinh^2\beta + 2\beta \sinh\beta \cosh\beta) \tag{15}$$

So, the rotating Frenet-Serret frame defines an Einstein space-time with the riemannian metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, and in this Einstein space-time the Maxwell equations have the tensor representation[14, 15]

$$\partial_\sigma G_{\mu\nu} + \partial_\mu G_{\nu\sigma} + \partial_\nu G_{\sigma\mu} = 0 \quad \text{a)} \quad \partial_\nu (|g|^{1/2} G^{\mu\nu}) = 0 \quad \text{b)} \tag{16}$$

in which, using the cylindrical coordinates r,ϕ.z ,t with $x_1 = r$, $x_2 = \phi$, $x_3 = z$, $x_4 = ct$; $\partial_1 = \partial_r$, $\partial_2 = \partial_\phi$, $\partial_3 = \partial_z$, $\partial_4 = 1/c\, \partial_t$, the components of the electromagnetic tensors are

$$G_{12} = rB_z, \quad G_{13} = -B_\phi, \quad G_{23} = rB_r; \quad G_{14} = -E_z, \quad G_{24} = -rE_\phi, \quad G_{34} = -E_z$$

$$G^{12} = H_z/r, \quad G^{13} = -H_\phi, \quad G^{23} = H_r/r; \quad G^{14}\, D_z, \quad G_2{}^4 = D_\phi/r, \quad G^{34} = D_z \tag{17}$$

To work with the differential forms, we introduce the exterior derivative

$$\underline{d} = (\partial_r dr + \partial_\phi d\phi + \partial_z dz + \partial_t dt) \wedge \tag{18}$$

(underlined expressions mean that they are defined with the cylindrical coordinates r, ϕ,z,t) and the two-forms $\underline{F} = \underline{E} + \underline{B}$ with

$$\underline{E} = E_r\,(dr \wedge cdt) + E_\phi\,(rd\phi \wedge cdt) + E_z\,(dz \wedge cdt)$$

$$\underline{B} = B_r\,(rd\phi \wedge dz) + B_\phi\,(dz \wedge dr) + B_z\,(dr \wedge rd\phi) \qquad (19a)$$

and writing $|g|^{1/2} = rq$, $q = [5(A^2-B^2)-1]^{1/2}$ the two-form $\underline{G} = \underline{D} + \underline{H}$

$$\underline{D} = -\,q\,[D_r\,(rd\phi \wedge dz) + D_\phi\,(dz \wedge dr) + D_z\,(dr \wedge rd\phi)]$$

$$\underline{H} = q[H_r\,(dr \wedge cdt) + H_\phi\,(rd\phi \wedge cdt) + H_z\,(dz \wedge cdt)] \qquad (19b)$$

Then, the Maxwell equatios have the 3-form representation

$$d\underline{F} = 0\,, \qquad d\,\underline{G} = 0 \qquad (20)$$

A simple calculation gives

$$
\begin{aligned}
d\,\underline{F} =\ & [(\partial_r(rB_r) + \partial_\phi B_\phi + \partial_z(rB_z)]\,(dr \wedge d\phi \wedge dz)\ + \\
& [\partial_t\,(rB_r) + c\{\partial_\phi E_z - \partial_z(rE_\phi)\}]\,(d\phi \wedge dz \wedge dt)\ + \\
& [\partial_t B_\phi + c(\partial_z E_r - \partial_r E_z)]\,(dz \wedge dr \wedge dt)\ + \\
& [\partial_t\,(rB_z) + c\{\partial_r\,(rE_\phi) - \partial_\phi E_r\}]\,(dr \wedge d\phi \wedge dt) \qquad (21a)
\end{aligned}
$$

$$
\begin{aligned}
d\underline{G} =\ & -\,[\partial_r(qrD_r) + \partial_\phi(qD_\phi) + \partial_z(qD_z)]\,(dr \wedge d\phi \wedge dz)\ + \\
& [-\partial_t(qrD_r) + c\,\{\partial_\phi(qH_z) - \partial_z(qrH_\phi)\}]\,(d\phi \wedge dz \wedge dt)\ + \\
& [-\partial_t(qD_\phi) + c\,\{\partial_z(qH_r) - \partial_r(qH_z)\}]\,(dz \wedge dr \wedge dt)\ + \\
& [-\partial_t(qrD_z) + c\,\{\partial_r(qrH_\phi - \partial_\phi(qH_r)\}]\,(dr \wedge d\phi \wedge dt) \qquad (21b)
\end{aligned}
$$

The Hodge star operator needed to take into account the constitutive relations (5a) in vacuum is defined by the relation

$$* \,(dr \wedge cdt) = -qc^{-1}\,(rd\phi \wedge dz), \qquad * \,(rd\phi \wedge dz) = q^{-1}c\,(dr \wedge cdt)$$

$$* \,(rd\phi \wedge cdt) = -qc^{-1}\,(dz \wedge dr)\,, \qquad * \,(dz \wedge dr) = q^{-1}c\,(rd\phi \wedge cdt)$$

$$* \,(dz \wedge cdt) = -qc^{-1}\,(dr \wedge rd\phi), \qquad * \,(dr \wedge rd\phi) = q^{-1}c\,(dz \wedge cdt) \qquad (22)$$

Applying (22) to $\underline{F}$ gives $*\boldsymbol{\underline{F}} = {}^*\underline{E} + {}^*\underline{B}$ and it is easily checked that $\underline{G} = \lambda_0\,{}^*\boldsymbol{F}$ with $\lambda_0 = (\varepsilon_0/\mu_0)^{1/2}$ so that in vacuum $d\underline{F} = 0$, $d\,{}^*\boldsymbol{\underline{F}} = 0$.

4. Wave equations in vacuum

4.1 Minkowski space-time

The wave equations satisfied by the electromagnetic field (in absence of charges and currents) are obtained from differential forms with the help of the Laplace-De Rham operator [6,8]

$$L = (d* \,d* + *\,d*\,d) \wedge \qquad (23)$$

requiring the Hodge star operators for the n-forms, n = 1,2,3. They are given in Appendix A where, using the exterior derivative (6), and assuming $E_x = E_y = 0$ so that the two-form (7) becomes $E_z = E_z$ $(dz \wedge cdt)$, we get

$$\text{L } E_z = (\Delta - c^{-2}\partial_t^2) E_z (dz \wedge cdt), \qquad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 \tag{24}$$

A similar relation exists for E_x, E_y and for the components of the **B**-field so that, we get finally for the 2-form F:

$$\text{L } F = (\Delta - c^{-2}\partial_t^2) F_{\mu\nu} (dx^\mu \wedge dx^\nu) \tag{25}$$

so that the wave equation has the 2-form representation L F = 0 .

4.2 Einstein space-time
4.2.1 Cartesian frame
In a riemannian cartesian frame, the components of the electromagnetic field tensor $F_{\mu\nu}$ are solutions of the tensor wave equation [16]

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta F_{\mu\nu} - 2R_{\mu\nu\alpha\rho}F^{\rho\alpha} + R_\mu{}^\rho F_{\rho\nu} - R_\nu{}^\rho F_{\rho\mu} = 0 \tag{26}$$

∇_α is the covariant derivative, $R_{\mu\nu\alpha\rho}$ and $R_\mu{}^\rho$ the Riemann curvature and Ricci tensors defined in terms of Christtoffel symbols $\Gamma_{\alpha,\beta\mu}$,$\Gamma^\rho{}_{\alpha\mu}$

$$\Gamma_{\beta,\mu\nu} = \tfrac{1}{2}(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu}), \qquad \Gamma^\alpha{}_{\mu\nu} = g^{\alpha\beta}\Gamma_{\beta,\mu\nu} \tag{27}$$

by the relations in which $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, $\partial_3 = \partial_y$, $\partial_4 = 1/c\partial_t$,

$$R_{\alpha\beta\mu\nu} = \partial_\nu\Gamma_{\alpha,\beta\mu} - \partial_\mu\Gamma_{\alpha,\beta\nu} + \Gamma^\rho{}_{\alpha\mu}\Gamma_{\rho,\beta\nu} - \Gamma^\rho{}_{\alpha\nu}\Gamma_{\rho,\beta\mu}$$

$$R^{\alpha\beta}{}_{\mu\nu} = g^{\alpha\rho}g^{\beta\sigma}R_{\rho\sigma\mu\nu}, \qquad R_\mu{}^\nu = R^{\nu\alpha}{}_{\alpha\mu}. \tag{28}$$

Now, it is proved [6] that the Laplace-De Rham operator L = d*d* + *d*d applied to the two form (7) written

$$\text{E} = F_{14}(dx^1 \wedge dx^4) + F_{24}(dx^2 \wedge dx^4) + F_{34}(dx^3 \wedge dx^4) \tag{29}$$

in which $dx^1 = dx$, $dx^2 = dy$, $dx^3 = dz$, $dx^4 = cdt$ gives for the component F_{i4}

$$\text{L } E = \tfrac{1}{2}(g^{\alpha\beta}\nabla_\alpha\nabla_\beta F_{i4} - 2R_{i4\alpha\rho}F^{\rho\alpha} + R_i{}^\rho F_{\rho4} - R_4{}^\rho F_{\rho i})((dx^i \wedge dx^4) \tag{30}$$

L E = 0 gives the 2-form representation of the wave equation in the Einstein space-time with cartesian coordinates
A similar result is obtained for **B** writing $-1/2F_{ij}$ $(dx^i \wedge dx^j)$ the B magnetic two-form (7) so that

$$\text{LB} = -1/4(g^{\alpha\beta}\nabla_\alpha\nabla_\beta F_{ij} - 2R_{ij\alpha\rho}F^{\rho\alpha} + R_i{}^\rho F_{\rho j} - R_j{}^\rho F_{\rho i}) (dx^i \wedge dx^j) \tag{30a}$$

Summing (30) and (30a) gives

$$\text{LF} = \tfrac{1}{2}[g^{\alpha\beta}\nabla_\alpha\nabla_\beta F_{\mu\nu} - 2R_{\mu\nu\alpha\rho}F^{\rho\alpha} + R_\mu{}^\rho F_{\rho\nu} - R_\nu{}^\rho F_{\rho\mu}] (dx^\mu \wedge dx^\nu) \tag{31}$$

In the Minkowski cartesian frame where $g_{ij} = \delta_{ij}$, $g_{44} = -1$, Eq.(31) reduces to (25).

4.2.2 Frenet-Serret frame

In the Frenet-Serret frame, the Laplace-De Rham operator is defined with the exterior derivative operator (18) and to get a relation such as (30) on the components of the electric field requires some care. First with the greek indices associated to the polar coordinates as previously, one has first to get the Christoffel symbols needed to define the covariant derivative and according to (28), the Riemann curvature and Ricci tensors, a job performed in Appendix B, we are now in position to transpose (30) to a rotating cyindrical frame. To this end, the electric two-form (19a) with

$$\underline{dx^1} \wedge \underline{dx^4} = dr \wedge cdt, \qquad \underline{dx^2} \wedge \underline{dx^4} = d\phi \wedge cdt, \quad \underline{dx^3} \wedge \underline{dx^4} = dz \wedge cdt \tag{32}$$

is written

$$\underline{E} = E_r \, (\underline{dx^1} \wedge \underline{dx^4}) + rE_\phi \, (\underline{dx^2} \wedge \underline{dx^4}) + E_z \, (\underline{dx^3} \wedge \underline{dx^4}) \tag{33}$$

but E_r, rE_ϕ, E_z are the G_{i4} components of the $G_{\mu\nu}$ tensor (17) so that leaving aside a minus sign

$$\underline{E} = G_{14} \, (\underline{dx^1} \wedge \underline{dx^4}) + G_{24} \, (\underline{dx^2} \wedge \underline{dx^4}) + G_{34} \, (\underline{dx^3} \wedge \underline{dx^4}) \tag{34}$$

and we get

$$\underline{L}\,\underline{E} = \tfrac{1}{2}(\, (g^{\alpha\beta} \, \underline{\nabla}_\alpha \underline{\nabla}_\beta G_{i4} - 2\underline{R}_{i4\alpha\rho} G^{\rho\alpha} + \underline{R}_i{}^\rho G_{\rho 4} - \underline{R}_4{}^\rho G_{\rho i})) \, (\underline{dx^i} \wedge d\underline{x^4}) \tag{35}$$

so that the components of the electric field are solutions of the two-form equation $\underline{L}\,\underline{E} = 0$ in the Frenet-Serret rotating frame. For the other components of the electromagnetic field, it comes

$$\underline{L}\,\underline{F} = \tfrac{1}{2}(\, (g^{\alpha\beta} \, \underline{\nabla}_\alpha \underline{\nabla}_\beta G_{\mu\nu} - 2\underline{R}_{\mu\nu\alpha\rho} G^{\rho\alpha} + \underline{R}_\mu{}^\rho G_{\rho\nu} - \underline{R}_\nu{}^\rho G_{\rho\mu}) \, (\underline{dx^\mu} \wedge d\underline{x}^\nu) \tag{36}$$

We have only considered the two-form F because in vacuum $G = \lambda_0{}^* F$.

5. To solve differential form equations

The local 2-form representation (2) of Maxwell's equations follows, as a consequence of the Stokes's theorem, from the Maxwell-Ampère and Maxwell-Faraday integral relations. Then, coming back to these theorems, to solve differential form equations is tantamount to perform the integrals

$$I = \int_M \omega \tag{37}$$

in which ω is a n-form, for instance F ou G, and M an oriented manifold with the same n-dimension as the degree of the ω form [5].

In the 3D-space, the numerical evaluation of (37) is based on the finite element technique, largely used [17, 18] in the simulation of partial differential equations. The manifold M is described by a chain of simplexes made for instance of triangular surfaces, tetrahedral volumes… on which the Whitney forms [5,19,20] gives a manageable description of the n-form ω. A simple example may be found in [19] and a through discussion of the technique in [20]. These solutions may be called weak in opposition to the strong solutions of the Maxwell's equations (1).

The numerical process just described is limited to the 3D-space but in the 4D space-time, in particular for the Frenet-Serret frame, ω depends on dt so that M has to be defined in terms of 2-cells, 3-cells, 4-cells of space-time [21] and the Whitney forms must be generalized accordingly. It does not seem that computational works have been made in this domain.

6. Discussion

Differential electromagnetic forms are usually managed in a Newton space and more rarely in a Minkowski space-time although, in this case, the comparison between tensors and differential forms is very enlightning [7]. This formalism is analyzed here in an Einstein space-time with a Riemann metric, particularly that of a Frenet-Serret frame. From a theoretical point of view, except for some more intricate relations due to Riemann, Ricci tensors and Christoffel symbols there is no difficulty to go from Newton to Einstein differential forms.. The situation is different from a computational point of view, since as mentionned in Sec.5, an important worrk has still to be performed to get the solutions of the differential form equation in an Einstein space-time.

This work may be considered as a first step in a complete analysis of electromagnetic differential forms in an Einstein space-time. The subjects to be discussed go from the presence of charges and currents (left aside here) to boundary conditions with between the introduction of potentials, the energy conveyed by the electromagnetic field and so on. This extension could be performed in the syle used in [7] to analyze the electromagnetic differential forms in a Minkowski space-time. In addition, it would make possible an interesting comparison (al-ready sketched in Sec.4.1) with the electromagnetic tensor formalism of the General Relati-vity [15].

Now, why to take an interest in rotating frames? A first response could have been ''Universe'' assumed cylindrical. But, although Einstein and Romer (also Levi Civita) have obtained some exact cylindrical wave solutions of the general relativity equations [22], this cosmos has been superseded by a spherical world (nevertheless, because of its particular properties, some works are still devoted to the Levi Civita world [23]. A second response comes from the analysis of the Wilsons' experiments in which was measured the electric potential between the inner and outer surfaces of a cylinder rotating in an external axially directed magnetic field: an analysis with many different approaches [6,23,24,25] (the Trocheris-Takeno des-cription of rotations is used in [25]). Finally, a third response is provided by the increasing at-tention paid to paraxial optical beams with an helicoidal geometrical structure [26], [27] lea-ding to a discussion of light propagation in rotating media: a problem object of some dispu-tes [28-32].The relativistic theory of geometrical optics [15] is still a challenge to which it would be interesting to see what could be the differential form contribution.

Appendix A: Minkowski space-time in vacuum

The four dimensional Hodge operator for Minkowski space-time is defined as follows [8]:

zero-forms and four-forms

$$* (dx \wedge dy \wedge dz \wedge cdt) = -1, \qquad *1 = (dx \wedge dy \wedge dz \wedge cdt) \qquad (A.1)$$

one-forms and three-forms

$$* (dx \wedge dy \wedge dz) = -cdt, \qquad * c\, dt = -(dx \wedge dy \wedge dz)$$

$$*(dy \wedge dz \wedge cdt) = -dx, \qquad *dx = - (cdt \wedge dy \wedge dz)$$

$$*(dz \wedge dx \wedge cdt) = -dy, \qquad *dy = - (cdt \wedge dz \wedge dx)$$

$$*(dx \wedge dy \wedge cdt) = -dz, \qquad *dz = - (cdt \wedge dx \wedge dy) \qquad (A.2)$$

two forms

$$*(dy \wedge dz) = - (dx \wedge cdt), \qquad *(dx \wedge cdt) = (dy \wedge dz)$$

$$*(dz \wedge dx) = - (dy \wedge cdt), \qquad *(dy \wedge cdt) = (dz \wedge dx)$$

$$*(dx \wedge dy) = - (dz \wedge cdt) , \qquad *(dz \wedge cdt) = (dx \wedge dy) \qquad (A.3)$$

Let us assume $E_x = E_y = 0$, then the electric two-form (7) becomes

$$\mathrm{E} = E_z(dz \wedge cdt) \qquad (A.4)$$

Applying the exterior derivative operator (6) to (B.4) and using (B.2) give

$$*d\, \mathrm{E} = \partial_x E_z \wedge dy - \partial_y E_z \wedge dx \qquad (A.5)$$

and

$$d*d\, \mathrm{E} = \partial_x{}^2 E_z\, (dx \wedge dy) + \partial_z \partial_x E_z\, (dz \wedge dy) + \partial_t \partial_x E_z\, (dt \wedge dy) +$$

$$-[\partial_y{}^2 E_z\, (dy \wedge dx) + \partial_z \partial_y E_z\, (dz \wedge dx) + \partial_t \partial_y E_z\, (dt \wedge dx)] \qquad (A.6)$$

so that according to (A.3)

$$*d*d\, \mathrm{E} = - \partial_z{}^2 E_z\, (dz \wedge cdt) + \partial_z \partial_x E_z\, (dx \wedge cdt) - 1/c \partial_t \partial_x E_z\, (dz \wedge dx) +$$

$$-[\partial_y{}^2 E_z\, (dz \wedge cdt) - \partial_z \partial_y E_z\, (dy \wedge cdt) - 1/ c \partial_t \partial_y E_z\, (dy \wedge dz)] \qquad (A.7)$$

1ow, using (A.3) and (A.2), we also have

$$*d\, *\mathrm{E} = -\partial_z E_z \wedge cdt - \partial_t E_z \wedge dz \qquad (A.8)$$

and

$$d*d\, *\mathrm{E} = -\partial_z \partial_x E_z\, (dx \wedge cdt) - \partial_z \partial_y E_z\, (dy \wedge cdt) - \partial_z{}^2 E_z\, (dz \wedge cdt) +$$

$$- 1/c\, [\partial_t{}^2 E_z (dt \wedge dz) + \partial_t \partial_x E_z\, (dx \wedge dz) + \partial_t \partial_y E_z\, (dy \wedge dz)] \qquad (A.9)$$

Summing (A.7) and (A.9) gives

$$(*d*d + d*d\, *)\mathrm{E} = - \; (\partial_x{}^2 + \partial_y{}^2 + \partial_z{}^2 - c^{-2}\partial_t{}^2)\, E_z(dz \wedge cdt) \qquad (A.10)$$

Appendix B: Christoffel symbols

The Christoffel symbols are defined in terms of the $g_{\mu\nu}$'s by the well known relations [14-16]

$$\Gamma_{\beta,\mu\nu} = \tfrac{1}{2} (\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu}), \qquad \Gamma^\alpha{}_{\mu\nu} = g^{\alpha\beta}\Gamma_{\beta,\mu\nu} \qquad (B.1)$$

In these expressions, the greek indices take the values 1,2,3,4 corresponding in a cylindrical frame to the coordinates x^1 = r, x^2 = ϕ, x^3 = z, x^4 = ct, while $\partial_1 = \partial_r$, $\partial_2 = 1/r\, \partial_\phi$, $\partial_3 = \partial_z$, $\partial_4 = 1/c\, \partial_t$.

The relations (13) give the components $g_{\mu\nu}$ of the metric tensor for a Frenet-Serret rotating frame and :

$$g_{44} = 1,\ g_{33} = -1,\ g_{22} = -r^2,\ g_{11} = u(r,\phi,t),\ g_{12} = g_{21} = v(r,\phi,t)\ ,\ g_{14} = g_{41} = w(r,\phi,t) \quad (B.2)$$

the explicit expressions of the functions u,v,w are to be found in (13), no $g_{\mu\nu}$ depends on z.
Then, the non-null components of the Christoffel symbols are given for $\mu \leq \nu$ (because of the $\mu\nu$-symmetry)

$$\underline{\Gamma}_{1,11} = \tfrac{1}{2}\, \partial_r u \quad , \quad \underline{\Gamma}_{1,12} = \tfrac{1}{2}\, \partial_\phi u \quad , \quad \underline{\Gamma}_{1,14} = \tfrac{1}{2}c\, \partial_t u \quad , \quad \underline{\Gamma}_{1,22} = 1/r\, \partial_\phi v - r,$$

$$\underline{\Gamma}_{1,24} = \tfrac{1}{2}c\, \partial_t v + 1/2r\partial_\phi w \quad , \quad \underline{\Gamma}_{1,44} = 1/c\partial_t w \quad , \quad \underline{\Gamma}_{2,11} = \partial_r v - 1/2r\partial_\phi u$$

$$\underline{\Gamma}_{2,12} = -r \quad , \quad \underline{\Gamma}_{2,14'} = 1/2c\, \partial_t v - 1/2\partial_r w \quad ,$$

$$\underline{\Gamma}_{4,11} = \partial_r w - 1/2c\, \partial_t u, \quad \underline{\Gamma}_{4,12} = \tfrac{1}{2}\, \partial_r w - 1/2c\, \partial_t v \quad (B.3)$$

The latin indices taking the values 1,2,3, the covariant derivatives of the components $E_1 = E_r$, $E_2 = E_\phi$, $E_3 = E_z$ of the electric field are

$$\underline{\nabla}_1 E_i = \partial_r E_i - \underline{\Gamma}_{1i}{}^k E_k$$

$$\underline{\nabla}_2 E_i = 1/r\partial_\phi E_i - \underline{\Gamma}_{2i}{}^k E_k$$

$$\underline{\nabla}_3 E_i = \partial_z E_i - \underline{\Gamma}_{3i}{}^k E_k \quad (B.4)$$

Underlined expressions mean they are defined with the cylindrical ccordinates r,ϕ, z,t.
Making in (B.2), u = v = w = 0, gives the metric of the Minkowski frame with polar coordinates and according to (B.3) the only nonnull Christoffeel symbols are

$$\Gamma_{1,22} = -r \quad , \quad \Gamma_{2,12} = \Gamma_{2,21} = -r \quad (B5)$$

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Wave Propagation in Materials for Modern Applications

Edited by Andrey Petrin

In the recent decades, there has been a growing interest in micro- and nanotechnology. The advances in nanotechnology give rise to new applications and new types of materials with unique electromagnetic and mechanical properties. This book is devoted to the modern methods in electrodynamics and acoustics, which have been developed to describe wave propagation in these modern materials and nanodevices. The book consists of original works of leading scientists in the field of wave propagation who produced new theoretical and experimental methods in the research field and obtained new and important results. The first part of the book consists of chapters with general mathematical methods and approaches to the problem of wave propagation. A special attention is attracted to the advanced numerical methods fruitfully applied in the field of wave propagation. The second part of the book is devoted to the problems of wave propagation in newly developed metamaterials, micro- and nanostructures and porous media. In this part the interested reader will find important and fundamental results on electromagnetic wave propagation in media with negative refraction index and electromagnetic imaging in devices based on the materials. The third part of the book is devoted to the problems of wave propagation in elastic and piezoelectric media. In the fourth part, the works on the problems of wave propagation in plasma are collected. The fifth, sixth and seventh parts are devoted to the problems of wave propagation in media with chemical reactions, in nonlinear and disperse media, respectively. And finally, in the eighth part of the book some experimental methods in wave propagations are considered. It is necessary to emphasize that this book is not a textbook. It is important that the results combined in it are taken "from the desks of researchers". Therefore, I am sure that in this book the interested and actively working readers (scientists, engineers and students) will find many interesting results and new ideas.

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