Chapter 2

Recent Advances in Variable Digital Filters

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Abstract

Variable digital filters are widely used in a number of applications of signal processing because of their capability of self-tuning frequency characteristics such as the cutoff frequency and the bandwidth. This chapter introduces recent advances on variable digital filters, focusing on the problems of design and realization, and application to adaptive filtering. In the topic on design and realization, we address two major approaches: one is the frequency transformation and the other is the multi-dimensional polynomial approximation of filter coefficients. In the topic on adaptive filtering, we introduce the details of adaptive band-pass/band-stop filtering that include the well-known adaptive notch filtering.

Keywords: variable digital filter, frequency transformation, polynomial approximation, adaptive notch filtering, adaptive band-pass/band-stop filtering

1. Introduction

Digital filter is well known as one of the essential and fundamental components in signal processing devices. In addition, many signal processing applications such as digital audio equipment and telecommunication systems sometimes require simultaneous realization of digital filtering and real-time control of filter characteristics. Such requirements can be fulfilled by means of variable digital filters (VDFs). Research on VDFs emerged in the 1970s and since then, many results have been reported. Among them, details of the results until the 1990s are widely reviewed in [1].

The problems that should be solved in development of VDFs are essentially the same as those in digital filters of fixed characteristics. Hence, research topics on VDFs as well as fixed characteristic filters are broadly classified into three categories [2]: the approximation problem, the realization problem, and the implementation problem. Moreover, in the field of VDFs, application-oriented results have also been actively reported. One of the famous applications is adaptive notch filters, which have been studied since the 1980s and the details will be reviewed in this chapter.

In the sequel, fundamentals of VDFs are first reviewed. Then, recent results on VDFs are introduced and discussed with focus on the approximation problem, the realization problem, and the applications. Such topics include some results proposed by the authors of this chapter.

2. Fundamentals of VDFs

2.1. Definition

VDFs are defined as the frequency selective digital filters (e.g., low-pass filters and band-pass filters) of which frequency characteristics can be changed in real time by means of controlling some parameters. A popular example of such VDFs is shown in **Figure 1**, which is the variable low-pass filter (VLPF) of which cutoff frequency can be changed by controlling the single parameter η. Another example shown in **Figure 2** is the variable band-pass filter (VBPF), where the bandwidth is fixed and the pass-band center frequency can be changed by the single parameter ξ.

It should be noted that VDFs are different from "filters with variable (adjustable) coefficients" which are used in adaptive filtering. Details of the differences are as follows:

- In the case of general adaptive filtering, all filter coefficients are changed by an adaptive algorithm. On the other hand, most of the coefficients of a VDF are fixed or given as some functions of a few variable parameters. For example, in the VLPF of **Figure 1**, only the single parameter η can be changed, and the other coefficients are fixed or given as functions of η.
- VDFs are different from general adaptive filters with respect to the mechanism of changing the frequency characteristics. In VDFs, the characteristics are changed but the frequency selectivity such as the low-pass and the band-pass shape is preserved. In other words, VDFs control the frequency characteristics under the constraint of preservation of frequency selectivity. On the other hand, general adaptive filters do not require this constraint. This means that such adaptive filters converge to optimal ones of which characteristics do not necessarily possess frequency selectivity.

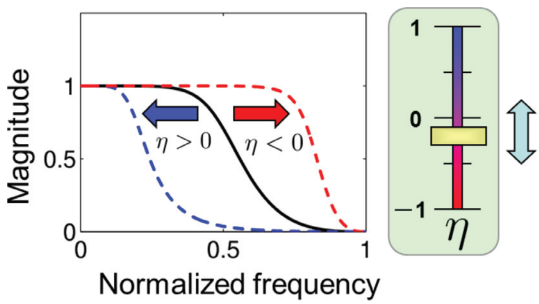

Figure 1. Example of VLPF.

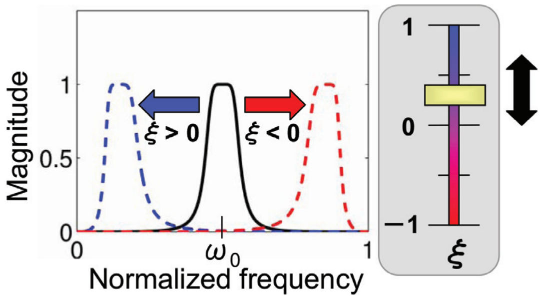

Figure 2. Example of VBPF.

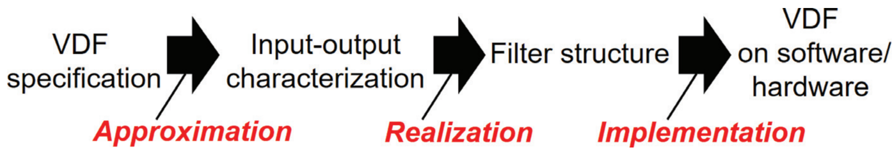

Figure 3. Procedure to obtain VDF.

2.2. How to obtain VDFs

This subsection reviews the procedure to obtain VDFs. The required procedure is basically the same as that in the case of fixed characteristic filters, where three important problems must be considered as shown in **Figure 3**: *approximation*, *realization*, and *implementation* [2]. In this chapter, we pay special attention to the approximation problem and the realization problem. The approximation problem is to obtain an input-output characterization such as transfer function from a prescribed specification of a VDF. The realization problem is to determine a structure (i.e., an appropriate set of adders and multipliers or an appropriate list of primitive operations for filtering) corresponding to the input-output characterization.

In the approximation problem for VDFs, the required task is to describe an input-output relationship (e.g., transfer function) of the VDF in such a manner that the description includes variable parameters. For example, consider the approximation problem for the VLPF shown in **Figure 1**. If one wishes to obtain this VLPF as an FIR filter, the approximation problem is to describe the transfer function in the form of

$$H(z,\eta) = \sum_{k=0}^{N} h_k(\eta) z^{-k} \tag{1}$$

and it is also necessary to describe each coefficient $h_k(\eta)$ as a function of η. Therefore, the approximation problem for this VLPF is to determine a set of functions $\{h_k(\eta)\}$ $(0 \le k \le N)$. Similarly, if one wishes to obtain IIR-type VLPF, it is necessary to describe the transfer function in the form of

$$H(z,\eta) = \frac{\sum_{k=0}^{M} b_k(\eta) z^{-k}}{1 + \sum_{m=1}^{N} a_m(\eta) z^{-m}} \tag{2}$$

and to determine the filter coefficients as the functions $\{a_m(\eta)\}$ $(1 \le m \le N)$ and $\{b_k(\eta)\}$ $(1 \le k \le M)$.

3. Research topics on VDFs

This section introduces research topics on VDFs from the viewpoints of the approximation problem and the realization problem. Two methods have been widely used for approximation and realization of VDFs: one is based on the variable transformation of transfer functions and the other is based on the multi-dimensional (M-D) polynomial approximation of filter coefficients. In the sequel details of these two methods are reviewed and some recent results on these two methods are introduced.

3.1. VDFs based on variable transformation of transfer functions

In this method, we first need to design the transfer function of "prototype filter," which is usually low pass, and its coefficients are fixed (i.e., variable parameters are not included in this transfer function). Next, we apply a variable transformation to this prototype transfer function and obtain a desired VDF, where the variable transformation makes use of a function which includes variable parameters that are associated with the components to be changed in frequency characteristics. Many approaches exist for variable transformations, and the most famous approach is the frequency transformation [3]. The frequency transformation makes use of all-pass functions for the variable transformation. Although details of the frequency transformation are well reviewed in [1], this chapter will also review this topic with some additional discussions. This is because many results using the frequency transformation have been still reported in recent years and some of such results include the authors' works.

Now, consider again the VLPF shown in **Figure 1**. If frequency transformation is used to obtain this VLPF, the first step is to prepare the transfer function of a prototype low-pass filter. Such a transfer function is denoted by $H_{\mathrm{p}}(z)$. Then, applying the following frequency transformation to $H_{\mathrm{p}}(z)$, we can obtain the desired VLPF with the transfer function $H(z,\eta)$:

$$\begin{aligned} H(z,\eta) &= H_{\mathrm{p}}(z)\big|_{z^{-1} \leftarrow T(z,\eta)} \\ T(z,\eta) &= \frac{z^{-1} - \eta}{1 - \eta z^{-1}} \end{aligned} \tag{3}$$

where $T(z,\eta)$ is the first-order all-pass function. By changing the value of η in $H(z,\eta)$, we can control the cutoff frequency of the VLPF. If $\eta > 0$, the cutoff frequency becomes lower than that of the prototype filter. The converse holds if $\eta < 0$. Stability of this VLPF is guaranteed if the prototype filter is stable and $|\eta| < 1$ is satisfied. Also, note that $|T(e^{j\omega}, \eta)| = 1$ holds for any η and ω because $T(z, \eta)$ is all-pass.

We next discuss the realization problem for this VLPF. From the realization point of view, Eq. (3) means that a block diagram of this VLPF can be obtained by replacing each delay element z^{-1} in the prototype filter with the all-pass filter $T(z, \eta)$. However, in most cases, such replacement causes delay-free loops and results in $H(z, \eta)$ with unrealizable block diagram. To explain this problem, consider a second-order IIR prototype filter with the transfer function given by

$$H_{\mathrm{p}}(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{4}$$

and the block diagram given by the direct form as in **Figure 4(a)**. Applying the aforementioned replacement of delay elements with $T(z, \eta)$ yields the VLPF of which the block diagram corresponds to **Figure 4(b)**. It is now clear that **Figure 4(b)** includes delay-free loops, and hence it is impossible to implement this block diagram. It is well known that delay-free loops can be avoided by means of mathematical manipulations of transfer function or difference equation. However, such manipulations are not good solutions in the case of VDF realization. For example, applying $z^{-1} \leftarrow T(z, \eta)$ to $H_{\mathrm{p}}(z)$ given by Eq. (4) and then performing mathematical manipulations, we obtain the transfer function of the second-order VLPF as follows:

$$\begin{aligned}
H(z,\eta) &= \frac{b_0'(\eta) + b_1'(\eta) + b_2'(\eta)}{1 + a_1'(\eta) + a_2'(\eta)} \\
a_1'(\eta) &= \frac{-2\eta + a_1\left(1+\eta^2\right) - 2a_2\eta}{1 - a_1\eta + a_2\eta^2} \\
a_2'(\eta) &= \frac{\eta^2 - a_1\eta + a_2}{1 - a_1\eta + a_2\eta^2} \\
b_0'(\eta) &= \frac{b_0 - b_1\eta + b_2\eta^2}{1 - a_1\eta + a_2\eta^2} \\
b_1'(\eta) &= \frac{-2b_0\eta + b_1\left(1+\eta^2\right) - 2b_2\eta}{1 - a_1\eta + a_2\eta^2} \\
b_2'(\eta) &= \frac{b_0\eta^2 - b_1\eta + b_2}{1 - a_1\eta + a_2\eta^2}.
\end{aligned} \tag{5}$$

If we implement the VLPF using this description, the computational cost significantly increases because the filter coefficients $a_1'(\eta), a_2'(\eta), b_0'(\eta), b_1'(\eta)$ and $b_2'(\eta)$ must be recalculated according to the change of η. In particular, the filter coefficients in Eq. (5) are rational polynomials that require divisions for recalculation of filter coefficients, causing very high implementation cost.

One of the popular methods to overcome this problem is the Taylor approximation-based description [4]. This method applies the first-order Taylor series approximation to all of the rational polynomials of filter coefficients in VDFs, under the assumption that the absolute

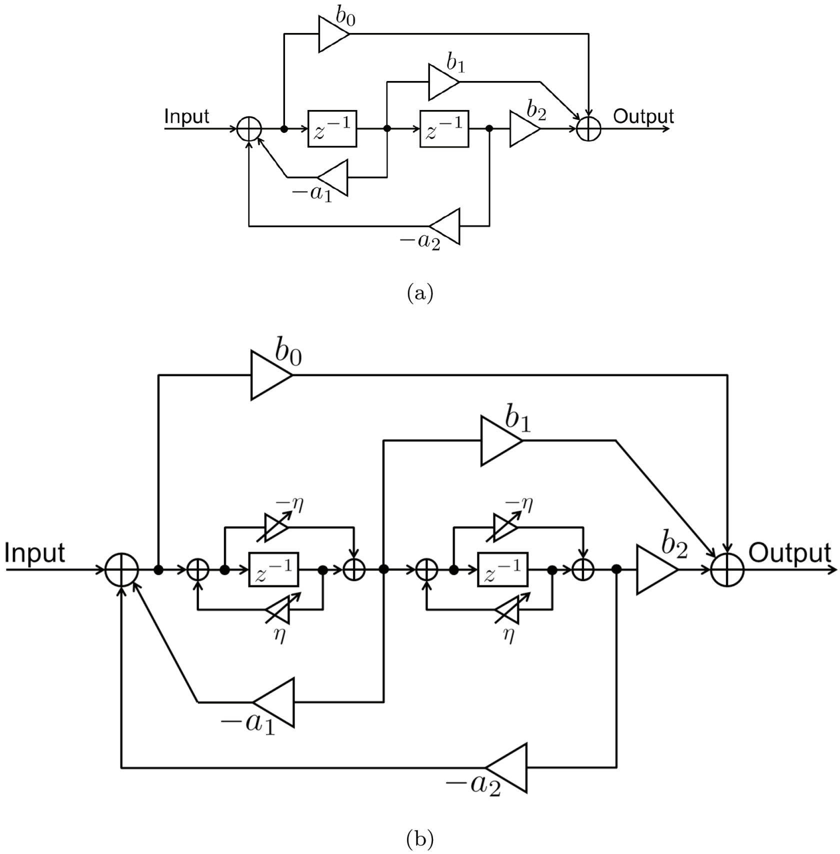

Figure 4. Problem in realization of VLPF based on the frequency transformation: (a) second-order prototype filter, and (b) VLPF given by applying $z^{-1} \leftarrow T(z, \eta)$ to the prototype filter.

values of all variable parameters are small. For example, in the case of Eq. (5), it is assumed that $|\eta| \ll 1$ and the filter coefficients are approximated to

$$\begin{aligned}
a'_1(\eta) &\approx a_1 + \left(a_1^2 - 2 - 2a_2\right)\eta \\
a'_2(\eta) &\approx a_2 + (a_1 a_2 - a_1)\eta \\
b'_0(\eta) &\approx b_0 + (a_1 b_0 - b_1)\eta \\
b'_1(\eta) &\approx b_1 + (a_1 b_1 - 2b_0 - 2b_2)\eta \\
b'_2(\eta) &\approx b_2 + (a_1 b_2 - b_1)\eta.
\end{aligned} \tag{6}$$

These new coefficients do not require divisions, and hence the VLPF can be realized in terms of additions and multiplications, as shown in **Figure 5**. In addition, this realization does not

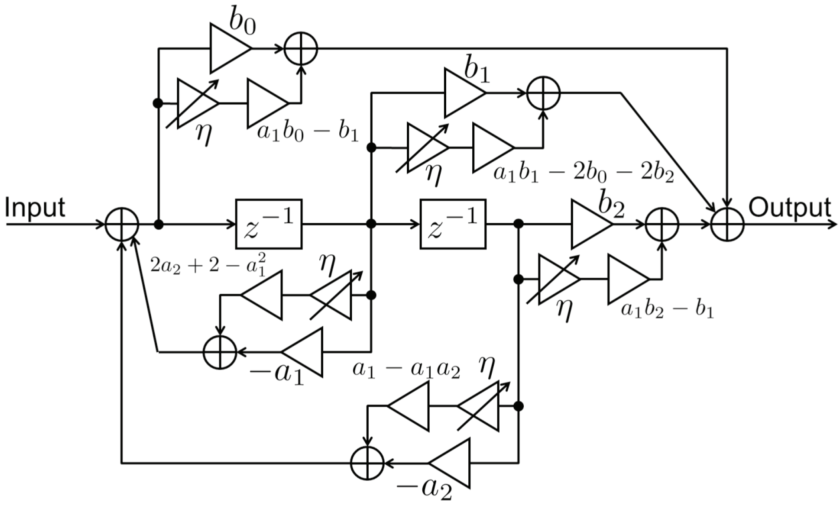

Figure 5. Second-order VLPF based on the frequency transformation and first-order Taylor series approximation.

require recalculation of filter coefficients even if the value of η is changed. This is because all of the multipliers except for η in this block diagram are realized as fixed coefficients.

Although the VLPFs based on the Taylor approximation provide an effective realization method, they have a serious drawback that the range of variable cutoff frequency is quite limited. This limitation is due to the assumption of $|\eta| \ll 1$, which means that the approximation error becomes larger as the cutoff frequency of the VLPFs goes far from that of the prototype filter. In addition, the VLPFs may become unstable if the value of $|\eta|$ is inappropriately large. In order to overcome these problems, some alternative methods are proposed [4–6]. All of these methods make use of low sensitivity structures for realization of block diagrams for the prototype filter. Then the replacement $z^{-1} \leftarrow T(z, \eta)$ and the Taylor approximation are applied to such block diagrams, leading to the desired VDFs. Although the methods given by [4–6] can be applied to the limited classes of transfer functions, the Taylor approximation error becomes smaller than the standard VDFs based on the direct form. This approach is also extended to the 2-D VDFs [7].

There are some other approaches for the reduction of the Taylor approximation error. In [8], the approach based on wave digital filters is presented. Although this approach requires the knowledge of analog filter theory, very high precision is attained in the resultant VDFs, and hence the variable cutoff frequency can be controlled in relatively wide range. In [9], state-space representation is used for construction of the block diagram of the prototype filter, and series approximations are applied to avoid the significant increase of the implementation cost of frequency transformation-based VDFs. This approach does not need any restriction that appeared in the conventional methods, and hence the method of [9] can be applied to arbitrary transfer functions and arbitrary state-space structures. Furthermore, in [10], the VDFs based on the combination of frequency transformation and coefficient decimation are proposed, and it is shown through FPGA implementation and performance evaluation that the proposed method attains very low cost for hardware implementation.

As discussed above, the problem of delay-free loops is an important issue in the approximation/realization of frequency transformation-based VDFs. It should be noted that, however, this problem does not always happen. In general, this problem happens if the all-pass function in the frequency transformation has a nonzero constant term in the numerator. This case corresponds to the VDFs with variable bandwidth. In other words, the problem of delay-free loops does not happen when the VDFs have fixed bandwidth, as shown in **Figure 2**.

We conclude this subsection with a summary of the merits and the drawbacks of the frequency transformation-based VDFs. The merits are as follows:

- Variable characteristics can be easily obtained because the theory of controlling cutoff frequency is based on the simple variable transformations.
- If Taylor approximation is not carried out, the frequency transformation preserves many useful properties on the shape of magnitude responses. For example, when a prototype low-pass filter is the Butterworth filter that possesses the monotonic and maximally flat magnitude response, the VDFs given by applying frequency transformations to this prototype filter also possess the monotonic and maximally flat magnitude responses.
- The aforementioned merit facilitates the design of adaptive band-pass or band-stop filters because the cost function for adaptive filtering becomes unimodal, leading to an adaptive algorithm that converges to the globally optimal solution. Details will be discussed in the next section.
- Compared with the VDFs based on the M-D polynomial approximation, the frequency transformation-based VDFs require much less computational cost in the filtering.

Next, the drawbacks are summarized as follows:

- As stated earlier, if the bandwidth needs to be variable in VDFs, the frequency transformation causes delay-free loops and this problem must be appropriately solved.
- If one wishes to obtain VDFs with multiple passbands or stopbands such as VBPFs, VBSFs, and variable multi-band filters, it is necessary to use high-order all-pass functions for the frequency transformation. As a result, the order of VDFs becomes higher than that of the prototype filter. For example, the order of the frequency transformation-based VBPFs and VBSFs becomes doubled as compared with the order of the prototype filter.
- Linear-phase VDFs cannot be obtained because the all-pass functions to be used in the frequency transformation are IIR filters. Even if a prototype filter is FIR, applying the frequency transformations simply results in IIR-type VDFs.
- Realization of variable characteristics is quite limited. To be specific, the frequency transformation can provide only the VDFs with variable cutoff frequencies. In other words, other components such as the transition bandwidth and the stopband attenuation cannot be controlled.

3.2. VDFs based on M-D polynomial approximation of filter coefficients

To the authors' best knowledge, the VDFs based on the M-D polynomial approximation of filter coefficients have been most actively studied [11–23] in the field of VDFs. One of the significant benefits of this approach over the frequency transformation-based VDFs is that many kinds of variable characteristics as well as variable cutoff frequencies can be attained. For example, this approach can provide VLPFs with variable transition bandwidth and variable stopband attenuation, as shown in **Figure 6**. In addition, since this approach is applicable to FIR filters as well as IIR filters, linear-phase characteristics and variable group delay can be attained in VDFs.

The first step to obtain this type of VDFs is to determine a set of K variable parameters $(\psi_1, \psi_2, \cdots, \psi_K)$ which correspond to the desired variable components of frequency characteristics such as cutoff frequency, transition bandwidth, and stopband attenuation. Such variable parameters are referred to as spectral parameters. After this step, filter coefficients of the desired VDFs are described as M-D polynomials with respect to these variable parameters. For example, the transfer function of an N-th order VDF with K variable parameters is described by

$$H(z, \psi_1, \psi_2, \cdots, \psi_K) = \sum_{n=0}^{N} h_n(\psi_1, \psi_2, \cdots, \psi_K) z^{-n} \tag{7}$$

and each filter coefficient $h_n(\psi_1, \psi_2, \cdots, \psi_K)$ is described in terms of the following M-D polynomial:

$$h_n(\psi_1, \psi_2, \cdots, \psi_K) = \sum_{m_{\psi_1}=0}^{M_{\psi_1}} \sum_{m_{\psi_2}=0}^{M_{\psi_2}} \cdots \sum_{m_{\psi_K}=0}^{M_{\psi_K}} c_n(m_{\psi_1}, m_{\psi_2}, \cdots m_{\psi_K}) \psi_1^{m_{\psi_1}} \psi_2^{m_{\psi_2}} \cdots \psi_K^{m_{\psi_K}}. \tag{8}$$

The approximation problem for this kind of VDFs is to determine the set of coefficients $\{c_n(m_{\psi_1}, m_{\psi_2}, \cdots m_{\psi_K})\}$ for $0 \le n \le N$. Here, it should be noted that $M_{\psi_1}, M_{\psi_2}, \cdots, M_{\psi_K}$ denote the orders of the M-D polynomials that, respectively, correspond to the variables $\psi_1, \psi_2, \cdots, \psi_K$. In order to obtain the set $\{c_n(m_{\psi_1}, m_{\psi_2}, \cdots m_{\psi_K})\}$, the standard approach is based on the

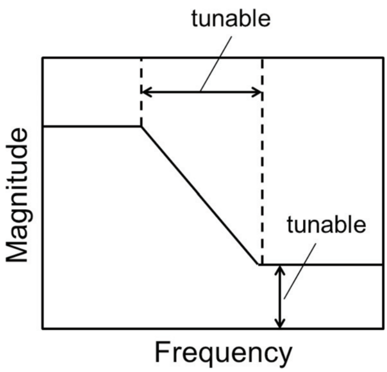

Figure 6. Example of VLPF based on the M-D polynomial approximation of filter coefficients.

minimization of an error function with respect to approximation of a prescribed ideal characteristic of the desired VDF and a curve fitting method to describe the desired M-D polynomials.

In realization of the VDFs given as above, Farrow structure [24] is widely used. To explain this, consider a simple VDF with a single variable parameter ψ_1. The transfer function of this VDF is given by

$$\begin{aligned} H(z,\psi_1) &= \sum_{n=0}^{N} h_n(\psi_1) z^{-n} \\ &= \sum_{n=0}^{N} \sum_{m_{\psi_1}=0}^{M_{\psi_1}} c_n(m_{\psi_1}) \psi_1^{m_{\psi_1}} z^{-n} \end{aligned} \tag{9}$$

which can be rewritten as

$$H(z,\psi_1) = \sum_{m_{\psi_1}=0}^{M_{\psi_1}} \left(\sum_{n=0}^{N} c_n(m_{\psi_1}) z^{-n} \right) \psi_1^{m_{\psi_1}}. \tag{10}$$

Now, by using the following definition

$$H_{m_{\psi_1}}(z) = \sum_{n=0}^{N} c_n(m_{\psi_1}) z^{-n}, \quad 0 \le m_{\psi_1} \le M_{\psi_1}, \tag{11}$$

the description of the VDF $H(z,\psi_1)$ becomes

$$H(z,\psi_1) = \sum_{m_{\psi_1}=0}^{M_{\psi_1}} H_{m_{\psi_1}}(z) \psi_1^{m_{\psi_1}}. \tag{12}$$

Using this description, we can realize $H(z,\psi_1)$ by means of the Farrow structure as shown in **Figure 7**. The block diagram of **Figure 7** is interpreted as the parallel combination of the set of N-th order FIR filters with fixed coefficients and the weights ψ_1. Since these N-th order FIR

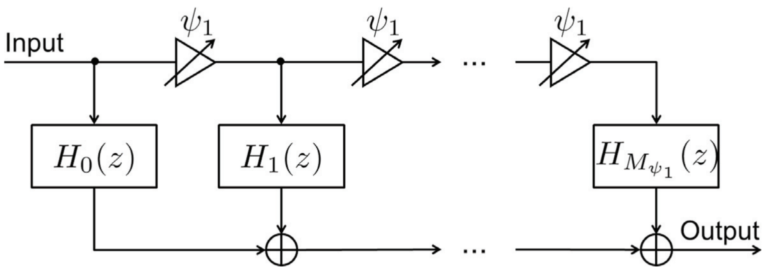

Figure 7. Realization of M-D polynomial approximation-based VDF based on the Farrow structure.

filters do not include ψ_1, recalculation of their coefficients according to the change of ψ_1 is not required. In this sense, the Farrow structure is suitable for the implementation of M-D polynomial approximation-based VDFs.

A drawback of the M-D polynomial approximation-based VDFs is the high computational cost in the filtering because the filter coefficients are described by M-D polynomials. In addition, this approach limits the range of variable characteristics. As in the case of frequency transformation with Taylor approximation, this limitation comes from the M-D polynomial approximation. Furthermore, since this approach requires a number of filters with fixed coefficients, their hardware implementation may cause an increase of characteristic degradations that comes from finite wordlength effects such as coefficient sensitivity and roundoff noise. However, such degradations can be suppressed by using high accuracy filter structures, and this approach has been recently proposed by the authors [23].

3.3. VDFs based on other approaches

In addition to the aforementioned two approaches, many other methods have also been presented in the literature. In [25], VDFs with variable bandwidth without delay-free loops can be achieved at low cost by means of cascade connection of a single subfilter. In [26–28], by applying the frequency response masking and the fast filterbank to design of VDFs, significant reduction of implementation cost over the VDFs with the Farrow structure is attained.

Also, VDFs for adaptive filtering have been widely studied. One of the famous methods in such VDFs is the variable notch filters with second-order IIR transfer functions. All of these variable notch filters successfully provide the variable characteristics by simple mechanism without delay-free loops or increase of computational cost. Other adaptive-filter-oriented VDFs include notch filters with variable attenuation at the notch frequency, comb filters with variable bandwidth, and variable attenuation. Details of these topics will be addressed in the next section.

4. Research topics on VDFs for adaptive filtering

In this section, we first pay attention to adaptive notch filters (ANFs) that are the special case of adaptive band-stop or band-pass filters. The ANFs are the most famous application of VDFs to adaptive signal processing, and many results on the ANFs have been reported since the 1980s. In addition to the ANFs, this section also introduces some other types of VDFs that are applied to adaptive filtering.

4.1. ANF based on all-pass filter

As shown in **Figure 8**, an ANF plays a central role in automatic detection and suppression of an unknown sinusoid immersed in a wide-band signal such as white noise. In order to detect and suppress the sinusoid, the ANF is controlled by an adaptive algorithm in such a manner that the notch frequency ω_0 of the ANF converges to the unknown frequency ω_s of the sinusoid. Hence, the ANF can be considered as the VDF with variable notch frequency, and the value of ω_0 at the steady state becomes the estimate of the frequency ω_s of the sinusoid. Therefore, ANFs are used not only for the detection/suppression of a sinusoid, but also for the frequency estimation.

Although the ANF shown in **Figure 8** is intended to suppress a sinusoid, the ANF is also capable of enhancement of the sinusoid and suppression of the white noise. This can be achieved by using a peaking filter, which is also called a resonator or an inverse notch filter, as an adaptive filter instead of using a notch filter. Alternatively, the notch filter can also be used: in this case, the sinusoid can be enhanced by subtracting the output of the notch filter from the input signal.[1] Such systems together with the ones shown in **Figure 8** are widely used in many practical applications such as radar, sonar, telecommunication system with the suppression of narrowband interference and howling suppressor in speech processing.

In the sequel, we explain the fundamentals of ANFs, that is, their problem statement and the mechanism of control of the notch frequency. As shown in **Figure 8**, the problem statement of ANFs usually describes the input signal as the sum of a sinusoid and a white noise. Hence, the input signal, denoted by $u(n)$, is given by

$$u(n) = A \sin\left(\omega_s n + \phi\right) + w(n) \tag{13}$$

where A and ω_s are, respectively, the amplitude and frequency of the unknown sinusoid, and ϕ is the random initial phase uniformly distributed in $[0, 2\pi)$. The signal $w(n)$ is a zero-mean white noise, and it is uncorrerated to ϕ. Based on this setup, let $y(n)$ be the output signal of the ANF.

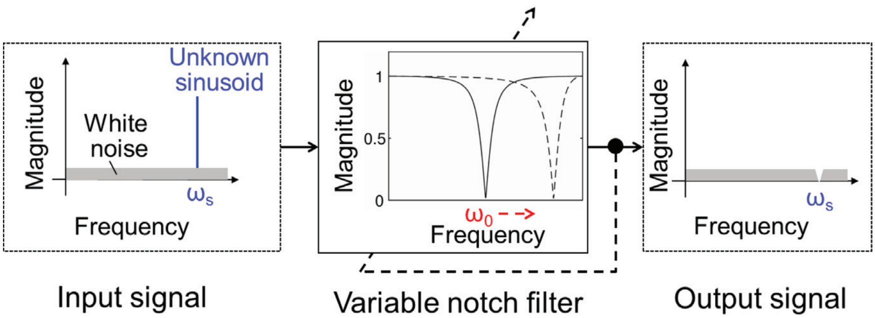

Figure 8. Detection and suppression of sinusoid using ANF.

[1]Note that this approach depends on the characteristic of a notch filter, and hence the use of an inappropriate notch filter may result in failure of enhancement of a sinusoid. The reason of this lies in the fact that the signal which is obtained by subtracting the output of a band-stop filter from the input is not necessarily equivalent to the output of a band-pass filter. However, in the case of the ANF based on a second-order all-pass filter, the frequency characteristic of the notch filter satisfies complementary properties that allow us to successfully obtain a signal equivalent to the band-pass-filtered signal by subtracting the notch-filtered signal from the input.

There are some methods to describe the transfer function of the notch filter for adaptive filtering. In this chapter we focus on the one based on the second-order all-pass filter [29]. This notch filter is described by the following transfer function

$$H(z,\eta,\xi) = \frac{1}{2}(1 + T(z,\eta,\xi)) \tag{14}$$

where $T(z,\eta,\xi)$ is the second-order all-pass filter of the form

$$T(z,\eta,\xi) = \frac{\eta - (1+\eta)\xi z^{-1} + z^{-2}}{1 - (1+\eta)\xi z^{-1} + \eta z^{-2}}. \tag{15}$$

Hence Eq. (14) is described as

$$H(z,\eta,\xi) = \frac{1+\eta}{2}\frac{1 - 2\xi z^{-1} + z^{-2}}{1 - (1+\eta)\xi z^{-1} + \eta z^{-2}}. \tag{16}$$

In this notch filter, the parameter η determines the 3-dB notch width, and the parameter ξ determines the notch frequency ω_0. This means that the notch filter given in this way can control the notch width and the notch frequency independently. Also, it is interesting to note that this notch filter can be interpreted as a VDF given by the frequency transformation [30]: it is clear that this notch filter is obtained by applying the frequency transformation $z^{-1} \leftarrow T(z,\eta,\xi)$ to the prototype filter of the form

$$H_{\mathrm{p}}(z) = \frac{1}{2}\left(1 + z^{-1}\right). \tag{17}$$

To be more precise, this notch filter has the same transfer function as that of the second-order Butterworth band-stop filter [31]. Therefore this notch filter has unity gain at $\omega = 0$ and $\omega = \pi$, and zero gain at ω_0. In addition, the magnitude response of this notch filter is monotonically decreasing in $0 < \omega < \omega_0$ and monotonically increasing in $\omega_0 < \omega < \pi$.

Figure 9 shows the block diagram of ANF based on this notch filter. As stated earlier, when the ANF attains steady state, the component of the sinusoid in the input $u(n)$ is suppressed at

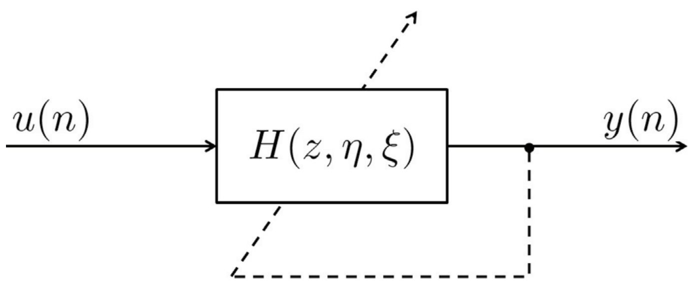

Figure 9. ANF based on the second-order all-pass filter.

the output signal $y(n)$. Here, it should be noted that many adaptive algorithms assume that the notch width is fixed, and that only the notch frequency ω_0 is controlled to estimate the frequency of the sinusoid. For this reason, we focus on how to control ω_0.

The most standard method to control ω_0 is based on the minimization of a cost function by means of the gradient descent method. Although this is similar to general adaptive filters, ANFs differ from the general adaptive filters in that the cost function to be used in ANFs is the mean square output, that is, $E[y^2(n)]$. In other words, ANFs do not usually deal with the error signal between a reference signal and the filter output.[2] Since ANFs control ω_0, the cost function $E[y^2(n)]$ must be formulated as a function of ω_0. This can be successfully achieved and, in addition, $E[y^2(n)]$ becomes unimodal if the input signal is given as in Eq. (13) and the ANF has monotonic magnitude response. Therefore, in such a case, the optimal notch frequency that minimizes $E[y^2(n)]$ can be successfully found by the gradient descent method. In fact, the optimal value of ω_0 coincides with ω_s if the all-pass-based ANF is used [32–34]. Hence, using the gradient of $E[y^2(n)]$ with respect to ω_0 in an adaptive algorithm allows ω_0 to converge to ω_s, leading to detection/suppression of the sinusoid.

Remark 1 *If the transfer function of the ANF is not based on the all-pass function, the optimal value of ω_0 may slightly deviate from ω_s. In other words, the frequency estimation is biased. This topic will be addressed in the next subsection.*

However, the gradient descent method has a serious drawback that the convergence speed becomes very slow when the initial value of ω_0 is distant from ω_s. To overcome this problem, many strategies have been proposed. In [32–34], the normalized lattice structure is applied to construct the notch filter, and the adaptive algorithm makes use of the state variable of the normalized lattice structure instead of the information of the gradient. This approach is called the Simplified Lattice Algorithm (SLA) and successfully accelerates the convergence speed at low computational cost. Furthermore, in [35], the authors have extended the SLA and proposed a new algorithm called the Affine Combination Lattice Algorithm (ACLA), and it has been proved that the ACLA achieves faster convergence than the SLA. Other approaches to improve the convergence speed include the methods based on the least square algorithm with forgetting factor [36], parallel combination of multiple notch filters with different notch width [37, 38], and construction of additional monotonically increasing function for the gradient [39, 40].

There are many other important research topics on the ANFs. One of them is the theoretical analysis of the behavior of ANFs at steady state. In [41], a steady-state analysis is presented for ANFs based on the one-multiplier lattice structure. This analysis enables us to evaluate the performance of ANFs such as the accuracy of frequency estimation. Also, in [42], the authors propose a unified method on the steady-state analysis of frequency estimation MSE (mean square error) for the SLA and the ACLA. As another research topic, in [43] fundamental frequency estimation using inverse notch filter is proposed.

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[2] Although some literature refers to $y(n)$ as the error signal, in the authors' opinion this terminology is incorrect. This is because the error in ANFs should be defined as the difference between the frequency of the sinusoid and its estimate, i.e. $\omega_s - \omega_0$. This quantity clearly differs from $y(n)$.

4.2. ANFs based on other approaches

Other types of ANFs have also been well studied. For example, the following second-order notch filter [44] is very well known:

$$H(z, r, a) = \frac{1 + az^{-1} + z^{-2}}{1 + arz^{-1} + r^2 z^{-2}} \tag{18}$$

where a and r correspond to the parameters that, respectively, control the notch frequency and the notch width. Hence, in this case, the parameter a is controlled by an adaptive algorithm to estimate ω_s. This notch filter is designed by the famous method called the constrained poles and zeros (CPZ), and this notch filter has been most widely used for ANFs [44–51].

Since the transfer function of this notch filter is different from that of the all-pass-based notch filter, the properties of these notch filters are also somewhat different. For example, the all-pass-based notch filter has the unity peak gain, whereas the peak gain of the CPZ-based notch filter depends on the notch width. This also makes the difference with respect to the value of $E\left[y^2(n)\right]$, see [52] for the details. Another difference between these two notch filters is that the all-pass-based ANFs provide unbiased frequency estimation, whereas the CPZ-based ANFs do not. Although this fact shows a drawback of the CPZ-based ANFs, many adaptive algorithms to reduce the bias have been proposed for the CPZ-based ANFs.

In addition to the CPZ-based notch filters, there exist many other types of notch filters. In [53, 54], the specific second-order transfer function is constructed in such a manner that it corresponds to a lattice structure. In [55–58], the bilinear transformation to a second-order analog filter is applied to the notch filter design. In [59], the frequency transformation is used to design a notch filter, but the prototype filter used here is different from Eq. (17).

4.3. Adaptive filtering based on high-order VBPFs/VBSFs

All of the adaptive filters that were addressed in previous subsections are based on second-order VDFs. On the other hand, there exist some results on high-order VDFs in adaptive signal processing. Needless to say, second-order ANFs have a drawback that it is difficult to realize sharp cutoff characteristics, causing insufficient frequency selectivity and relatively poor signal-to-noise ratio (SNR) at the output signal. On the other hand, in [31], the authors improve the output SNR by means of higher-order VBPFs or VBSFs instead of using second-order notch filters in the adaptive filtering. As shown in **Figure 10**, high-order filters can realize sharper cutoff characteristics than second-order filters and provide higher output SNR.

Compared with ANFs, little has been studied on the adaptive filtering based on the high-order VDFs. To the authors' best knowledge, the most significant work is found in [60–63], where fourth-order Butterworth VBPF and VBSF are applied to adaptive filtering, and their center frequencies are controlled by adaptive algorithms. Furthermore, the convergence characteristics are also theoretically analyzed. In this work, it is also claimed that the use of much higher-order VDFs for adaptive filtering is almost impossible because higher-order transfer functions involve mathematically more complicated descriptions, and hence it is conjectured that formulations of filter coefficients with variable characteristics and adaptive

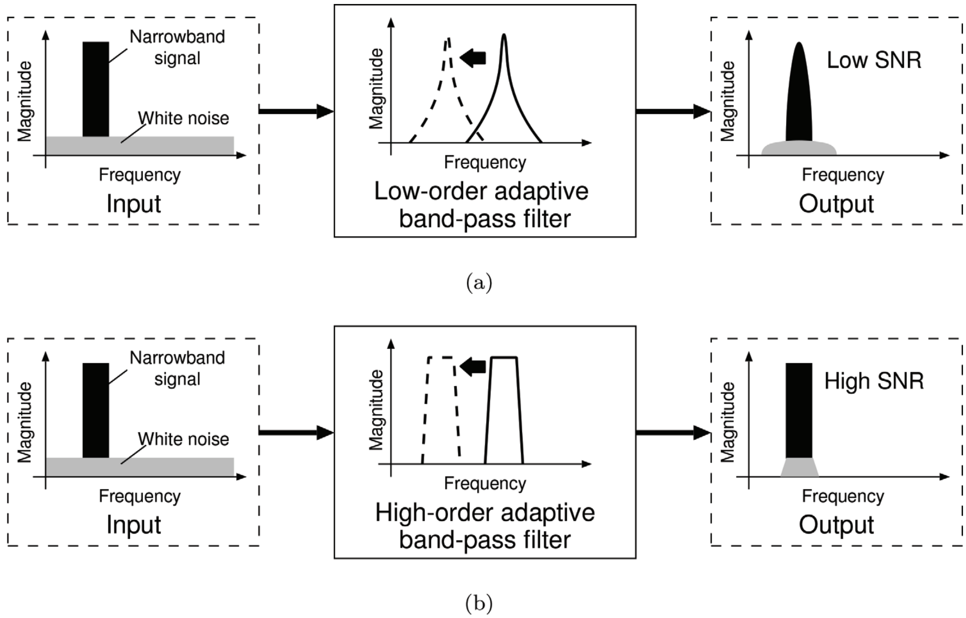

Figure 10. Example of adaptive band-pass filtering: (a) using second-order VBPF, and (b) using high-order VBPF.

control of them become very complicated. However, in the authors' recent work [31], we have successfully realized adaptive filtering based on higher-order VBPFs/VBSFs, where we have derived a gradient descent method-based adaptive algorithm for arbitrary-order VBPFs/VBSFs in a simple form by means of frequency transformation in terms of the block diagram as well as the mathematical description. As a result, it is demonstrated in [31] that the use of higher-order VBPFs/VBSFs for adaptive filtering leads to higher output SNR than the use of ANFs.

As stated above, adaptive band-pass/band-stop filtering based on high-order VBPFs/VSFs can be realized in a simple manner. However, there are still many open problems such as mathematical discussion of convergence of the adaptive algorithm, improvement of convergence speed, and suppression of large quantization errors that are generated due to the nature of high-order narrowband filters. Although the problem of quantization errors can be solved by means of the state-space-based VBPFs/VBSFs [64], further investigations are need to cope with the other problems.

4.4. Other VDFs for adaptive filtering

In addition to the ANFs and higher-order adaptive band-pass/band-stop filtering, many applications of other VDFs to adaptive filtering have been presented. In [65], adaptive filtering based on the cascade connection of second-order all-pass filters is proposed. This method is shown to be superior to the standard ANF-based methods for the detection of multiple sinusoids. Another approach for the detection of multiple sinusoids is also proposed in [66–68], where comb filters with variable bandwidth and variable notch gain are applied to adaptive filtering.

Furthermore, adaptive filtering based on VLPFs can be found in the literature [69]. It should be noted that, in general, realization of adaptive low-pass filtering is much more difficult than adaptive notch filtering or adaptive band-pass/band-stop filtering. The reason of this lies in the difficulty in the problem setup that can describe a unimodal cost function. However in the work of [69], a unimodal cost function is successfully obtained by considering the detection of passband-edge frequency of a low-pass filtered signal and using the approach of weighted cost function.

5. Conclusion

This chapter has reviewed recent research activities on VDFs with focus on the approximation problem, the realization problem, and the applications to adaptive filtering. Since this chapter has paid attention to 1-D VDFs with variable magnitude responses, the introduction of other types of VDFs such as M-D VDFs and variable fractional-delay filters has been omitted. For a similar reason, VDF applications other than adaptive filtering have also been omitted. Although VDFs have been studied for a long time, many elegant results are still being proposed, and hence the research on VDFs will continue to be an active area of investigation.

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