

# Linear Lyapunov Cone-Systems

Przemysław Przyborowski and Tadeusz Kaczorek  
*Warsaw University of Technology – Faculty of Electrical Engineering,  
Institute of Control and Industrial Electronics,  
Poland*

## 1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs (Farina L. & Rinaldi S., 2000; Kaczorek T., 2001). The realization problem for positive linear systems without and with time delays has been considered in (Benvenuti L. & Farina L., 2004; Farina L. & Rinaldi S., 2000; Kaczorek T., 2004a; Kaczorek T., 2006a; Kaczorek T., 2006b; Kaczorek T. & Busłowicz M., 2004a).

The reachability, controllability to zero and observability of dynamical systems have been considered in (Klamka J., 1991). The reachability and minimum energy control of positive linear discrete-time systems have been investigated in (Busłowicz M. & Kaczorek T., 2004). The positive discrete-time systems with delays have been considered in (Kaczorek T., 2004b; Kaczorek T. & Busłowicz M., 2004b; Kaczorek T. & Busłowicz M., 2004c). The controllability and observability of Lyapunov systems have been investigated by Murty Apparao in the paper (Murty M.S.N. & Apparao B.V., 2005). The positive discrete-time and continuous-time Lyapunov systems have been considered in (Kaczorek T., 2007b; Kaczorek T. & Przyborowski P., 2007a; Kaczorek T. & Przyborowski P., 2008; Kaczorek T. & Przyborowski P., 2007e). The positive linear time-varying Lyapunov systems have been investigated in (Kaczorek T. & Przyborowski P., 2007b). The continuous-time Lyapunov cone systems have been considered in (Kaczorek T. & Przyborowski P., 2007c). The positive discrete-time Lyapunov systems with delays have been investigated in (Kaczorek T. & Przyborowski P., 2007d).

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century (Nishimoto K., 1984; Miller K. S. & Ross B., 1993; Podlubny I., 1999). This idea by engineers has been used for modelling different process in the late 1960s (Bologna M. & Grigolini P., 2003; Vinagre B. M. et al., 2002; Vinagre B. M. & Feliu V., 2002; Zaborowsky V. & Meylanov R., 2001). Mathematical fundamentals of fractional calculus are given in the monographs (Miller K. S. & Ross B., 1993; Nishimoto K., 1984; Oldham K. B. &

Spanier J, 1974; Podlubny I, 1999; Oustaloup A., 1995). The fractional order controllers have been developed in (Oldham K. B. & Spanier J., 1974; Oustaloup A., 1993; Podlubny I., 2002). A generalization of the Kalman filter for fractional order systems has been proposed in (Sierociuk D. & Dzieliński D., 2006). Some others applications of fractional order systems can be found in (Ostalczyk P., 2000; Ostalczyk P., 2004a; Ostalczyk P., 2004b; Ferreira N.M.F. & Machado I.A.T., 2003; Gałkowski K., 2005; Moshrefi-Torbati M. & Hammond K., 1998; Reyes-Melo M.E. et al., 2004; Riu D. et al., 2001; Samko S. G. et al., 1993; Dzieliński A. & Sierociuk D., 2006). In (Ortigueira M. D., 1997) a method for computation of the impulse responses from the frequency responses for the fractional standard (non-positive) discrete-time linear systems is proposed. The reachability and controllability to zero of positive fractional systems has been considered in (Kaczorek T., 2007c; Kaczorek T., 2007d). The reachability and controllability to zero of fractional cone-systems has been considered in (Kaczorek T., 2007e). The fractional discrete-time Lyapunov systems has been investigated in (Przyborowski P., 2008a) and the fractional discrete-time cone-systems in (Przyborowski P., 2008b).

The chapter is organized as follows, In the Section 2, some basic notations, definitions and lemmas will be recalled. In the Section 3, the continuous-time linear Lyapunov cone-systems will be considered. For the systems, the necessary and sufficient conditions for being the cone-system, the asymptotic stability and sufficient conditions for the reachability and observability will be established. In the Section 4, the discrete-time linear Lyapunov cone-systems will be considered. For the systems, the necessary and sufficient conditions for being the cone-system, the asymptotic stability, reachability, observability and controllability to zero will be established. In the Section 5, the fractional discrete-time linear Lyapunov cone-systems will be considered. For the systems, the necessary and sufficient conditions for being the cone-system, the reachability, observability and controllability to zero and sufficient conditions for the stability will be established. In the Section 6, the considerations will be illustrated by numerical examples.

## 2. Preliminaries

Let  $R^{nm}$  be the set of real  $n \times m$  matrices,  $R^n = R^{n \times 1}$  and let  $R_+^{nm}$  be the set of real  $n \times m$  matrices with nonnegative entries. The set of nonnegative integers will be denoted by  $Z_+$ .

### Definition 1.

The Kronecker product  $A \otimes B$  of the matrices  $A = [a_{ij}] \in R^{m \times n}$  and  $B \in R^{p \times q}$  is the block matrix (Kaczorek T., 1998):

$$A \otimes B = [a_{ij} B]_{\substack{i=1, \dots, m \\ j=1, \dots, n}} \in R^{mp \times nq} \quad (1)$$

### Lemma 1.

Let us consider the equation:

$$AXB = C \quad (2)$$

where:  $A \in R^{m \times n}$ ,  $B \in R^{q \times p}$ ,  $C \in R^{m \times p}$ ,  $X \in R^{n \times q}$

Equation (2) is equivalent to the following one:

$$(A \otimes B^T)x = c \tag{3}$$

where  $x := [x_1 \ x_2 \ \dots \ x_n]^T$ ,  $c := [c_1 \ c_2 \ \dots \ c_m]^T$ , and  $x_i$  and  $c_i$  are the  $i$ th rows of the matrices  $X$  and  $C$  respectively.

Proof: See (Kaczorek T., 1998)

**Lemma 2.**

If  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of the matrix  $A$  and  $\mu_1, \mu_2, \dots, \mu_n$  the eigenvalues of the matrix  $B$ , then  $\lambda_i + \mu_j$  for  $i, j = 1, 2, \dots, n$  are the eigenvalues of the matrix:

$$\bar{A} = A \otimes I_n + I_n \otimes B^T$$

Proof: See (Kaczorek T., 1998)

**Definition 2.**

Let  $P = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \in R^{n \times n}$  be nonsingular and  $p_k$  be the  $k$ th ( $k = 1, \dots, n$ ) its row. The set:

$$P := \left\{ X(t) \in R^{n \times n} : \bigcap_{k=1}^n p_k X_i(t) \geq 0 \right\} \tag{4}$$

where  $X_i(t)$ ,  $i = 1, \dots, n$  is the  $i$ th column of the matrix  $X(t)$ , is called a linear cone of the state variables generated by the matrix  $P$ . In the similar way we may define the linear cone of the inputs:

$$Q := \left\{ U(t) \in R^{m \times n} : \bigcap_{k=1}^m q_k U_i(t) \geq 0 \right\} \tag{5}$$

generated by the nonsingular matrix  $Q = \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} \in R^{m \times m}$  and the linear cone of the outputs

$$V := \left\{ Y(t) \in R^{p \times n} : \bigcap_{k=1}^p v_k Y_i(t) \geq 0 \right\} \tag{6}$$

generated by the nonsingular matrix  $V = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix} \in R^{p \times p}$ .

### 3. Continuous-time linear Lyapunov cone-systems

Consider the continuous-time linear Lyapunov system (Kaczorek T. & Przyborowski P., 2007a) described by the equations:

$$\dot{X}(t) = A_0 X(t) + X(t) A_1 + BU(t) \quad (7a)$$

$$Y(t) = CX(t) + DU(t) \quad (7b)$$

where,  $X(t) \in R^{n \times n}$  is the state-space matrix,  $U(t) \in R^{m \times n}$  is the input matrix,  $Y(t) \in R^{p \times n}$  is the output matrix,  $A_0, A_1 \in R^{n \times n}$ ,  $B \in R^{n \times m}$ ,  $C \in R^{p \times n}$ ,  $D \in R^{p \times m}$ .

The solution of the equation (1a) satisfying the initial condition  $X(t_0) = X_0$  is given by (Kaczorek T. & Przyborowski P., 2007a):

$$X(t) = e^{A_0(t-t_0)} X_0 e^{A_1(t-t_0)} + \int_{t_0}^t e^{A_0(t-\tau)} BU(\tau) e^{A_1(t-\tau)} d\tau \quad (8)$$

#### Lemma 3.

The Lyapunov system (7) can be transformed to the equivalent standard continuous-time,  $nm$ -inputs and  $pn$ -outputs, linear system in the form:

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t) \quad (9a)$$

$$\tilde{y}(t) = \tilde{C}\tilde{x}(t) + \tilde{D}\tilde{u}(t) \quad (9b)$$

where,  $\tilde{x}(t) \in R^{n^2}$  is the state-space vector,  $\tilde{u}(t) \in R^{(nm)}$  is the input vector,  $\tilde{y}(t) \in R^{(pn)}$  is the output vector,  $\tilde{A} \in R^{n^2 \times n^2}$ ,  $\tilde{B} \in R^{n^2 \times (nm)}$ ,  $\tilde{C} \in R^{(pn) \times n^2}$ ,  $\tilde{D} \in R^{(pn) \times (nm)}$ .

Proof:

The transformation is based on Lemma 1. The matrices  $X, U, Y$  are transformed to the vectors:

$$\tilde{x} = [X_1 \quad X_2 \quad \dots \quad X_n]^T, \tilde{u} = [U_1 \quad U_2 \quad \dots \quad U_m]^T, \tilde{y} = [Y_1 \quad Y_2 \quad \dots \quad Y_p]^T$$

where  $X_i, U_i, Y_i$  denotes the  $i$ th rows of the matrices  $X, U, Y$ , respectively.

The matrices of (9) are:

$$\tilde{A} = (A_0 \otimes I_n + I_n \otimes A_1^T), \tilde{B} = B \otimes I_n, \tilde{C} = C \otimes I_n, \tilde{D} = D \otimes I_n \quad (10)$$

#### 3.1 Cone-systems

##### Definition 3.

The Lyapunov system (7) is called  $(P, Q, V)$ -cone-system if  $X(t) \in P$  and  $Y(t) \in V$  for every  $X_0 \in P$  and for every input  $U(t) \in Q$ ,  $t \geq t_0$ .

Note that for  $\mathbf{P} = R_+^{n \times n}$ ,  $\mathbf{Q} = R_+^{m \times n}$ ,  $\mathbf{V} = R_+^{p \times n}$  we obtain  $(R_+^{n \times n}, R_+^{m \times n}, R_+^{p \times n})$ -cone system which is equivalent to the positive Lyapunov system (Kaczorek T. & Przyborowski P., 2007c).

**Theorem 1.**

The Lyapunov system (7) is  $(\mathbf{P}, \mathbf{Q}, \mathbf{V})$ -cone-system if and only if :

$$\hat{A}_0 = PA_0P^{-1}, \hat{A}_1 = A_1 \tag{11}$$

are the Metzler matrices and

$$\hat{B} = PBQ^{-1} \in R_+^{n \times m}, \hat{C} = VCP^{-1} \in R_+^{p \times n}, \hat{D} = VDQ^{-1} \in R_+^{p \times m}. \tag{12}$$

Proof:

Let:

$$\hat{X}(t) = PX(t), \hat{U}(t) = QU(t), \hat{Y}(t) = VY(t) \tag{13}$$

From definition 2 it follows that if  $X(t) \in \mathbf{P}$  then  $\hat{X}(t) \in R_+^{n \times n}$ , if  $U(t) \in \mathbf{Q}$  then  $\hat{U}(t) \in R_+^{m \times n}$ , and if  $Y(t) \in \mathbf{V}$  then  $\hat{Y}(t) \in R_+^{p \times n}$ .

From (7) and (13) we have:

$$\begin{aligned} \dot{\hat{X}}(t) &= P\dot{X}(t) = PA_0X(t) + PX(t)A_1 + PBV(t) = PA_0P^{-1}\hat{X}(t) + PP^{-1}\hat{X}(t)A_1 + \\ &+ PBQ^{-1}\hat{U}(t) = \hat{A}_0\hat{X}(t) + \hat{X}(t)A_1 + \hat{B}\hat{U}(t) \end{aligned} \tag{14a}$$

and

$$\hat{Y}(t) = VY(t) = VCX(t) + VDU(t) = VCP^{-1}\hat{X}(t) + VDQ^{-1}\hat{U}(t) = \hat{C}\hat{X}(t) + \hat{D}\hat{U}(t) \tag{14b}$$

It is known (Kaczorek T. & Przyborowski P., 2007a) that the system (14) is positive if and only if the conditions (11) and (12) are satisfied. □

**3.2 Asymptotic stability**

Consider the autonomous Lyapunov  $(\mathbf{P}, \mathbf{Q}, \mathbf{V})$ -cone-system:

$$\dot{X}(t) = A_0X(t) + X(t)A_1, \quad X(t_0) = X_0 \tag{15}$$

where,  $X(t) \in \mathbf{P}$  and  $PA_0P^{-1}, A_1 \in R^{n \times n}$  are the Metzler matrices.

**Definition 4.**

The Lyapunov  $(\mathbf{P}, \mathbf{Q}, \mathbf{V})$ -cone-system (15) is called asymptotically stable if:

$$\lim_{t \rightarrow \infty} X(t) = 0 \text{ for every } X_0 \in \mathbf{P}$$

**Theorem 2.**

Let us assume that  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of the matrix  $A_0$  and  $\mu_1, \mu_2, \dots, \mu_n$  the eigenvalues of the matrix  $A_1$ . The system (15) is stable if and only if:

$$\operatorname{Re}(\lambda_i + \mu_j) < 0 \text{ for } i, j = 1, 2, \dots, n \quad (16)$$

Proof:

The theorem results directly from the theorem for asymptotic stability of standard systems (Kaczorek T., 2001), since by Lemma 2 eigenvalues of matrix  $\tilde{A}$  are the sums of eigenvalues of the matrices  $A_0$  and  $A_1$ .  $\square$

**3.3 Reachability****Definition 5.**

The state  $X_f \in \mathbf{P}$  of the the Lyapunov  $(\mathbf{P}, \mathbf{Q}, \mathbf{V})$ -cone-system (7) is called reachable at time  $t_f$ , if there exists an input  $U(t) \in \mathbf{Q}$  for  $t \in [t_0, t_f]$ , which steers the system from the initial state  $X_0 = 0$  to the state  $X_f$ .

**Definition 6.**

If for every state  $X_f \in \mathbf{P}$  there exists  $t_f > t_0$ , such that the state is reachable at time  $t_f$ , then the system is called reachable.

**Theorem 3.**

The  $(\mathbf{P}, \mathbf{Q}, \mathbf{V})$ -cone-system (7) is reachable if the matrix:

$$R_f := \int_{t_0}^{t_f} e^{PA_0P^{-1}(t_f-\tau)} (PBQ^{-1})(PBQ^{-1})^T e^{(PA_0P^{-1})^T(t_f-\tau)} d\tau \quad (17)$$

is a monomial matrix (only one element in every row and in every column of the matrix is positive and the remaining are equal to zero).

The input, that steers the system from initial state  $X_0 = 0$  to the state  $X_f$  is given by:

$$U(t) = Q^{-1}[(PBQ^{-1})^T e^{(PA_0P^{-1})^T(t_f-t)} R_f^{-1} P X_f e^{A_1(t-t_f)}] \quad (18)$$

for  $t \in [t_0, t_f]$ .

Proof:

If  $R_f$  is the monomial matrix, then there exists  $R_f^{-1} \in R_+^{n \times n}$  and the input (18) is well-defined.

Using (8) and (18) we obtain:

$$\begin{aligned}
 X(t_f) &= P^{-1} \left[ \int_{t_0}^{t_f} e^{PA_0P^{-1}(t_f-\tau)} (PBQ^{-1})(PBQ^{-1})^T e^{(PA_0P^{-1})^T(t_f-\tau)} R_f^{-1} P X_f e^{A_1(\tau-t_f)} e^{A_1(t_f-\tau)} d\tau \right] = \\
 &= P^{-1} \left[ \int_{t_0}^{t_f} e^{PA_0P^{-1}(t_f-\tau)} (PBQ^{-1})(PBQ^{-1})^T e^{(PA_0P^{-1})^T(t_f-\tau)} d\tau R_f^{-1} P X_f \right] = P^{-1} P X_f = X_f
 \end{aligned}
 \tag{19}$$

since  $e^{A_1(\tau-t_f)} e^{A_1(t_f-\tau)} = I_n$ . □

**3.4 Dual Lyapunov cone-systems**

*Definition 7.*

The Lyapunov system described by the equations:

$$\dot{X}(t) = A_0^T X(t) + X(t)A_0^T + C^T U(t) \tag{20a}$$

$$Y(t) = B^T X(t) + D U(t) \tag{20b}$$

is called the dual system with respect to the system (7). The matrices  $A_0, A_1, B, C, D, X(t), U(t), Y(t)$  are the same as in the system (7).

**3.5 Observability**

*Definition 8.*

The state  $X_0$  of the Lyapunov (P,Q,V) -cone- system (7) is called observable at time  $t_f > 0$ , if  $X_0$  can be uniquely determined from the knowledge of the output  $Y(t)$  and input  $U(t)$  for  $t \in [0, t_f]$ .

*Definition 9.*

The Lyapunov (P,Q,V)-cone- system (7) is called observable, if there exists an instant  $t_f > 0$ , such that the system (7) is observable at time  $t_f$ .

*Theorem 4.*

The Lyapunov (P,Q,V)-cone-system (7) is observable if the dual system (20) is reachable i.e. if the matrix:

$$O_f := \int_{t_0}^{t_f} e^{(PA_0P^{-1})^T(t_f-\tau)} (VCP^{-1})^T (VCP^{-1}) e^{(PA_0P^{-1})(t_f-\tau)} d\tau \tag{21}$$

is a monomial matrix.

Proof:

The Lyapunov (P,Q,V)-cone-system (7) is observable if and only if the equivalent standard system (9) is observable and this implies that dual system with respect to the system (9) must be reachable thus the dual system (20) with respect to the system (7) also must be reachable. Using Theorem 3. we obtain the hypothesis of the Theorem 4. □

#### 4. Discrete-time linear Lyapunov cone-systems

Consider the discrete-time linear Lyapunov system (Kaczorek T., 2007b; Kaczorek T. & Przyborowski P., 2007e; Kaczorek T. & Przyborowski P., 2008) described by the equations:

$$X_{i+1} = A_0 X_i + X_i A_1 + B U_i \quad (22a)$$

$$Y_i = C X_i + D U_i \quad (22b)$$

where,  $X_i \in R^{n \times n}$  is the state-space matrix,  $U_i \in R^{m \times n}$  is the input matrix,  $Y_i \in R^{p \times n}$  is the output matrix,  $A_0, A_1 \in R^{n \times n}$ ,  $B \in R^{n \times m}$ ,  $C \in R^{p \times n}$ ,  $D \in R^{p \times m}$ ,  $i \in Z_+$ .

The solution of the equation (22a) satisfying the initial condition  $X_0$  is given by (Kaczorek T., 2007b):

$$X_i = \sum_{k=0}^i \frac{i!}{k!(i-k)!} A_0^k X_0 A_1^{i-k} + \sum_{j=0}^{i-1} \sum_{k=0}^j \frac{j!}{k!(j-k)!} A_0^k B U_{i-j-1} A_1^{i-k}, i \in Z_+ \quad (23)$$

##### Lemma 4.

The Lyapunov system (22) can be transformed to the equivalent standard discrete-time,  $nm$ -inputs and  $pn$ -outputs, linear system in the form:

$$\bar{x}_{i+1} = \bar{A} \bar{x}_i + \bar{B} u_i \quad (24a)$$

$$\bar{y}_i = \bar{C} \bar{x}_i + \bar{D} u_i \quad (24b)$$

where,  $\bar{x}_i \in R^{n^2}$  is the state-space vector,  $\bar{u}_i \in R^{(nm)}$  is the input vector,  $\bar{y}_i \in R^{(pn)}$  is the output vector,  $\bar{A} \in R^{n^2 \times n^2}$ ,  $\bar{B} \in R^{n^2 \times (nm)}$ ,  $\bar{C} \in R^{(pn) \times n^2}$ ,  $\bar{D} \in R^{(pn) \times (nm)}$ ,  $i \in Z_+$ .

Proof:

The proof is similar to the one of Lemma 3.

The matrices of (24) have the form:

$$\bar{A} = (A_0 \otimes I_n + I_n \otimes A_1^T), \bar{B} = B \otimes I_n, \bar{C} = C \otimes I_n, \bar{D} = D \otimes I_n \quad (25)$$

#### 4.1 Cone-systems

##### Definition 10.

The Lyapunov system (22) is called  $(P, Q, V)$ -cone-system if  $X_i \in P$  and  $Y_i \in V$  for every  $X_0 \in P$  and for every input  $U_i \in Q$ ,  $i \in Z_+$ .

Note that for  $P = R_+^{n \times n}$ ,  $Q = R_+^{m \times n}$ ,  $V = R_+^{p \times n}$  we obtain  $(R_+^{n \times n}, R_+^{m \times n}, R_+^{p \times n})$ -cone system which is equivalent to the positive Lyapunov system (Kaczorek T., 2007b).



**Theorem 5.**

The Lyapunov system (22) is  $(P, Q, V)$ -cone-system if and only if :

$$\hat{A}_0 = PA_0P^{-1} = \left[ \hat{a}_{ij}^0 \right]_{\substack{i=1, \dots, n \\ j=1, \dots, n}}, \hat{A}_1 = A_1 = \left[ \hat{a}_{ij}^1 \right]_{\substack{i=1, \dots, n \\ j=1, \dots, n}} \tag{26}$$

are the Metzler matrices satisfying

$$\hat{a}_{kk}^0 + \hat{a}_{ll}^1 \geq 0 \text{ for every } k, l = 1, \dots, n \tag{27}$$

and

$$\hat{B} = PBQ^{-1} \in R_+^{n \times m}, \hat{C} = VCP^{-1} \in R_+^{p \times n}, \hat{D} = VDQ^{-1} \in R_+^{p \times m} \tag{28}$$

Proof:

Let:

$$\hat{X}_i = PX_i, \hat{U}_i = QU_i, \hat{Y}_i = VY_i \tag{29}$$

From definition 2 it follows that if  $X_i \in P$  then  $\hat{X}_i \in R_+^{n \times n}$ , if  $U_i \in Q$  then  $\hat{U}_i \in R_+^{m \times n}$ ,

and if  $Y_i \in V$  then  $\hat{Y}_i \in R_+^{p \times n}$ .

From (22) and (29) we have:

$$\begin{aligned} \hat{X}_{i+1} = PX_{i+1} &= PA_0X_i + PX_iA_1 + PBQ^{-1}U_i = PA_0P^{-1}\hat{X}_i + PP^{-1}\hat{X}_iA_1 + \\ &+ PBQ^{-1}\hat{U}_i = \hat{A}_0\hat{X}_i + \hat{X}_iA_1 + \hat{B}\hat{U}_i \end{aligned} \tag{30a}$$

and

$$\hat{Y}_i = VY_i = VCP^{-1}\hat{X}_i + VDQ^{-1}\hat{U}_i = \hat{C}\hat{X}_i + \hat{D}\hat{U}_i \tag{30b}$$

The Lyapunov system (30) is positive if and only if, the equivalent standard system is positive. By the theorem for the positivity of the standard discrete-time systems, the matrices  $(\hat{A}_0 \otimes I_n + I_n \otimes \hat{A}_1^T), (\hat{B} \otimes I_n), (\hat{C} \otimes I_n), (\hat{D} \otimes I_n)$  have to be the matrices with nonnegative entries, so from (30) follows the hypothesis of the Theorem 5.  $\square$

**4.2 Asymptotic stability**

Consider the autonomous Lyapunov  $(P, Q, V)$ -cone-system:

$$X_{i+1} = A_0X_i + X_iA_1 \tag{31}$$

where,  $X_i \in P, i \in Z_+$ .

**Definition 11.**

The Lyapunov  $(P, Q, V)$ -cone-system (15) is called asymptotically stable if:

$$\lim_{i \rightarrow \infty} X_i = 0 \text{ for every } X_0 \in P$$

**Theorem 6.**

Let us assume that  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of the matrix  $A_0$  and  $\mu_1, \mu_2, \dots, \mu_n$  the eigenvalues of the matrix  $A_1$ . The system (31) is stable if and only if:

$$|\lambda_i + \mu_j| < 1 \text{ for } i, j = 1, 2, \dots, n \quad (32)$$

Proof:

The theorem results directly from the theorem for asymptotic stability of standard systems (Kaczorek T., 2001), since by Lemma 2 eigenvalues of matrix  $\bar{A}$  are the sums of eigenvalues of the matrices  $A_0$  and  $A_1$ .  $\square$

**4.3 Reachability****Definition 12.**

The Lyapunov  $(P, Q, V)$ -cone-system (22) is called reachable if for any given  $X_f \in \mathbf{P}$  there exist  $q \in \mathbf{Z}_+, q > 0$  and an input sequence  $U_i \in \mathbf{Q}, i = 0, 1, \dots, q-1$  that steers the state of the system from  $X_0 = 0$  to  $X_q = X_f$ , i.e.  $X_q = X_f$ .

**Theorem 7.**

The Lyapunov  $(P, Q, V)$ -cone-system (22) is reachable:

a) For  $A_1$  satisfying the condition  $XA_1 = A_1X$ , i.e.  $A_1 = aI_n, a \in R$ , if and only if the matrix:

$$R_n = [PBQ^{-1} \quad \bar{A}_0(PBQ^{-1}) \quad \dots \quad \bar{A}_0^{n-1}(PBQ^{-1})] \quad (33)$$

contains  $n$  linearly independent monomial columns,  $\bar{A}_0 = PA_0P^{-1} + A_1$ .

b) For  $A_1 \neq aI_n, a \in R$ , if and only if the matrix  $PBQ^{-1}$  contains  $n$  linearly independent monomial columns.

Proof:

From (26),(28),(29) and from the definitions 2 and 12, we have that the discrete-time Lyapunov  $(P, Q, V)$ -cone-system (22) is reachable if and only if the positive discrete-time Lyapunov system, with the matrices  $\hat{A}_0, \hat{A}_1, \hat{B}, \hat{C}, \hat{D}$ , is reachable – so from the theorem for the reachability of positive discrete-time Lyapunov systems (Kaczorek T. & Przyborowski P., 2007e; Kaczorek T. & Przyborowski P., 2008) follows the hypothesis of the theorem 7.  $\square$

**4.4 Controllability to zero****Definition 13.**

The Lyapunov  $(P, Q, V)$ -cone-system (22) is called controllable to zero if for any given nonzero  $X_0 \in \mathbf{P}$  there exist  $q \in \mathbf{Z}_+, q > 0$  and an input sequence  $U_i \in \mathbf{Q}, i = 0, 1, \dots, q-1$  that steers the state of the system from  $X_0$  to  $X_q = X_f = 0$ .

**Theorem 8.**

The Lyapunov  $(P, Q, V)$ -cone-system (22) is controllable to zero:

- a) in a finite number of steps (not greater than  $n^2$ ) if and only if the matrix  $(PA_0P^{-1} \otimes I_n + I_n \otimes A_1^T)$  is nilpotent, i.e. has all zero eigenvalues.
- b) in an infinite number of steps if and only if the system is asymptotically stable.

Proof:

From (26),(28),(29) and from the definitions 2 and 13, we have that the discrete-time Lyapunov  $(P, Q, V)$ -cone-system (22) is controllable to zero if and only if the positive discrete-time Lyapunov system, with the matrices  $\hat{A}_0, \hat{A}_1, \hat{B}, \hat{C}, \hat{D}$ , is controllable to zero – so from the theorem for the controllability to zero of positive discrete-time Lyapunov systems (Kaczorek T. & Przyborowski P., 2007e; Kaczorek T. & Przyborowski P., 2008) follows the hypothesis of the theorem 8. □

**Lemma 5.**

If the matrices  $PA_0P^{-1}$  and  $A_1$  are nilpotent then the matrix  $(PA_0P^{-1} \otimes I_n + I_n \otimes A_1^T)$  is also nilpotent with the nilpotency index  $\nu \leq 2n$ .

Proof: See (Kaczorek T. & Przyborowski P, 2008).

**4.5 Dual Lyapunov cone-systems**

**Definition 14.**

The Lyapunov system described by the equations:

$$X_{i+1} = A_0^T X_i + X_i A_1^T + C^T U_i \tag{34a}$$

$$Y_i = B^T X_i + D U_i \tag{35b}$$

is called the dual system respect to the system (22). The matrices  $A_0, A_1, B, C, D, X_i, U_i, Y_i$  are the same as in the system (22).

**4.6 Observability**

**Definition 15.**

The Lyapunov  $(P, Q, V)$ -cone-system (22) is called observable in  $q$ -steps, if  $X_0$  can be uniquely determined from the knowledge of the output  $Y_i$  and  $U_i = 0, i \in Z_+$  for  $i \in [0, q]$ .

**Definition 16.**

The Lyapunov  $(P, Q, V)$ -cone-system (22) is called observable, if there exists a natural number  $q \geq 1$ , such that the system (22) is observable in  $q$ -steps.

**Theorem 9.**

The Lyapunov  $(P, Q, V)$ -cone-system (22) is observable:

- a) For  $A_1$  satisfying the condition  $X A_1 = A_1 X$ , i.e.  $A_1 = a I_n, a \in R$ , if and only if the matrix:

$$O_n = \begin{bmatrix} VCP^{-1} \\ (VCP^{-1})\bar{A}_0 \\ \vdots \\ (VCP^{-1})\bar{A}^{n-1} \end{bmatrix} \tag{33}$$

contains  $n$  linearly independent monomial rows,  $\bar{A}_0 = PA_0P^{-1} + A_1$ .

b) For  $A_1 \neq aI_n, a \in R$ , if and only if the matrix  $VCP^{-1}$  contains  $n$  linearly independent monomial rows.

Proof:

From (26),(28),(29) and from the definitions 2 and 15, we have that the discrete-time Lyapunov  $(P, Q, V)$ -cone-system (22) is observable if and only if the positive discrete-time Lyapunov system, with the matrices  $\hat{A}_0, \hat{A}_1, \hat{B}, \hat{C}, \hat{D}$ , is observable - so from the theorem for the observability of positive discrete-time Lyapunov systems (Kaczorek T. & Przyborowski P., 2007e; Kaczorek T. & Przyborowski P., 2008) follows the hypothesis of the theorem 9. □

### 5. Fractional discrete-time linear Lyapunov cone-systems

Consider the fractional discrete-time linear Lyapunov system (Przyborowski P., 2008a; Przyborowski P., 2008b) described by the equations:

$$\Delta^N X_{i+1} = A_0 X_i + X_i A_1 + BU_i \tag{34a}$$

$$Y_i = CX_i + DU_i \tag{34b}$$

where,  $X_i \in R^{n \times n}$  is the state-space matrix,  $U_i \in R^{m \times n}$  is the input matrix,  $Y_i \in R^{p \times n}$  is the output matrix,  $A_0, A_1 \in R^{n \times n}, B \in R^{n \times m}, C \in R^{p \times n}, D \in R^{p \times m}, i \in Z_+$  and

$$\Delta^N X_i = \frac{1}{h^N} \sum_{j=0}^i (-1)^j \binom{N}{j} X_{i-j}, \quad \binom{N}{j} = \begin{cases} 1 & \text{for } j=0 \\ \frac{N(N-1)\dots(N-j+1)}{j!} & \text{for } j=1, 2, \dots \end{cases}$$

is the Grünwald-Letnikov  $N$ -order ( $N \in R, 0 < N \leq 1$ ) fractional difference, and  $h$  is the sampling interval.

The equations (34) can be written in the form:

$$X_{i+1} + \sum_{j=1}^{i+1} (-1)^j \binom{N}{j} X_{i-j+1} = A_0 X_i + X_i A_1 + BU_i \tag{35a}$$

$$Y_i = CX_i + DU_i \tag{35b}$$

**Lemma 6.**

The fractional Lyapunov system (34) can be transformed to the equivalent fractional discrete-time,  $nm$  -inputs and  $pn$  -outputs, linear system in the form:

$$\Delta^N \widehat{x}_{i+1} = \widehat{A}\widehat{x}_i + \widehat{B}\widehat{u}_i \tag{36a}$$

$$\widehat{y}_i = \widehat{C}\widehat{x}_i + \widehat{D}\widehat{u}_i \tag{36b}$$

where,  $\widehat{x}_i \in R^{n^2}$  is the state-space vector,  $\widehat{u}_i \in R^{(nm)}$  is the input vector,  $\widehat{y}_i \in R^{(pn)}$  is the output vector,  $\widehat{A} \in R^{n^2 \times n^2}$ ,  $\widehat{B} \in R^{n^2 \times (nm)}$ ,  $\widehat{C} \in R^{(pn) \times n^2}$ ,  $\widehat{D} \in R^{(pn) \times (nm)}$ ,  $i \in Z_+$ .

Proof:

The proof is similar to the one of Lemma 3.

The matrices of (36) have the form:

$$\widehat{A} = (A_0 \otimes I_n + I_n \otimes A_1^T), \widehat{B} = B \otimes I_n, \widehat{C} = C \otimes I_n, \widehat{D} = D \otimes I_n \tag{37}$$

**5.1 Cone-systems**

**Definition 17.**

The fractional Lyapunov system (22) is called  $(P, Q, V)$  -cone-system if  $X_i \in P$  and  $Y_i \in V$  for every  $X_0 \in P$  and for every input  $U_i \in Q$ ,  $i \in Z_+$ .

Note that for  $P = R_+^{n \times n}$ ,  $Q = R_+^{m \times n}$ ,  $V = R_+^{p \times n}$  we obtain  $(R_+^{n \times n}, R_+^{m \times n}, R_+^{p \times n})$  -cone system which is equivalent to the fractional positive Lyapunov system (Przyborowski P., 2008a).

**Theorem 10.**

The fractional Lyapunov system (34) is  $(P, Q, V)$  -cone-system if and only if :

$$\widehat{A}_0 = PA_0P^{-1} = [\widehat{a}_{ij}^0]_{\substack{i=1, \dots, n \\ j=1, \dots, n}}, \widehat{A}_1 = A_1 = [\widehat{a}_{ij}^1]_{\substack{i=1, \dots, n \\ j=1, \dots, n}} \tag{38}$$

are the Metzler matrices satisfying

$$\widehat{a}_{kk}^0 + \widehat{a}_{ll}^1 + N \geq 0 \text{ for every } k, l = 1, \dots, n \tag{39}$$

and

$$\widehat{B} = PBQ^{-1} \in R_+^{n \times m}, \widehat{C} = VCP^{-1} \in R_+^{p \times n}, \widehat{D} = VDQ^{-1} \in R_+^{p \times m} \tag{40}$$

Proof:

Let:

$$\widehat{X}_i = PX_i, \widehat{U}_i = QU_i, \widehat{Y}_i = VY_i \tag{41}$$

From definition 2 it follows that if  $X_i \in P$  then  $\widehat{X}_i \in R_+^{n \times n}$ , if  $U_i \in Q$  then  $\widehat{U}_i \in R_+^{m \times n}$ , and if  $Y_i \in V$  then  $\widehat{Y}_i \in R_+^{p \times n}$ .

From (34) and (41) we have:

$$\begin{aligned} \hat{X}_{i+1} + \sum_{j=1}^{i+1} (-1)^j \binom{N}{j} \hat{X}_{i-j+1} &= PX_{i+1} + \sum_{j=1}^{i+1} (-1)^j \binom{N}{j} PX_{i-j+1} = \\ &= PA_0 X_i + PX_i A_1 + PB U_i = PA_0 P^{-1} \hat{X}_i + PP^{-1} \hat{X}_i A_1 + PB Q^{-1} \hat{U}_i = \\ &= \hat{A}_0 \hat{X}_i + \hat{X}_i A_1 + \hat{B} \hat{U}_i \end{aligned} \quad (41a)$$

and

$$\hat{Y}_i = VY_i = V C X_i + V D U_i = V C P^{-1} \hat{X}_i + V D Q^{-1} \hat{U}_i = \hat{C} \hat{X}_i + \hat{D} \hat{U}_i \quad (42b)$$

It is known (Przyborowski P., 2008a) that the system (34) is positive if and only if the conditions (41a) and (42b) are satisfied.  $\square$

## 5.2. Stability

Consider the autonomous fractional Lyapunov  $(P, Q, V)$ -cone-system:

$$\Delta^N X_{i+1} = A_0 X_i + X_i A_1 \quad (43)$$

where,  $X_i \in \mathbf{P}$ ,  $i \in \mathbf{Z}_+$ .

**Definition 18.** (Dzieliński A. & Sierociuk D., 2006)

The fractional Lyapunov  $(P, Q, V)$ -cone- system (43) is called stable in finite relative time if for  $\alpha, \beta \in \mathbf{R}_+$ ,  $\alpha < \beta$ ,  $\alpha, \beta < \infty$ ,  $k = 1, \dots, n$ ;  $M, N \in \mathbf{Z}_+$ :

$$\|X_i^k\| < \alpha \quad \text{for } i = 0, -1, \dots, -N$$

implies

$$\|X_i^k\| < \beta \quad \text{for } i = 0, 1, \dots, M$$

where  $X_i^k$  is the  $k$  th column of the matrix  $X_i$ .

**Theorem 11.**

The fractional Lyapunov  $(P, Q, V)$ -cone- system (34) is stable in the meaning of the definition 18 if:

$$\|\hat{A} + I_{n^2} N\| + \sum_{j=2}^{i+1} \left\| (-1)^j \binom{N}{j} I_{n^2} \right\| < 1 \quad (44)$$

where  $\hat{A} = A_0 \otimes I_n + I_n \otimes A_1^T$  and  $\|W\|$  denotes the norm of the matrix  $W$ , defined as  $\max_l |\lambda_l|$ , where  $\lambda_l$  is the  $l$  th eigenvalue of the matrix  $W$ .

Proof:

The theorem results directly from the theorem of asymptotic stability of standard fractional systems (Dzieliński A. & Sierociuk D., 2006).  $\square$

**5.3 Reachability**

*Definition 19.*

The fractional Lyapunov  $(P, Q, V)$ -cone-system (34) is called reachable if for any given  $X_f \in P$  there exists  $q \in Z_+, q > 0$  and an input sequence  $U_i \in Q, i = 0, 1, \dots, q - 1$  that steers the state of the system from  $X_0 = 0$  to  $X_f$ , i.e.  $X_q = X_f$ .

*Theorem 12.*

The fractional Lyapunov  $(P, Q, V)$ -cone-system (34) is reachable:

a) For  $A_1$  satisfying the condition  $XA_1 = A_1X$ , i.e.  $A_1 = aI_n, a \in R$ , if and only if the matrix:

$$R_n = [PBQ^{-1}, (\bar{A}_0 + I_n N)PBQ^{-1}] \tag{45}$$

contains  $n$  linearly independent monomial columns,  $\bar{A}_0 = PA_0P^{-1} + A_1$ .

b) For  $A_1 \neq aI_n, a \in R$ , if and only if the matrix  $PBQ^{-1}$  contains  $n$  linearly independent monomial columns.

Proof:

From (38),(39),(40) and from the definitions 2 and 19, we have that the fractional discrete-time Lyapunov  $(P, Q, V)$ -cone-system (34) is reachable if and only if the fractional positive discrete-time Lyapunov system, with the matrices  $\hat{A}_0, \hat{A}_1, \hat{B}, \hat{C}, \hat{D}$ , is reachable - so from the theorem for the reachability of positive discrete-time Lyapunov systems (Przyborowski P.,2008a). follows the hypothesis of the theorem 12.  $\square$

**5.4 Controllability to zero**

*Definition 20.*

The fractional Lyapunov  $(P, Q, V)$ -cone-system (34) is called controllable to zero if for any given nonzero  $X_0 \in P$  there exist  $q \in Z_+, q > 0$  and an input sequence  $U_i \in Q, i = 0, 1, \dots, q - 1$  that steers the state of the system from  $X_0$  to  $X_q = X_f = 0$ .

*Theorem 13.*

The fractional Lyapunov  $(P, Q, V)$ -cone-system (34) is controllable to zero if and only if  $q = 2$  and:

$$PA_0P^{-1} \otimes I_n + I_n \otimes A_1^T + (I_n N) \otimes I_n = 0 \tag{46}$$

Proof:

From (38),(39),(40) and from the definitions 2 and 19, we have that the fractional discrete-time Lyapunov  $(P, Q, V)$ -cone-system (34) is controllable to zero if and only if the fractional positive discrete-time Lyapunov system, with the matrices  $\hat{A}_0, \hat{A}_1, \hat{B}, \hat{C}, \hat{D}$ , is controllable - so from the theorem for the controllability of positive discrete-time Lyapunov systems (Przyborowski P., 2008a). follows the hypothesis of the theorem 13.  $\square$

### 5.5 Dual fractional Lyapunov cone-systems

#### Definition 21.

The fractional Lyapunov system described by the equations:

$$\Delta^N X_{i+1} = A_0^T X_i + X_i A_1^T + C^T U_i \quad (47a)$$

$$Y_i = B^T X_i + D U_i \quad (47b)$$

is called the dual system respect to the system (34). The matrices  $A_0, A_1, B, C, D, X_i, U_i, Y_i$  are the same as in the system (34).

### 5.6 Observability

#### Definition 22.

The fractional Lyapunov  $(P, Q, V)$ -cone-system (34) is called observable in  $q$ -steps, if  $X_0$  can be uniquely determined from the knowledge of the output  $Y_i$  and  $U_i = 0, i \in Z_+$  for  $i \in [0, q]$ .

#### Definition 23.

The fractional Lyapunov  $(P, Q, V)$ -cone-system (34) is called observable, if there exists a natural number  $q \geq 1$ , such that the system (34) is observable in  $q$ -steps.

#### Theorem 14.

The fractional Lyapunov  $(P, Q, V)$ -cone-system (34) is observable:

a) For  $A_1$  satisfying the condition  $X A_1 = A_1 X$ , i.e.  $A_1 = a I_n, a \in R$ , if and only if the matrix:

$$O_n = \begin{bmatrix} V C P^{-1} \\ V C P^{-1} (\bar{A}_0 + I_n N) \end{bmatrix} \quad (48)$$

contains  $n$  linearly independent monomial rows,  $\bar{A}_0 = P A_0 P^{-1} + A_1$ .

b) For  $A_1 \neq a I_n, a \in R$ , if and only if the matrix  $V C P^{-1}$  contains  $n$  linearly independent monomial rows.

Proof:

From (38),(39),(40) and from the definitions 2 and 20, we have that the fractional discrete-time Lyapunov  $(P, Q, V)$ -cone-system (34) is controllable to zero if and only if the fractional positive discrete-time Lyapunov system, with the matrices  $\hat{A}_0, \hat{A}_1, \hat{B}, \hat{C}, \hat{D}$ , is observable - so from the theorem for the observability of positive discrete-time Lyapunov systems (Przyborowski P., 2008a). follows the hypothesis of the theorem 14.  $\square$

## 6. Examples

Consider the state, input and output cones generated by the matrices

$$P = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (49)$$



**6.1 Example 1**

Consider the continuous-time Lyapunov system (7) with the matrices

$$\begin{aligned}
 A_0 &= \frac{1}{3} \begin{bmatrix} -7 & -4 \\ -2 & -5 \end{bmatrix}, A_1 = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}, B = \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}, \\
 C &= \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
 \end{aligned}
 \tag{50}$$

This system is (P,Q,V)-cone-system with  $P, Q$  and  $V$  defined by (49) since:

$$\begin{aligned}
 \hat{A}_0 &= \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -7 & -4 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \\
 \hat{A}_1 &= \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}
 \end{aligned}$$

are the Metzler matrices and

$$\begin{aligned}
 \hat{B} &= \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 \hat{C} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\
 \hat{D} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
 \end{aligned}$$

are matrices with nonnegative entries.

$A_0$  has the eigenvalues:  $\lambda_1 = -1, \lambda_2 = -3$  and  $A_1$  has the eigenvalues:  $\mu_1 = -4, \mu_2 = -1$  therefore the system is asymptotically stable, since all the sums of the eigenvalues:

$$(\lambda_1 + \mu_1) = -5, (\lambda_1 + \mu_2) = -2, (\lambda_2 + \mu_1) = -7, (\lambda_2 + \mu_2) = -4$$

have negative real parts.

For this system the reachability matrix

$$R_f = \int_0^{t_f} \begin{bmatrix} e^{(-t_f+\tau)^2} & 0 \\ 0 & 4e^{3(-t_f+\tau)^2} \end{bmatrix} d\tau$$

and the observability matrix

$$O_f = \int_0^{t_f} \begin{bmatrix} 4e^{(-t_f+\tau)^2} & 0 \\ 0 & e^{3(-t_f+\tau)^2} \end{bmatrix} d\tau$$

are the monomial matrices for every  $t_f > 0$ . Therefore, the system is reachable and observable.

## 6.2 Example 2

Consider the discrete-time Lyapunov system (22) with the matrices

$$\begin{aligned} A_0 &= \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}, A_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix}, B = \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}, \\ C &= \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (50)$$

This system is  $(P, Q, V)$ -cone-system with  $P, Q$  and  $V$  defined by (49) since:

$$\hat{A}_0 = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.4 \end{bmatrix}, \hat{A}_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

are the Metzler matrices satisfying conditions:

$$\begin{aligned} \hat{a}_{11}^0 + \hat{a}_{11}^1 &= 0.3 + 0.2 = 0.5 > 0, \quad \hat{a}_{22}^0 + \hat{a}_{11}^1 = 0.2 + 0.2 = 0.4 > 0 \\ \hat{a}_{11}^0 + \hat{a}_{22}^1 &= 0.3 + 0.5 = 0.8 > 0, \quad \hat{a}_{22}^0 + \hat{a}_{22}^1 = 0.2 + 0.5 = 0.7 > 0 \end{aligned}$$

and

$$\begin{aligned} \hat{B} &= \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \hat{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\ \hat{D} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

are matrices with nonnegative entries.

$A_0$  has the eigenvalues:  $\lambda_1 = 0.1, \lambda_2 = 0.4$  and  $A_1$  has the eigenvalues:  $\mu_1 = 0.2, \mu_2 = 0.5$  therefore the system is asymptotically stable, since all the eigenvalues:

$$(\lambda_1 + \mu_1) = 0.3, (\lambda_1 + \mu_2) = 0.6, (\lambda_2 + \mu_1) = 0.6, (\lambda_2 + \mu_2) = 0.9$$

have moduli less than one.

The system is reachable and observable because the matrix  $PBQ^{-1}$  has  $n = 2$  monomial columns, and the matrix  $VCP^{-1}$  has  $n = 2$  monomial rows.

The system is not controllable to zero in finite number of steps since the matrix

$$(PA_0P^{-1} \otimes I_n + I_n \otimes A_1^T) = \begin{bmatrix} 0.3 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.9 \end{bmatrix}$$

is not a the nilpotent matrix, but the system is controllable to zero in the infinite number of steps since it is asymptotically stable.

**6.3 Example 3**

Consider the discrete-time Lyapunov system (34) with  $N = \frac{1}{4}$  and the matrices

$$A_0 = \begin{bmatrix} -0.5 & -0.8 \\ 0.6 & 0.9 \end{bmatrix}, A_1 = \begin{bmatrix} 0.17 & 0 \\ 1 & 0.4 \end{bmatrix}, B = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \tag{51}$$

$$C = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (n = 2)$$

This system is  $(P,Q,V)$ -cone-system with  $P, Q$  and  $V$  defined by (49) since:

$$\hat{A}_0 = \begin{bmatrix} 0.3 & 2 \\ 0 & 0.1 \end{bmatrix}, \hat{A}_1 = \begin{bmatrix} 0.17 & 0 \\ 1 & 0.4 \end{bmatrix}, \hat{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \hat{C} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \hat{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The system is  $(P,Q,V)$  -cone-system because:

$$\hat{a}_{11}^0 + \hat{a}_{11}^1 + N = 0.3 + 0.17 + 0.25 = 0.72 > 0, \hat{a}_{22}^0 + \hat{a}_{11}^1 + N = 0.1 + 0.17 + 0.25 = 0.52 > 0$$

$$\hat{a}_{11}^0 + \hat{a}_{22}^1 + N = 0.3 + 0.4 + 0.25 = 0.95 > 0, \hat{a}_{22}^0 + \hat{a}_{22}^1 + N = 0.1 + 0.4 + 0.25 = 0.75 > 0$$

and the matrices  $\hat{B}, \hat{C}, \hat{D}$  have nonnegative entries.

For the instant  $i = 100$  we have

$$\left\| \bar{A} + I_n N + \sum_{j=2}^{i+1} (-1)^j \binom{N}{j} I_n \right\| = 0.8268 < 1$$

so the system is stable in the meaning of the the definition 18.

The system is reachable and observable because the matrix  $PBQ^{-1}$  has  $n = 2$  monomial columns, and the matrix  $VCP^{-1}$  has  $n = 2$  monomial rows.

The system is not controllable to zero in finite number of steps since the matrix

$$(PA_0P^{-1} \otimes I_n + I_n \otimes A_1^T + (I_n N) \otimes I_n) = \begin{bmatrix} 0.72 & 0 & 0 & 0 \\ 0 & 0.52 & 0 & 0 \\ 0 & 0 & 0.95 & 0 \\ 0 & 0 & 0 & 0.75 \end{bmatrix}$$

is not a zero matrix.

## 7. Conclusions

In this paper three types of systems have been considered. For the continuous-time linear Lyapunov cone-systems, the necessary and sufficient conditions for being the cone-system, the asymptotic stability and sufficient conditions for the reachability and observability have been established. For the discrete-time linear Lyapunov cone-systems, the necessary and sufficient conditions for being the cone-system, the asymptotic stability, reachability, observability and controllability to zero have been established. For the fractional discrete-time linear Lyapunov cone-systems, the necessary and sufficient conditions for being the cone-system, the reachability, observability and controllability to zero and sufficient conditions for the stability have been established. The considerations have been illustrated on the numerical examples.

## 8. References

- Benvenuti L. & Farina L. (2004). A tutorial on positive realization problem, *IEEE Trans. Autom. Control*, vol. 49, No 5, pp. 651-664.
- Bologna M. & Grigolini P. (2003). *Physics of Fractal Operators*. Springer-Verlag, New York.
- Busłowicz M. & Kaczorek T. (2004). Reachability and minimum energy control of positive linear discrete-time systems with one delay, *Proceedings of 12th Mediterranean Conference on Control and Automation*, June 6-9, Kusadasi, Izmir, Turkey.
- Dzieliński A. & Sierociuk D.. Stability of discrete fractional order state-space systems., *Proceedings of 2nd IFAC Workshop on Fractional Differentiation and its Applications. IFAC FDA'06*.
- Farina L. & Rinaldi S. (2000). *Positive Linear Systems; Theory and Applications*, J. Wiley, New York.
- Ferreira N.M.F. & Machado I.A.T.(2003). Fractional-order hybrid control of robotic manipulators. *Proceedings of 11th Int. Conf. Advanced Robotics, ICAR'2003*, Coimbra, Portugal, pp. 393-398.
- Gałkowski K. (2005). Fractional polynomials and nD systems. *Proceedings of IEEE Int. Symp. Circuits and Systems, ISCAS'2005*, Kobe, Japan.
- Kaczorek T. (1998). *Vectors and Matrices in Automation and Electrotechnics*, Wydawnictwo Naukowo-Techniczne, Warszawa (in Polish).
- Kaczorek T. (2001). *Positive 1D and 2D systems*, Springer Verlag, London.

- Kaczorek T. (2004a). Realization problem for positive discrete-time systems with delay, *Systems Science*, vol. 30, No. 4, pp.117-130.
- Kaczorek T. (2004b). Stability of positive discrete-time systems with time-delay, *Proceedings of The 8th World Multi-Conference on Systemics, Cybernetics and Informatics*, July 18-21, Orlando, Florida, USA.
- Kaczorek T. (2006a). Realization problem for positive multivariable discrete-time linear systems with delays in the state vector and inputs, *Int. J. Appl. Math. Comp. Sci.*, vol. 16, No. 2, pp. 101-106.
- Kaczorek T. (2006b). A realization problem for positive continuous-time systems with reduced number of delays, *Int. J. Appl. Math. Comp. Sci.*, vol. 16, No. 3, pp. 101-117.
- Kaczorek T. (2007a). New reachability and observability tests for positive linear discrete-time systems, *Bull. Pol. Acad. Sci. Techn.* vol. 55, No. 1.
- Kaczorek T. (2007b). Positive discrete-time linear Lyapunov systems, *Proceedings of The 15th Mediterranean Conference of Control and Automation – MED 2007*, 27-29 June, Athens.
- Kaczorek T. (2007c). Reachability and controllability to zero of positive fractional discrete-time systems, *Machine Intelligence and Robotics Control*, vol. 6, No. 4, 2007.
- Kaczorek T. (2007d). Reachability and controllability to zero tests for standard and positive fractional discrete-time systems, *Submitted to Journal European Systems Analysis*.
- Kaczorek T. (2007e). Reachability and controllability to zero of cone fractional linear systems, *Archives of Control Sciences*, vol. 17, 2007, No.3, pp. 357-367.
- Kaczorek T. & Busłowicz M. (2004a). Minimal realization problem for positive multivariable linear systems with delay, *Int. J. Appl. Math. Comp. Sci.*, vol. 14, No. 2, pp. 181-187.
- Kaczorek T. & Busłowicz M. (2004b). Recent developments in theory of positive discrete-time linear systems with delays – stability and robust stability, *Pomiary, Automatyka, Kontrola*, No 9, pp. 9-12
- Kaczorek T. & Busłowicz M. (2004c). Recent developments in theory of positive discrete-time linear systems with delays – reachability, minimum energy control and realization problem. *Pomiary, Automatyka, Kontrola*, No 9, pp. 12-15.
- Kaczorek T. & Przyborowski P. (2007a). Positive continuous-time linear Lyapunov systems, *Proceedings of The International Conference on Computer as a Tool EUROCON 2007*, 9-12 September, Warsaw , pp. 731-737.
- Kaczorek T. & Przyborowski P. (2007b). Positive continuous-time linear time-varying Lyapunov systems, *Proceedings of The XVI International Conference on Systems Science*, 4-6 September 2007, Wrocław – Poland , vol. I, pp. 140-149.
- Kaczorek T. & Przyborowski P. (2007c). Continuous-time linear Lyapunov cone-systems, *Proceedings of The 13th IEEE IFAC International Conference on Methods and Models in Automation and Robotics*, 27 - 30 August 2007, Szczecin – Poland, pp. 225-229.
- Kaczorek T. & Przyborowski P. (2007d). Positive discrete-time linear Lyapunov systems with delays, *Proceedings of The International Workshop "Computational Problems of Electrical Engineering September 14th-16th, 2007*, Wilkasy, Poland, *Electrical Review (Przegląd Elektrotechniczny)*, 2k/2007, pp. 12-15.
- Kaczorek T. & Przyborowski P. (2007e). Positive linear Lyapunov systems, *FNA-ANS International Journal - Problems Of Nonlinear Analysis In Engineering Systems Journal*, No.2(28), vol.13, 2007, pp. 35-60.
- Kaczorek T. & Przyborowski P. (2008) Reachability, controllability to zero and observability of the positive discrete-time Lyapunov systems, *Submitted to The Control and Cybernetics Journal*.
- Klamka J. (1991). *Controllability of dynamical systems*, Kluwer Academic Publ. Dordrecht.
- Miller K. S. & Ross B. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York.

- Moshrefi-Torbati M. & Hammond K. (1998). Physical and geometrical interpretation of fractional operators., *J. Franklin Inst.*, Vol. 335B, no. 6, 1998, pp. 1077-1086.
- Murty M.S.N. & Apparao B.V. (2005). Controllability and observability of Lyapunov systems, *Ranchi University Mathematical Journal*, vol. 32, pp. 55-65.
- Nishimoto K. (1984). *Fractional Calculus*. Koriyama: Decartes Press.
- Oldham K. B. & Spanier J. (1974). *The Fractional Calculus*. New York: Academic Press.
- Ortigueira M. D. (1997). Fractional discrete-time linear systems, *Proceedings of the IEE-ICASSP 97*, Munich, Germany, IEEE, New York, vol. 3, 2241-2244.
- Ostalczyk P. (2000). The non-integer difference of the discrete-time function and its application to the control system synthesis. *Int. J. Syst. Sci.* vol. 31, no. 12, pp. 1551-1561.
- Ostalczyk P. (2004a). Fractional-Order Backward Difference Equivalent Forms Part I – Horner’s Form. *Proceedings of 1-st IFAC Workshop Fractional Differentiation and its Applications, FDA’04*, Enseirb, Bordeaux, France, pp. 342-347.
- Ostalczyk P. (2004b). Fractional-Order Backward Difference Equivalent Forms Part II – Polynomial Form. *Proceedings of 1st IFAC Workshop Fractional Differentiation and its Applications, FDA’04*, Enseirb, Bordeaux, France, pp. 348-353.
- Oustaloup A. (1993). *Commande CRONE*. Paris, Hermés.
- Oustaloup A. (1995). *La dérivation non entière*. Paris: Hermés.
- Podlubny I. (1999). *Fractional Differential Equations*. San Diego: Academic Press.
- Podlubny I. (2002). Geometric and physical interpretation of fractional integration and fractional differentiation. *Fract. Calc. Appl. Anal.* Vol. 5, no. 4, 2002, pp. 367-386.
- Podlubny I., Dorcak L., Kostial I. (1997). On fractional derivatives, fractional-order systems and PID $\mu$ -controllers. *Proceedings of 36th IEEE Conf. Decision and Control*, San Diego, CA, 1997, pp. 4985-4990.
- Przyborowski P. (2008a). Positive Fractional Discrete-time Lyapunov Systems, *Archives of Control Sciences*, vol.18(LIV), 2008, No. 1, pp. 5-18.
- Przyborowski P. (2008b). Fractional discrete-time Lyapunov cone-systems, *Electrical Review (Przegląd Elektrotechniczny)*, 5/2008, pp. 47-52.
- Reyes-Melo M.E., Martinez-Vega J.J., Guerrero-Salazar C.A., Ortiz-Mendez U. (2004). Modelling and relaxation phenomena in organic dielectric materials. Application of differential and integral operators of fractional order. *J. Optoelect. Adv. Mat.* Vol. 6, no. 3, 2004, pp. 1037-1043.
- Riu D., Retière N., Ivanes M. (2001). Turbine generator modeling by non-integer order systems. *Proceedings of IEEE Int. Electric Machines and Drives Conference, IEMDC 2001*, Cambridge, MA, 2001, pp. 185-187.
- Samko S. G., Kilbas A.A., Marichev O.I. (1993). *Fractional Integrals and Derivative. Theory and Applications*. London: Gordon&Breach.
- Sierociuk D. & Dzieliński D. (2006). Fractional Kalman filter algorithm for the states, parameters and order of fractional system estimation. *Int. J. Appl. Math. Comp. Sci.*, 2006, vol. 16, no. 1, pp. 129-140.
- Vinagre B. M., Monje C. A., Calderon A.J. (2002). Fractional order systems and fractional order control actions. *Lecture 3 IEEE CDC’02 TW#2: Fractional calculus Applications in Automatic Control and Robotics*.
- Vinagre B. M. & Feliu V. (2002). Modeling and control of dynamic system using fractional calculus: Application to electrochemical processes and flexible structures. *Proceedings of 41st IEEE Conf. Decision and Control*, Las Vegas, NV, 2002, pp. 214-239.



## **Automation and Robotics**

Edited by Juan Manuel Ramos Arreguin

ISBN 978-3-902613-41-7

Hard cover, 388 pages

**Publisher** I-Tech Education and Publishing

**Published online** 01, May, 2008

**Published in print edition** May, 2008

In this book, a set of relevant, updated and selected papers in the field of automation and robotics are presented. These papers describe projects where topics of artificial intelligence, modeling and simulation process, target tracking algorithms, kinematic constraints of the closed loops, non-linear control, are used in advanced and recent research.

### **How to reference**

In order to correctly reference this scholarly work, feel free to copy and paste the following:

Przemyslaw Przyborowski and Tadeusz Kaczorek (2008). Linear Lyapunov Cone-Systems, Automation and Robotics, Juan Manuel Ramos Arreguin (Ed.), ISBN: 978-3-902613-41-7, InTech, Available from: [http://www.intechopen.com/books/automation\\_and\\_robotics/linear\\_lyapunov\\_cone-systems](http://www.intechopen.com/books/automation_and_robotics/linear_lyapunov_cone-systems)

# **INTECH**

open science | open minds

### **InTech Europe**

University Campus STeP Ri  
Slavka Krautzeka 83/A  
51000 Rijeka, Croatia  
Phone: +385 (51) 770 447  
Fax: +385 (51) 686 166  
[www.intechopen.com](http://www.intechopen.com)

### **InTech China**

Unit 405, Office Block, Hotel Equatorial Shanghai  
No.65, Yan An Road (West), Shanghai, 200040, China  
中国上海市延安西路65号上海国际贵都大饭店办公楼405单元  
Phone: +86-21-62489820  
Fax: +86-21-62489821

© 2008 The Author(s). Licensee IntechOpen. This chapter is distributed under the terms of the [Creative Commons Attribution-NonCommercial-ShareAlike-3.0 License](#), which permits use, distribution and reproduction for non-commercial purposes, provided the original is properly cited and derivative works building on this content are distributed under the same license.