

# The Symmetric Circulant Traveling Salesman Problem

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## 1. Introduction

An  $n \times n$  matrix  $D = d[i, j]$  is said to be circulant, if the entries  $d[i, j]$  verifying  $(j - i) = k \pmod n$ , for some  $k$ , have the same value (for a survey on circulant matrix properties, see Davis (1979)). A directed (respectively, undirected) graph is circulant, if its adjacency matrix is circulant (respectively, symmetric, and circulant). Similarly, a weighted graph is circulant, if its weighted adjacency matrix is circulant.

In the last years, it had been often investigated if a graph problem becomes easier when it is restricted to the circulant graphs. For example, the Maximum Clique problem, and the Minimum Graph Coloring problem remain NP-hard, and not approximable within a constant factor, when the general instance is forced to be a circulant undirected graphs, as shown by Codenotti, et al. (1998). On the other hand, Muzychuk (2004) has proved that the Graph Isomorphism problem restricted to circulant undirected graphs is in P, while the general case is, probably, harder.

It is still an open question whether the Directed Hamiltonian Circuit problem, restricted to circulant (directed) graphs, remains NP-hard, or not. A solution in some special cases has been found by Garfinkel (1977), Fan Yang, et al. (1997), and Bogdanowicz (2005). The Hamiltonian Circuit problem admits, instead, a polynomial time algorithm on the circulant undirected graphs, as shown by Burkard, and Sandholzer (1991). It leads to a polynomial time algorithm for the Bottleneck Traveling Salesman Problem on the symmetric circulant matrices.

Finally, in Gilmore, et al. (1985) it is shown that the Shortest Hamiltonian Path problem is polynomial time solvable on the circulant matrices, while the general case is NP-hard. The positive results contained in Burkard, and Sandholzer (1991), and in Gilmore, et al. (1985) have encouraged the research on the Symmetric Circulant Traveling Salesman problem, that is, the Sum Traveling Salesman Problem restricted to the symmetric, and circulant matrices.

In this chapter we deal with such problem, called for short SCTSP. In §1-§3 the problem is introduced, and the notation is fixed. In §4-§6 an overview is given on the last 16 year results. Firstly, an upper bound (§4.1), a lower bound (§4.2), and a polynomial time 2-approximation algorithm for the general case of SCTSP (§4.3) are discussed. No better result concerning the computational complexity of SCTSP is known. Secondly, some sufficient theorems solving particular cases of SCTSP are presented (§5). Finally, §6 is devoted to a recently introduced subcase of SCTSP. §7 completes the chapter by presenting open problems, remarks, and future developments.

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We list here some abbreviations used throughout the chapter:

- $n$  denotes a positive integer greater than 1;
- $[m]$  denotes the set  $\{1, 2, \dots, m\}$ , for any positive integer  $m$ ;
- $a \equiv_m b$  denotes the relation  $a \equiv b \pmod m$ , and  $\langle a \rangle_m$  denotes the integer  $(a \pmod m)$ , for any positive integer  $m$ , and for any two integers  $a, b$ ;
- $(x_t)_{t=s'}^s$  denotes the tuple  $(x_{s'}, x_{s'+1}, \dots, x_s)$ , for any two integers  $s, s'$  such that  $s \geq s'$ , and for any  $(s - s' + 1)$  integers  $x_{s'}, x_{s'+1}, \dots, x_s$ .

## 2. The symmetric circulant traveling salesman problem

Let  $D = (d[i, j])$  be an  $n \times n$  matrix. Assume that  $d[i, j] = 0$ , if  $i = j$ , and that  $d[i, j]$  is a positive integer, if  $i \neq j$ . Let  $\mathbb{Z}_n$  denote both its row index set, and its column index set. A **Hamiltonian tour**  $T$  for  $D$  is a cyclic permutation  $T : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ . The **(sum) cost** of  $T$  is the integer

$$sum_D(T) = \sum_{i=0}^{n-1} d[i, T(i)]. \tag{1}$$

The optimal sum cost of  $D$  is the integer

$$opt(D) = \min\{sum_D(T) : T \text{ is a Hamiltonian tour for } D\}. \tag{2}$$

The Sum Traveling Salesman Problem asks for finding  $opt(D)$ . It is a well known NP-hard problem. Moreover, no performance guarantee polynomial time approximation algorithm for it is known.

An  $n \times n$  matrix  $D = (d[i, j])$  with entries in  $\mathbb{N} \cup \{\infty\}$  is said to be **circulant**, if  $d[i, j] = d[0, \langle j - i \rangle_n]$ , for any  $i, j \in \mathbb{Z}_n$ . A **symmetric circulant matrix** is a circulant matrix which is also symmetric. As *Example 1* below suggests, a symmetric circulant matrix has a strong algebraic structure: It is fully determined by the entries in the first half of its first row.

**Example 1** *The following two matrices are symmetric circulant.*

$$D_0 = \begin{pmatrix} 0 & 4 & 1 & 6 & 1 & 4 \\ 4 & 0 & 4 & 1 & 6 & 1 \\ 1 & 4 & 0 & 4 & 1 & 6 \\ 6 & 1 & 4 & 0 & 4 & 1 \\ 1 & 6 & 1 & 4 & 0 & 4 \\ 4 & 1 & 6 & 1 & 4 & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & \infty & 2 & 6 & 2 & \infty \\ \infty & 0 & \infty & 2 & 6 & 2 \\ 2 & \infty & 0 & \infty & 2 & 6 \\ 6 & 2 & \infty & 0 & \infty & 2 \\ 2 & 6 & 2 & \infty & 0 & \infty \\ \infty & 2 & 6 & 2 & \infty & 0 \end{pmatrix}$$

Let  $SC(\mathbb{N}^{n \times n})$  denote the set of all  $n \times n$  symmetric circulant matrices with null principal diagonal entries, and positive integer entries otherwise. Note that  $D_0 \in SC(\mathbb{N}^{6 \times 6})$ , while  $D_1 \notin SC(\mathbb{N}^{6 \times 6})$ .

The Symmetric Circulant Traveling Salesman problem (for short, SCTSP) asks for finding  $\text{opt}(D)$ , when  $D$  is a matrix in  $SC(\mathbb{N}^{n \times n})$ .

### 3. Definitions, and preliminaries

Let  $D = (d[i, j])$  be a matrix in  $SC(\mathbb{N}^{n \times n})$ . For any  $a \in [\lfloor n/2 \rfloor]$ , the  $a$ -**stripe** of  $D$  is the set

$$D(a) = \{ \{i, j\} \subset \mathbb{Z}_n : \langle j - i \rangle_n = a, \text{ or } \langle i - j \rangle_n = a \}. \tag{3}$$

The integer  $d[0, a]$  is denoted by  $d(a)$ . It is called the  $a$ -**stripe cost** of  $D$ . Note that two different stripes have empty intersection.

If  $T : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  is a Hamiltonian tour for  $D$ , then  $\text{sum}_D(T)$  depends just on the stripe costs of  $D$ :

For any  $i \in \mathbb{Z}_n$ ,  $\{i, T(i)\}$  belongs to  $D(a_i)$ , and costs  $d(a_i)$ , where  $a_i = \min\{\langle i - T(i) \rangle_n, \langle T(i) - i \rangle_n\}$ .

Indeed,  $a_i \leq \lfloor n/2 \rfloor$  holds by definition, and  $a_i > 0$  holds, as  $T$  is a cyclic permutation. Thus,  $T(i) \neq i$ . Finally, the following statement holds:

$$\{i, j\} \in D(a) \Rightarrow d[i, j] = d(a), \quad \forall i, j \in \mathbb{Z}_n, \forall a \in [\lfloor n/2 \rfloor]. \tag{4}$$

Indeed, if  $\{i, j\} \in D(a)$ , then either  $\langle j - i \rangle_n = a$ , or  $\langle i - j \rangle_n = a$ . In the first case, (4) holds, as  $D$  is circulant, and, thus,  $d[i, j] = d[0, \langle j - i \rangle_n] = d[0, a]$ . In the second case, (4) holds, as  $D$  is symmetric, and circulant, and, thus,  $d[i, j] = d[j, i] = d[0, \langle i - j \rangle_n] = d[0, a]$ .

**Definition 2** Let  $D = (d[i, j])$  be a matrix in  $SC(\mathbb{N}^{n \times n})$ . The  $\lfloor n/2 \rfloor$ -tuple  $\alpha_D = (a_t)_{t=1}^{\lfloor n/2 \rfloor}$  is a **presentation** for  $D$ , if  $d(a_t) \leq d(a_{t+1})$ , for any integer  $1 \leq t < \lfloor n/2 \rfloor$ , and  $\{a_1, \dots, a_{\lfloor n/2 \rfloor}\} = [\lfloor n/2 \rfloor]$ .

A presentation sorts the stripes of a matrix  $D \in SC(\mathbb{N}^{n \times n})$  in non decreasing order with respect to their cost. Clearly, there exists just a presentation for  $D$  if and only if any two stripes have different stripe cost, and, thus, also the converse of (4) holds. In this case, we say that  $D$  has **distinct stripe costs**.

**Example 3** Let  $\alpha_D = (a_t)_{t=1}^{\lfloor n/2 \rfloor}$  be a presentation for  $D \in SC(\mathbb{N}^{n \times n})$ . As observed by Garfinkel in (1977), the permutation  $T_1 : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ , defined as  $T_1(i) = \langle i + a_1 \rangle_n$ , for any  $i \in \mathbb{Z}_n$ , is a Hamiltonian tour for  $D$  if and only if  $\text{gcd}(n, a_1) = 1$ . In this case  $T_1$  is, clearly, optimal.

**Example 4** Let  $\alpha_D = (a_t)_{t=1}^{\lfloor n/2 \rfloor}$  be a presentation for  $D \in SC(\mathbb{N}^{n \times n})$  such that  $\text{gcd}(n, a_1, a_2) > 1$ . A Hamiltonian tour  $T : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  for  $D$  such that  $\{i, T(i)\} \in D(a_1) \cup D(a_2)$ , for any  $i \in \mathbb{Z}_n$ , cannot exist since the set  $\{a_1, a_2\}$  does not generate  $\mathbb{Z}_n$ .

The previous examples suggest the following definition, that will play a crucial role in the next sections.

**Definition 5** Let  $\alpha_D = (a_t)_{t=1}^{\lfloor n/2 \rfloor}$  be a presentation for  $D \in SC(\mathbb{N}^{n \times n})$ . The  $g$ -**sequence** of  $\alpha_D$  is the tuple  $g(\alpha_D) = (g_t(\alpha_D))_{t=0}^{\lfloor n/2 \rfloor}$  defined as follows:

$$\begin{cases} g_0(\alpha_D) = n \\ g_t(\alpha_D) = \gcd(g_{t-1}(\alpha_D), a_t) = \gcd(n, a_1, \dots, a_t), \quad \text{if } t \in \llbracket n/2 \rrbracket. \end{cases} \tag{5}$$

Note that the  $g$ -sequence verifies the following properties:

$$g_t(\alpha_D) \text{ divides } g_{t'}(\alpha_D), \quad \forall 0 \leq t' \leq t \leq \llbracket n/2 \rrbracket \tag{6}$$

$$g_t(\alpha_D) \leq g_{t-1}(\alpha_D), \quad \forall 1 \leq t \leq \llbracket n/2 \rrbracket \tag{7}$$

$$g_{\llbracket n/2 \rrbracket}(\alpha_D) = 1 \tag{8}$$

In particular (8) holds as  $a_{\bar{t}} = 1$ , for some  $\bar{t} \in \llbracket n/2 \rrbracket$ . In the following, we write  $g_t$  instead of  $g_t(\alpha_D)$  if the context is clear.

#### 4. The circulant weighted undirected graph $G_\tau(\alpha_D)$

An usual way of representing a weighted undirected graph  $G$  with node set  $\{0, 1, \dots, m - 1\}$  is its weighted adjacency matrix: An  $m \times m$  symmetric matrix  $D_G$  whose general entry  $d_G[i, j]$  corresponds either to 0, if  $i = j$ , or to the cost of  $\{i, j\}$ , if  $\{i, j\}$  is an edge in  $G$ , or to  $\infty$ , in the other cases. If  $D_G$  is symmetric circulant, then  $G$  is said to be **circulant**.

On the converse, a matrix  $D = (d[i, j])$  in  $SC(\mathbb{N}^{n \times n})$  can be thought as the weighted adjacency matrix of a complete circulant weighted undirected graph. More precisely, any  $A \subset \llbracket \llbracket n/2 \rrbracket \rrbracket$  determines a unique circulant weighted undirected graph having the following weighted adjacency matrix  $D_A = (d_A[i, j])$ :

$$d_A[i, j] = \begin{cases} 0, & \text{if } i = j; \\ d[i, j], & \text{if } \{i, j\} \in D(a), \text{ for some } a \in A; \\ \infty, & \text{otherwise.} \end{cases}$$

$D_A$  is symmetric circulant, since  $D \in SC(\mathbb{N}^{n \times n})$ . Suppose, now, that a presentation  $\alpha_D = (a_t)_{t=1}^{\llbracket n/2 \rrbracket}$  for  $D$  is known. Since we are interested on a Hamiltonian tour for  $D$  with least possible cost, and  $\alpha_D$  sorts the stripes in non decreasing order with respect to their cost, it is advisable to study the weighted undirected graph associated to the set  $\{a_1, a_2, \dots, a_\tau\}$ , for any  $\tau \in \llbracket \llbracket n/2 \rrbracket \rrbracket$ .

**Definition 6** Let  $D$  be a matrix in  $SC(\mathbb{N}^{n \times n})$ , and let  $\alpha_D = (a_t)_{t=1}^{\llbracket n/2 \rrbracket}$  be a presentation for it.

Let us fix  $\tau \in \llbracket \llbracket n/2 \rrbracket \rrbracket$ .  $G_\tau(\alpha_D)$  is the weighted undirected graph having  $\mathbb{Z}_n$  as node set,  $D(a_1) \cup \dots \cup D(a_\tau)$  as edge set, and, finally,  $d(a_t)$  as edge  $\{i, j\}$  cost, if  $\{i, j\} \in D(a_t)$ , for some  $t \in [\tau]$ .

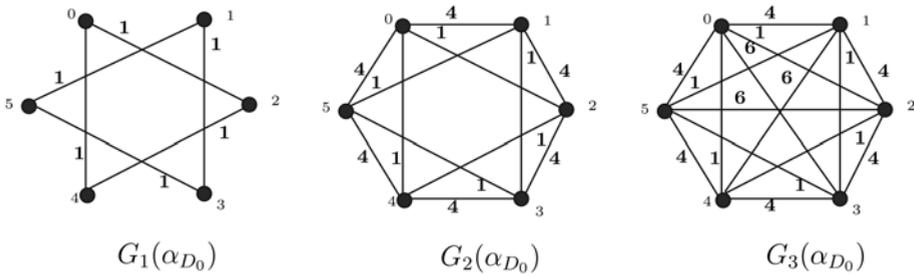


Fig. 1. The circulant weighted undirected graphs  $G_1(\alpha_{D_0})$ ,  $G_2(\alpha_{D_0})$ ,  $G_3(\alpha_{D_0})$

**Example 7** Let us consider the matrix  $D_0 \in SC(\mathbb{N}^{6 \times 6})$  defined in Example 1. The stripes of  $D_0$  have the following costs:  $d_0(1) = 4$ ,  $d_0(2) = 1$ ,  $d_0(3) = 6$ . Hence, there exists a unique presentation  $\alpha_{D_0} = (2, 1, 3)$ . In Figure 1 the circulant weighted undirected graphs  $G_1(\alpha_{D_0})$ ,  $G_2(\alpha_{D_0})$ ,  $G_3(\alpha_{D_0})$  are depicted.

The path in  $G_\tau(\alpha_D)$  of length  $l$  passing through the nodes  $v_0, v_1, \dots, v_l$  is denoted by  $[v_0, v_1, \dots, v_l]$ . Say  $P$  such a path.  $v_0$ , and  $v_l$  are called, respectively, the starting point, and the ending point of  $P$ . The (sum) cost of  $P$  is

$$c_D(P) = \sum_{\lambda=1}^l d[v_{\lambda-1}, v_\lambda]. \tag{9}$$

The path  $[u, u']$  is an **arc of  $P$**  if  $u = v_{\lambda-1}$ , and  $u' = v_\lambda$ , for some  $\lambda \in [l]$ . Let  $P^*$  denote the path  $[v_l, v_{l-1}, \dots, v_0]$ , and, for any  $v \in \mathbb{Z}_n$ , let  $(P + v)$  denote the path  $[v_0 + v, v_1 + v, \dots, v_l + v]$ , where each sum is performed modulo  $n$ . Note that both  $P^*$  and  $(P + v)$  are well defined. Moreover, for any  $\lambda \in [l]$ ,  $d[v_{\lambda-1}, v_\lambda] = d[v_\lambda, v_{\lambda-1}]$  holds as  $G_\tau(\alpha_D)$  is undirected, and  $d[v_{\lambda-1}, v_\lambda] = d[v_{\lambda-1} + v, v_\lambda + v]$  holds as  $G_\tau(\alpha_D)$  is circulant. Hence, both  $c_D(P) = c_D(P^*)$ , and  $c_D(P) = c_D(P + v)$  hold.

Finally, the path  $[v_0, v_l]$  is an **arc in  $D(a_t)$** , if  $\{v_0, v_l\} \in D(a_t)$ , for some  $t \in [\tau]$ . A well known theorem due to Boesch, and Tindell (1984), and concerning the connectivity of a circulant weighted undirected graph can be restated for  $G_\tau(\alpha_D)$  as follows.

**Theorem 8** Let  $\alpha_D = (a_t)_{t=1}^{\lfloor n/2 \rfloor}$  be a presentation for  $D \in SC(\mathbb{N}^{n \times n})$ . Let us fix  $\tau \in [\lfloor n/2 \rfloor]$ .  $G_\tau(\alpha_D)$  has  $g_\tau$  pairwise isomorphic connected components. In particular, the set  $\{v \in \mathbb{Z}^n : v \equiv_{g_\tau} i\}$  induces a different connected component, for any  $i = 0, 1, \dots, g_\tau - 1$ . Finally, any connected component forms itself a circulant weighted undirected graph.

**PROOF.** (Sketch) Let us fix a node  $v_0 \in \mathbb{Z}_n$ . A node  $v \in \mathbb{Z}_n$  belongs to the same connected component of  $v_0$  if and only if there exists a path in  $G_\tau(\alpha_D)$  starting at  $v_0$ , and ending at  $v$ . Let  $P$  be a path starting at  $v_0$ . As the edge set of  $G_\tau(\alpha_D)$  is  $D(a_1/g_\tau) \cup \dots \cup D(a_\tau/g_\tau)$ , any arc  $[u, u']$  of  $P$  is an arc in  $D(a_t)$ , and, thus, verifies  $u \equiv_n u' \pm a_t$ , for some  $t \in [\tau]$  (see (3)). It follows that  $v$  is the ending point of a path starting at  $v_0$  if and only if there exists integers  $y_1, \dots, y_\tau$  such that

$$v - v_0 \equiv_n \sum_{t=1}^{\tau} y_t \cdot a_t \equiv_n g_{\tau} \left( \sum_{t=1}^{\tau} y_t \cdot a_t / g_{\tau} \right). \tag{*}$$

As  $g_{\tau}$  divides  $n$  by Definition 5, (\*) implies that  $v \equiv_{g_{\tau}} v_0$  holds.

On the other hand, if  $v \equiv_{g_{\tau}} v_0$ , then  $(v - v_0) \equiv_n g_{\tau} b$ , for some  $b \in \mathbb{Z}$ . It follows by definition of  $g_{\tau}$  that  $\gcd(n/g_{\tau}, a_1/g_{\tau}, \dots, a_{\tau}/g_{\tau}) = 1$ . Thus, by Euclid's lemma, there exists integers  $y_1, \dots, y_{\tau}$  such that  $b \equiv_{n/g_{\tau}} \sum_{t=1}^{\tau} y_t \cdot a_t / g_{\tau}$ . By substituting it in  $(v - v_0) \equiv_n g_{\tau} \cdot b$ , (\*)

follows. Hence, two nodes are in the same connected component if and only if they are equivalent modulo  $g_{\tau}$ . Finally, any connected component is isomorphic to the circulant weighted undirected graph having  $\mathbb{Z}_{n/g_{\tau}}$  as node set,  $D(a_1/g_{\tau}) \cup \dots \cup D(a_{\tau}/g_{\tau})$  as edge set, and  $d[g_{\tau} \cdot i, g_{\tau} \cdot j]$  as edge  $\{i, j\}$  cost.  $\square$

A Hamiltonian path for a graph is a path passing exactly once through an node in the graph. A shortest Hamiltonian path starting at a node  $v$  is a least possible cost one among those having  $v$  as starting point. The next theorem is a direct consequence of a result of Bach, et al. (see Chapter 4 in Gilmore, et al. (1985)).

**Theorem 9** Let  $\alpha_D = (a_t)_{t=1}^{\lfloor n/2 \rfloor}$  be a presentation for  $D \in SC(\mathbb{N}^{n \times n})$ . An algorithm setting  $v_0 = 0$ , and

$$v_{\lambda} \in \arg \min_{u \notin \{v_0, \dots, v_{\lambda-1}\}} \{t : \{v_{\lambda-1}, u\} \in D(a_t), t \in [\lfloor n/2 \rfloor]\}, \quad \forall 1 \leq \lambda < n,$$

finds a shortest Hamiltonian path for  $G_{\lfloor n/2 \rfloor}(\alpha_D)$  starting at the node 0. Such path costs

$$SHP(\alpha_D) = \sum_{t=1}^{\lfloor n/2 \rfloor} (g_{t-1} - g_t) \cdot d(a_t).$$

The algorithm described in Theorem 9 is a non deterministic one. For example, both choices  $v_1 = a_1$ , and  $v_1 = n - a_1$  are possible, as both arcs  $\{0, a_1\}$ , and  $\{0, n - a_1\}$  are in  $D(a_1)$ . Moreover, it is a nearest neighbor ruled one: For any  $1 \leq \lambda < n$ , and for any  $u \notin \{v_0, \dots, v_{\lambda-1}\}$ ,  $d[v_{\lambda-1}, v_{\lambda}] \leq d[v_{\lambda-1}, u]$  holds, as  $\alpha_D$  is a presentation. Example 10 below shows that the contribution given by  $\alpha_D$  is fundamental, as it forces to insert in the solution arcs belonging to the same stripe as far as possible.

**Example 10** Let  $D = (d[i, j])$  be a matrix in  $SC(\mathbb{N}^{6 \times 6})$  having as strip costs  $d(1) = d(2) = 1$ , and  $d(3) = 2$ . Clearly,  $[0, 1, 2, 3, 4, 5]$  is a shortest Hamiltonian path of cost 5. An algorithm setting  $v_0 = 0$ , and following the nearest neighbor rule

$$v_{\lambda} \in \arg \min_{u \notin \{v_0, \dots, v_{\lambda-1}\}} \{d[v_{\lambda-1}, u]\}, \quad \forall 1 \leq \lambda < n,$$

may return the Hamiltonian path  $[0, 2, 3, 5, 4, 1]$  of cost 6, since it indifferently inserts in the solution arcs in  $D(1)$  (i.e.,  $[0, 2]$ , and  $[3, 5]$ ), and arcs in  $D(2)$  (i.e.,  $[2, 3]$ , and  $[4, 5]$ ), since  $d(1) = d(2) = 1$  holds.

Let us compute  $SHP(\alpha_D)$  by the formula given in Theorem 9. It follows from Definition 5 that  $g_0 = n$ , and that  $g_1 = \gcd(n, a_1) < n$ , as  $a_1$  is a stripe, and, then,  $a_1 \leq \lfloor n/2 \rfloor$ . Hence, the first summand is always greater than 0. And what about the other summands? As (6) holds,

there exist at most  $r$  indexes  $t$ , for some  $r \leq \log_2 n$ , such that  $g_t < g_{t-1}$  holds. Hence, at most  $r$  summands in  $SHP(\alpha_D)$  are greater than 0. Finally, as (7), and (8) hold, there exists an index  $\bar{t}$  such that  $g_t = 1$  holds if and only if  $t \geq \bar{t}$ . Therefore, the  $t$ -th summand for any  $t > \bar{t}$  is equal to 0. Hence, just a few number of stripes could be involved in the construction of a shortest Hamiltonian path for  $G_{\lfloor n/2 \rfloor}(\alpha_D)$  starting at 0. It suggests the following definition.

**Definition 11** Let  $\alpha_D = (a_t)_{t=1}^{\lfloor n/2 \rfloor}$  be a presentation for  $D \in SC(\mathbb{N}^{n \times n})$ .

The  $r$ -tuple  $(\zeta_j)_{j=1}^r$  is the **stripe sequence** (for short, **s.s.**) of  $\alpha_D$ , if  $\zeta_{j+1} < \zeta_j$ , for any  $1 \leq j < r$ , and  $\{\zeta_1, \dots, \zeta_r\} = \{t \in [\lfloor n/2 \rfloor] : g_t(\alpha_D) < g_{t-1}(\alpha_D)\}$ .  $\zeta_j$  is called the  $j$ -**th s.s. index** of  $\alpha_D$ , for any  $j \in [r]$ .

Note that the higher is  $j$ , the lower is  $\zeta_j$ , and the higher is  $g_{\zeta_j}(\alpha_D)$ . In particular,

$$\begin{cases} \zeta_1 = \min\{t \in [\lfloor n/2 \rfloor] : g_t(\alpha_D) = g_{\lfloor n/2 \rfloor}(\alpha_D) = 1\} \\ \zeta_r = 1 \end{cases} \tag{10}$$

For any  $1 \leq j < r$ , the integer  $g_{\zeta_{j+1}}(\alpha_D)/g_{\zeta_j}(\alpha_D)$  is denoted by  $h_j(\alpha_D)$ . In the following, we write  $h_j$  instead of  $h_j(\alpha_D)$  if the context is clear.

### 5. Bounds for the general case of SCTSP

In this section the most remarkable results regarding the general case of SCTSP are reported. Unfortunately, such results do not allow to prove neither that SCTSP is in  $P$ , nor that it is an NP-hard problem.

#### 5.1 An upper bound for SCTSP

The first author explicitly dealing with SCTSP is Van der Veen (1992). Its heuristic **HT1** is a polynomial time algorithm for SCTSP in the case in which the matrix in input has distinct stripe costs. Van der Veen computes the cost of the Hamiltonian tour returned by **HT1** just in some cases. Gerace, and Greco (2008b) propose the procedure **H**, a restyling of Van der Veen’s procedure. The main difference is the input instance: While **HT1** accepts just matrices in  $SC(\mathbb{N}^{n \times n})$  with distinct stripe costs, **H** works on any matrix in  $SC(\mathbb{N}^{n \times n})$ , once a presentation for it is given. In the following, we explain how **H** works.

Let  $D$  be a matrix in  $SC(\mathbb{N}^{n \times n})$ , and let  $\alpha_D = (a_t)_{t=1}^{\lfloor n/2 \rfloor}$  be a presentation for it. For any  $\tau \in [\lfloor n/2 \rfloor]$ , let  $\Delta_\tau(\alpha_D)$  be the connected component of  $G_\tau(\alpha_D)$  containing the node 0. It follows by *Theorem 8* that its node set, say it  $V_\tau(\alpha_D)$ , is  $\{v \in \mathbb{Z}_n : v \equiv_{g_\tau} 0\}$

First of all, we describe a procedure **HP** returning on input  $(\alpha_D, \tau)$  a Hamiltonian path for  $\Delta_\tau(\alpha_D)$  starting at the node 0. **HP** corresponds to *Steps 2-3* of **HT1**.

Suppose that  $\tau = 1$ . For any  $0 \leq \lambda < n/g_1$ , let  $v_\lambda = \langle \lambda \cdot a_1 \rangle_n$ . Note that  $v_\lambda \equiv_{g_1} 0$ . Let  $\mathbf{HP}(\alpha_D, 1) = [v_0, v_1, \dots, v_{n/g_1-1}]$ . Since  $g_1 = \gcd(n, a_1)$  by *Definition 5*, it follows that **HP**( $\alpha_D, 1$ ) passes through any node in  $V_1(\alpha_D)$ . Thus, it is a Hamiltonian path for  $\Delta_1(\alpha_D)$ .

Suppose, now, that  $\tau > 1$ . Let  $P_0 = \mathbf{HP}(\alpha_D, \tau - 1)$ . We distinguish two cases. If  $g_{\tau-1} = g_\tau$ , then  $P_0$  is a Hamiltonian path also for  $\Delta_\tau(\alpha_D)$  by *Theorem 8*. In this case  $\mathbf{HP}(\alpha_D, \tau)$  returns  $P_0$ . Otherwise,  $g_{\tau-1} > g_\tau$  holds. As  $g_\tau = \gcd(g_{\tau-1}, a_\tau)$ , and  $v \in V_{\tau-1}(\alpha_D)$  if and only if  $v \equiv g_{\tau-1} \cdot 0$ , it follows that

$$V_\tau(\alpha_D) = \{ \langle v + \mu a_\tau \rangle_n : v \in V_{\tau-1}(\alpha_D), \mu = 0, 1, \dots, g_{\tau-1}/g_\tau - 1 \}. \quad (**)$$

Let  $z$  denote the ending point of  $P_0$ , and  $h$  the integer  $g_{\tau-1} / g_\tau$ . For any  $\mu \in [h-1]$ , let  $u_\mu$  denote the integer  $\langle \mu(z+a_\tau) \rangle_n$ , and  $P_\mu$  the path  $(P_0+u_\mu)$ . Finally, let  $P$  be the path obtained by linking  $P_0, P_1, \dots, P_{h-1}$  by the arcs  $[\langle u_\mu - a_\tau \rangle_n, u_\mu]$ , for any  $\mu \in [h-1]$ .  $\mathbf{HP}(\alpha_D, \tau)$  returns  $P$ . Note that  $P$  passes through any node in  $V_\tau(\alpha_D)$ , as  $P_0$  passes through any node in  $V_{\tau-1}(\alpha_D)$ , and **(\*\*)** holds. Hence, it is a Hamiltonian path for  $\Delta_\tau(\alpha_D)$ .

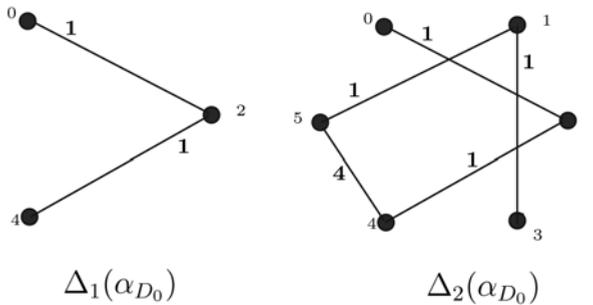


Fig. 2. Shortest Hamiltonian paths for  $\Delta_1(\alpha_{D_0})$ , and for  $\Delta_2(\alpha_{D_0})$  starting at 0

**Example 12** Let us consider the matrix  $D_0 \in SC(\mathbb{N}^{6 \times 6})$  defined in Example 1. Its unique presentation is  $\alpha_{D_0} = (2, 1, 3)$ , and  $G_1(\alpha_{D_0})$ , and  $G_2(\alpha_{D_0})$  are depicted in Figure 1. The path shown in Figure 2 are returned, respectively, by executing  $\mathbf{HP}(\alpha_{D_0}, 1)$ , and  $\mathbf{HP}(\alpha_{D_0}, 2)$ .

**Remark.** Let  $\tau \in [\lfloor n/2 \rfloor]$ . The path  $\mathbf{HP}(\alpha_{D_0}, \tau) = [v_0, v_1, \dots, v_{n_{g_\tau-1}}]$  verifies  $v_0 = 0$ , and  $v_\lambda \in \arg \min_{u \notin \{v_0, \dots, v_{\lambda-1}\}} \{t : \{v_{\lambda-1}, u\} \in D(a_t), t \in [\tau]\}$ , for any  $1 \leq \lambda < n/g_\tau$ . Thus,  $\mathbf{HP}$  is a deterministic nearest neighbor ruled algorithm. By applying Kruskal's algorithm to  $\Delta_\tau(\alpha_D)$ , a minimum spanning tree  $T$ , whose weight is equal to the cost of  $\mathbf{HP}(\alpha_D, \tau)$ , is obtained. Thus,  $\mathbf{HP}(\alpha_D, \tau)$  is a shortest Hamiltonian path for  $\Delta_\tau(\alpha_D)$  starting at the node 0 (see also Corollary 6 in Gilmore, et al. (1985)).

Let us define, now, the procedure **H**.

**Procedure H.**

**Instance.** A matrix  $D \in SC(\mathbb{N}^{n \times n})$ , and a presentation  $\alpha_D$  for  $D$ .

**Step a.** Execute  $\mathbf{Pr}(\alpha_D, 1)$ .

**Step b.** Let  $H = [v_0, v_1, \dots, v_{n-1}, v_0]$  be the Hamiltonian cycle obtained in Step a. Return the Hamiltonian tour  $T_H: \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  for  $D$ , defined as follows:  $T_H(v_\lambda) = v_{\langle \lambda+1 \rangle_n}$ , for any  $\lambda \in \mathbb{Z}_n$ .



$\mathbf{HP}(\alpha_D, \tau)$  contains an arc  $[u, u']$  such that  $(u' - u) \equiv_n a_\tau$  if and only if  $g_{\tau-1} > g_\tau$  holds, that is, if and only if  $\tau$  is a s.s. index of  $\alpha_D$ . Hence, *Step 3* of  $\mathbf{Pr}$  is well defined. Gerace, and Greco (2008b) prove that  $\mathbf{H}$  is a correct polynomial time procedure, and that the cost of  $\mathbf{H}(D, \alpha_D)$  is time  $O(n)$  computable (without running  $\mathbf{H}$ ) by the next theorem.

**Theorem 13** *Let  $\alpha_D$  be a presentation for a matrix  $D \in SC(\mathbb{N}^{n \times n})$ , let  $(a_{\zeta_j})_{j=1}^r$  be its s.s., and let  $\rho = \max\{j \in [r] : g_{\zeta_j} \text{ is odd}\}$ . If  $\hat{\rho}$  denotes the integer  $\min\{r - 1, \rho\}$ , then the Hamiltonian tour  $\mathbf{H}(D, \alpha_D)$  costs*

$$SHP(\alpha_D) + d(a_{\zeta_1}) + \sum_{j=1}^{\hat{\rho}} (h_j - 2) \cdot (d(a_{\zeta_j}) - d(a_{\zeta_{j+1}})).$$

As a consequence of *Theorem 13*, the integer

$$UB(D) = \min\{\text{sum}_D(\mathbf{H}(D, \alpha_D)) : \alpha_D \text{ is a presentation for } D\}. \tag{11}$$

is an upper bound for  $\text{opt}(D)$ . If there exists just a presentation  $\alpha_D$  for  $D$ , and  $\mathbf{Pr}(\alpha_D, 1)$  ends immediately with no more recursive calling,  $UB(D)$  is equal to the upper bound given in Van der Veen (1992), *Theorem 7.2.5*.

In the general case  $D$  admits more than a presentation. As *Example 14*, and *Example 15* below show, the cost of the Hamiltonian tour returned by  $\mathbf{H}$  depends on the presentation. Since the number of the presentations for  $D$  could be exponential in  $n$ ,  $UB(D)$  is not efficiently computable by determining  $\text{sum}_D(\mathbf{H}(D, \alpha_D))$ , for any presentation  $\alpha_D$ .

**Example 14** *Let  $n = 108$ , and let  $D$  be the matrix in  $SC(\mathbb{N}^{n \times n})$  having as stripe costs  $d(36) = 1, d(8) = d(16) = d(27) = 2$ , and  $d(k) = 3 + k$ , for any other  $k \in [54]$ . We consider just two of the six possible presentations for  $D$ : the one verifying  $a_1 = 36, a_2 = 27, a_3 = 16, a_4 = 8$  is denoted by  $\alpha_D = (a_i)_{i=1}^{54}$ ; the one verifying  $b_1 = 36, b_2 = 8, b_3 = 16, b_4 = 27$  is denoted by  $\beta_D = (b_i)_{i=1}^{54}$ . Let us denote by  $(a_{\zeta_j})_{j=1}^r$ , (respectively, by  $(b_{\zeta_k})_{k=1}^s$  the s.s. of  $\alpha_D$  (respectively, of  $\beta_D$ ). Let us compute  $\text{sum}_D(\mathbf{H}(D, \alpha_D))$ , and  $\text{sum}_D(\mathbf{H}(D, \beta_D))$  by following the arrows in the two schemes reported in *Figure 4* (the differences between them are pointed out in bold). Such schemes are obtained by making use of (5), of (10), of *Theorem 9*, and of *Theorem 13*. Note that  $\text{sum}_D(\mathbf{H}(D, \alpha_D)) > \text{sum}_D(\mathbf{H}(D, \beta_D))$ .*

**Example 15** *Let  $n = 135$ , and let  $D$  be the matrix in  $SC(\mathbb{N}^{n \times n})$  verifying  $d(45) = 1, d(5) = d(9) = 2$ , and  $d(k) = 3 + k$ , for any other  $k \in [52]$ . There exist exactly two presentations for  $D$ . Let  $\alpha_D = (a_i)_{i=1}^{67}$  be the one verifying  $a_1 = 45, a_2 = 5, a_3 = 9$ , and let  $\beta_D = (b_i)_{i=1}^{67}$  be the one verifying  $b_1 = 45, b_2 = 9, b_3 = 5$ . As above, let  $(a_{\zeta_j})_{j=1}^r$ , (respectively,  $(b_{\zeta_k})_{k=1}^s$  denotes the s.s. of  $\alpha_D$  (respectively, of  $\beta_D$ ), and let us compute  $\text{sum}_D(\mathbf{H}(D, \alpha_D))$ , and  $\text{sum}_D(\mathbf{H}(D, \beta_D))$  by following the arrows in the two schemes reported in *Figure 5* (the differences are pointed out in bold). Note that  $\text{sum}_D(\mathbf{H}(D, \alpha_D)) > \text{sum}_D(\mathbf{H}(D, \beta_D))$  also in this case.*

In both examples  $\mathbf{H}(D, \beta_D)$  costs less than  $\mathbf{H}(D, \alpha_D)$ . In the former, the presentation  $\beta_D$  sorts the stripes having the same cost in a way that  $g_i(\beta_D)$  remains even as long as possible. In fact,  $g_2(\alpha_D)$  is odd, while  $g_2(\beta_D)$  is even. In the latter,  $n$  is an odd number. Thus,  $g_i(\beta_D)$ , and  $g_i(\alpha_D)$

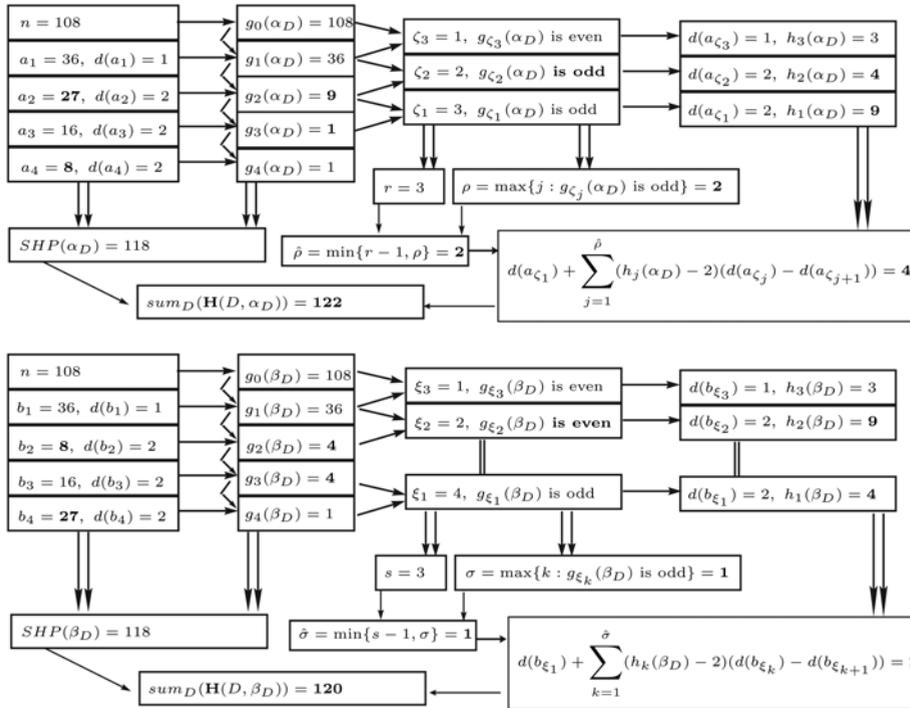


Fig. 4. How to compute  $sum_D(\mathbf{H}(D, \alpha_D))$ , and  $sum_D(\mathbf{H}(D, \beta_D))$  in Example 14.

are necessarily odd, for any  $t \in [\lfloor n/2 \rfloor]$ . Anyway,  $\beta_D$  sorts the stripes having the same cost in a way that  $g_2(\beta_D)$  is as great as possible. Such considerations suggest the following definition.

**Definition 16** Let  $D$  be a matrix in  $SC(\mathbb{N}^{n \times n})$ , and let  $\beta_D = (b_t)_{t=1}^{\lfloor n/2 \rfloor}$  be a presentation for  $D$ .  $\beta_D$  is sharp if  $g_t(\beta_D)$  odd implies that  $g_t(\alpha_D)$  is an odd integer less than, or equal to  $g_t(\beta_D)$ , for any  $t \in [\lfloor n/2 \rfloor]$ , and for any other presentation  $\alpha_D$  for  $D$ .

A sharp presentation for a matrix in  $SC(\mathbb{N}^{n \times n})$  is time  $O(n \log n)$  computable by the procedure SP reported below.

**Procedure SP.**

**Instance.** A matrix  $D$  in  $SC(\mathbb{N}^{n \times n})$ .

**Step 1.** Set  $S = [\lfloor n/2 \rfloor]$ ,  $g = n$ , and  $t = 1$ . Sort in non decreasing order the stripe costs of  $D$ . Let  $(d_t)_{t=1}^{\lfloor n/2 \rfloor}$  the tuple so obtained.

**Step 2.** While there exists  $a \in S$  such that  $d(a) = d_t$ , and  $\gcd(g, a)$  is even set  $b_t = a$ ,  $S = S \setminus a$ ,  $g = \gcd(g, a)$ , and  $t = t + 1$ .

**Step 3.** While  $S \neq \emptyset$ , extract from  $S \cap \{a' : d(a') = d_t\}$  the element  $a$  maximizing  $\gcd(g, a')$ . Set  $b_t = a$ ,  $S = S \setminus a$ ,  $g = \gcd(g, a)$ , and  $t = t + 1$ .

**Step 4.** Return the presentation  $(b_t)_{t=1}^{\lfloor n/2 \rfloor}$ . ■

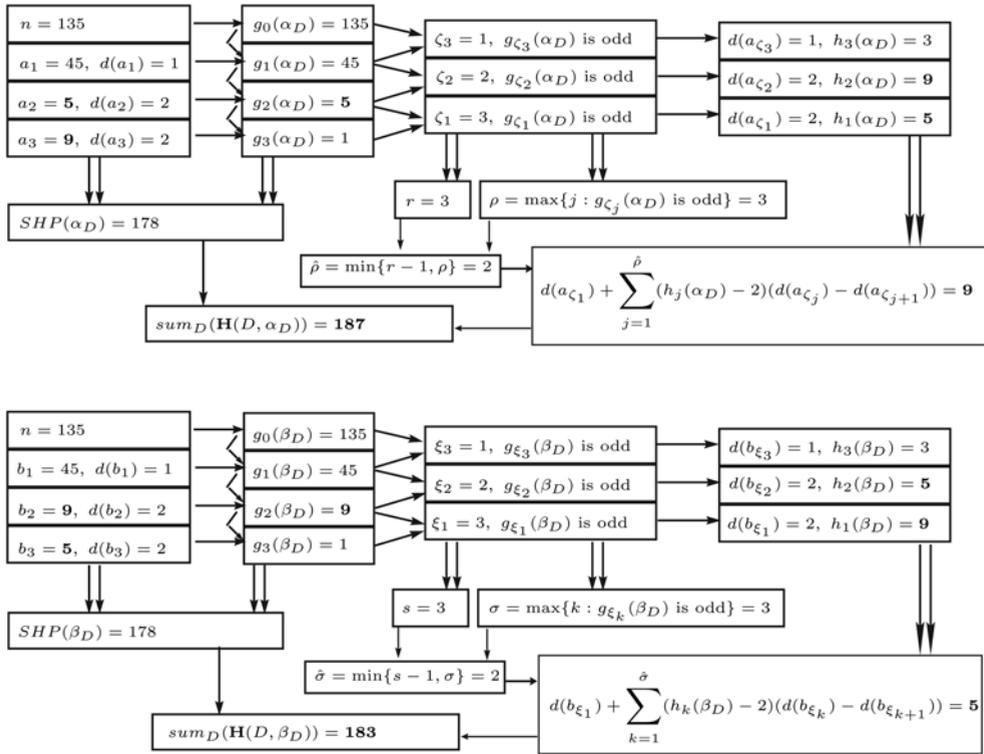


Fig. 5. How to compute  $sum_D(\mathbf{H}(D, \alpha_D))$ , and  $sum_D(\mathbf{H}(D, \beta_D))$  in Example 15.

Let  $\beta_D = \mathbf{SP}(D)$ . Gerace, and Greco (2008b) prove that  $UB(D) = sum_D(\mathbf{H}(D, \beta_D))$  holds, as  $\beta_D$  is sharp. Since  $sum_D(\mathbf{H}(D, \beta_D))$  is time  $O(n)$  computable (see Theorem 13), it follows that  $UB(D)$  is a time  $O(n \log n)$  computable upper bound for  $opt(D)$ .

### 5.2 A lower bound for SCTSP

Let  $D$  be a matrix in  $SC(\mathbb{N}^{n \times n})$ . If  $D$  has distinct stripe costs, Theorem 7.4.2 in Van der Veen (1992) gives a lower bound for  $opt(D)$ . By the same argument, Theorem 17 below shows that any presentation for  $D$  leads to a lower bound.

**Theorem 17** Let  $\alpha_D$  be a presentation for a matrix  $D \in SC(\mathbb{N}^{n \times n})$ , and let  $(a_{\zeta_j})_{j=1}^r$  be its s.s.. Then,  $SHP(\alpha_D) + d(a_{\zeta_1}) \leq opt(D)$  holds.

**PROOF.** Let us fix an optimal Hamiltonian tour  $T : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  for  $D$ . Setting  $v_0 = T(0)$ , and  $v_\lambda = T(v_{\lambda-1})$ , for any integer  $1 \leq \lambda < n$ , naturally induces a Hamiltonian cycle  $H_T = [v_0, v_1, \dots, v_{n-1}, v_0]$  for  $G_{\lfloor n/2 \rfloor}(\alpha_D)$ . It follows from (1), and from (9) that  $c_D(H_T) = sum_D(T)$ . If no arc  $[u, v]$  of  $H_T$  would verify  $\{u, v\} \in \bigcup_{i=\zeta_1}^{\lfloor n/2 \rfloor} D(a_i)$ , then  $H_T$  would be a Hamiltonian cycle also for  $G_{\zeta_1-1}(\alpha_D)$ , a weighted undirected graph having  $g_{\zeta_1-1} > 1$  connected components, as a consequence of

Theorem 8, and of Definition 11. Hence, there exists an arc  $[u, v]$  in  $H_T$  such that  $c_D([u, v]) = d[u, v] \geq d(a_{\zeta_1})$ . By deleting  $[u, v]$  from  $H_T$  a Hamiltonian path  $P$  for  $G_{\lfloor n/2 \rfloor}(\alpha_D)$  is obtained. Clearly,  $c_D(P) \geq SHP(\alpha_D)$  holds. Thus,

$$sum_D(T) = c_D(H_T) = c_D(P) + c_D([u, v]) \geq SHP(\alpha_D) + d(a_{\zeta_1}).$$

As  $sum_D(T) = opt(D)$ , the claim follows.  $\square$

Let  $\beta_D = (b_i)_{i=1}^{\lfloor n/2 \rfloor}$  be a presentations for  $D$ , possibly different from  $\alpha_D$ . Since  $\{a_1, \dots, a_{\lfloor n/2 \rfloor}\} = [\lfloor n/2 \rfloor] = \{b_1, \dots, b_{\lfloor n/2 \rfloor}\}$ , the weighted undirected graphs  $G_{\lfloor n/2 \rfloor}(\alpha_D)$ , and  $G_{\lfloor n/2 \rfloor}(\beta_D)$  coincide by Definition 6. It follows from Theorem 9 that  $SHP(\alpha_D) = SHP(\beta_D)$  holds. As shown by Gerace, and Greco (2008b),  $d(a_{\zeta_1}) = d(b_{\xi_1})$  also holds, where  $b_{\xi_1}$  denote the 1-st s.s. index of  $\beta_D$ .

It follows from Theorem 17 that the integer

$$LB(D) = SHP(\alpha_D) + d(a_{\zeta_1}) \tag{12}$$

is a well defined lower bound for  $opt(D)$  holds not depending on the chosen presentation

### 5.3 A 2-approximation algorithm for SCTSP

A first 2-approximation algorithm for the general case of SCTSP is reported Gerace, and Irwing (1998). For any matrix  $D \in SC(\mathbb{N}^{n \times n})$ , such algorithm makes use of the construction proposed by Burkard, and Sandholzer (1991) for solving the Hamiltonian circuit problem in a circulant undirected graph. The returned Hamiltonian tour has a costs greater than, or equal to  $UB(D)$ .

By the procedure **SP**, a sharp presentation  $\beta_D$  for  $D$  can be found in polynomial time. If we apply **H** on input  $(D, \beta_D)$ , a Hamiltonian tour for  $D$  of cost  $UB(D)$  is obtained in polynomial time. Let **H\*** denote the algorithm that, given  $D$ , returns  $\mathbf{H}(D, \beta_D)$ . Clearly, **H\*** is a 2-approximation algorithm for SCTSP. Gerace, and Greco (2008b) proves that the analysis of **H\*** is tight.

## 6. When the optimal cost is equal to the lower bound

Let  $D$  be a matrix in  $SC(\mathbb{N}^{n \times n})$ . Let  $\alpha_D$  be a presentation for it, and let  $(a_{\zeta_j})_{j=1}^r$  be its s.s..

Theorem 18 below extends some results appearing in Van der Veen (1992), and in Gerace, and Irwing (1998). It is inspired by the following remark: According to (12), there exists a Hamiltonian tour for  $D$  of cost  $LB(D)$  if and only if there exists a shortest Hamiltonian path for  $G_{\lfloor n/2 \rfloor}(\alpha_D)$  starting at the node 0, and ending at a node  $v$  such that the arc  $[v, 0]$  costs  $d(a_{\zeta_1})$ . Note that  $[v, 0]$  is not necessary an arc in  $D(a_{\zeta_1})$ , if more than a stripe costs  $d(a_{\zeta_1})$ .

**Theorem 18** Let  $D = (d[i, j])$  be a matrix in  $SC(\mathbb{N}^{n \times n})$ . Suppose that there exists a presentation  $\alpha_D$  for  $D$  having  $(a_{\zeta_j})_{j=1}^r$  as s.s., and that there exists  $v \in \mathbb{Z}_n$  verifying  $d[v, 0] = d(a_{\zeta_1})$ , and

$$v \equiv_n \sum_{j=r}^1 (g_{\zeta_j-1}/g_{\zeta_j} - 1)(2\gamma_j - g_{\zeta_j}) \cdot a_{\zeta_j},$$

for some integers  $\gamma_r, \dots, \gamma_2, \gamma_1$  such that  $0 \leq j \leq g_{\zeta_1}$  holds, for any  $j \in [r]$ . Then,  $\text{opt}(D) = LB(D)$  holds.

If  $D$  has distinct stripe costs, then the converse also holds.

**PROOF.** (Sketch) Let  $\alpha_D$  be a presentation satisfying the hypotheses for some suitable integers  $v, \gamma_r, \dots, \gamma_2, \gamma_1$ . Since  $\alpha_D$  is fixed, Theorem 7.3.1 in Van der Veen (1992) implies that there exists a shortest Hamiltonian path  $P$  for  $G_{\square n/2 \square}(\alpha_D)$  starting at 0, and ending at  $v$ . Let  $H$  be the Hamiltonian cycle for  $G_{\square n/2 \square}(\alpha_D)$  obtained by composing  $P$  with the arc  $[v, 0]$ . Since  $d[v, 0] = d(a_{\zeta_1})$ , and Theorem 9 holds,  $H$  costs  $SHP(\alpha_D) + d(a_{\zeta_1}) = LB(D)$ .  $H$  naturally induces a Hamiltonian tour  $T_H$  verifying  $c_D(H) = \text{sum}_D(T_H)$ . It follows from Theorem 17 that  $\text{opt}(D) = LB(D)$ .

Suppose that  $D$  has distinct stripe costs, and that  $\text{opt}(D) = LB(D)$ . Let  $\alpha_D$  be the unique presentation for  $D$ , and let  $(a_{\zeta_j})_{j=1}^r$  be its s.s.. Let  $T : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  be a Hamiltonian tour for  $D$  of cost  $LB(D)$ , and let  $i \in \mathbb{Z}_n$  be an integer maximizing  $d[i, T(i)]$ . Clearly,  $d[i, T(i)] \geq d(a_{\zeta_1})$  holds (see also the proof of Theorem 17). Let  $P$  be the Hamiltonian path obtained by deleting the arc  $[i, T(i)]$  from the Hamiltonian cycle for  $G_{\square n/2 \square}(\alpha_D)$  induced by  $T$ . Since  $P$  starts at the node  $T(i)$ , and ends at the node  $i$ ,  $(P - T(i))$  is a Hamiltonian path starting at 0, and ending at  $v = \langle i - T(i) \rangle_n$ . It follows from Theorem 9 that  $(P - T(i))$  is a shortest one, since  $c_D(P) = c_D(P - T(i))$ , and

$$c_D(P) = LB(D) - d[i, T(i)] = SHP(\alpha_D) + d(a_{\zeta_1}) - d[i, T(i)] \leq SHP(\alpha_D).$$

Moreover,  $d[i, T(i)] = LB(D) - SHP(\alpha_D) = d(a_{\zeta_1})$  is verified. As  $D$  is circulant,  $d[v, 0] = d[i, T(i)] = d(a_{\zeta_1})$  also holds. As  $D$  has distinct stripe costs, Theorem 7.3.1 in Van der Veen (1992) implies that  $v \equiv_n \sum_{j=r}^1 (g_{\zeta_j-1}/g_{\zeta_j} - 1)(2\gamma_j - g_{\zeta_j}) \cdot a_{\zeta_j}$ , for some integers  $\gamma_r, \dots, \gamma_2, \gamma_1$  such that  $0 \leq \gamma_j \leq g_j$  holds, for any  $j \in [r]$ . The second claim of the theorem is thus proved.  $\square$

As already observed, the number of presentation for a matrix  $D \in SC(\mathbb{N}^{n \times n})$  could be exponential in  $n$ . Hence, an algorithm based on the sufficient condition given in Theorem 18 cannot efficiently determine if  $\text{opt}(D) = LB(D)$  holds. Proposition 19 below gives some conditions implying  $\text{opt}(D) = LB(D)$ , once a presentation for  $D$  is fixed. In Garfinkel (1977) (respectively, in Van der Veen (1992)) appears a condition similar to condition (b) (respectively, to condition (c)). Finally, condition (d) is a consequence of Theorem 18.

**Proposition 19** Let  $D = (d[i, j])$  be a matrix in  $SC(\mathbb{N}^{n \times n})$ . Let  $\alpha_D$  be a presentation for it, and let  $(a_{\zeta_j})_{j=1}^r$  be its s.s.. If one of the following condition occurs, then  $\text{opt}(D) = LB(D)$  holds:

- a)  $d(a_{\zeta_r}) = d(a_{\zeta_1})$ ;
- b)  $r = 1$ ;

- c)  $r \geq 2$ , and  $g_{\zeta_2} = 2$ ;
- d)  $r \geq 2$ , and there exist  $r-1$  integers  $y_r, \dots, y_2$  verifying  $0 \leq y_j \leq g_{\zeta_j}$ , for any  $2 \leq j \leq r$ , and

$$\sum_{j=r}^2 (g_{\zeta_{j-1}}/g_{\zeta_j} - 1)(g_{\zeta_j} - 2y_j) \cdot a_{\zeta_j} + g_{\zeta_1-1}a_{\zeta_1} \equiv_n 0.$$

**PROOF.** (a) If  $d(a_{\zeta_r}) = d(a_{\zeta_1})$ , then  $d(a_t) = d(a_{\zeta_r})$ , for any  $\zeta_r \leq t \leq 1$ . In particular,  $d(a_{\zeta_j}) = d(a_{\zeta_{j+1}})$  holds, for any  $j \in [r-1]$ . It follows from *Theorem 13* that  $sum_D(\mathbf{H}(D, \alpha_D)) = SHP(\alpha_D) + d(a_{\zeta_1})$ . The claim thus follows by making use of (12), and of *Theorem 17*.

(b) It is a subcase of condition (a): If  $r = 1$ , then  $d(a_{\zeta_r}) = d(a_{\zeta_1})$ .

(c) It follows from *Theorem 13* that, if  $r \geq 2$ , and  $g_{\zeta_2} = 2$ , then  $\rho = 1$ , and  $\hat{\rho} = 1$ . Since  $g_{\zeta_1} = 1$  holds by (10), we have that  $h_1 = g_{\zeta_2} / g_{\zeta_1} = 2$ . Hence,  $sum_D(\mathbf{H}(D, \alpha_D)) = SHP(\alpha_D) + d(a_{\zeta_1})$  is verified. The claim thus follows by making use of (12), and of *Theorem 17*.

(d) Let us set  $\gamma_1 = 1$ , and  $\gamma_j = g_{\zeta_j} - y_j$ , for any  $2 \leq j \leq r$ . Trivially,  $g_{\zeta_j} - 2y_j = 2j - g_{\zeta_j}$  holds, for any  $2 \leq j \leq r$ . Since  $g_{\zeta_1} = 1$ , also  $g_{\zeta_1} = 2\gamma_1 - g_{\zeta_1} = 1$  is verified. It follows from the hypothesis that

$$\sum_{j=r}^2 (g_{\zeta_{j-1}}/g_{\zeta_j} - 1)(2\gamma_j - g_{\zeta_j}) \cdot a_{\zeta_j} + g_{\zeta_1-1}(2\gamma_1 - g_{\zeta_1}) \cdot a_{\zeta_1} \equiv_n 0.$$

$g_{\zeta_1-1} a_{\zeta_1}$  can be written as  $(g_{\zeta_1-1} / g_{\zeta_1} - 1)a_{\zeta_1} + a_{\zeta_1}$ . Hence,

$$\sum_{j=r}^1 (g_{\zeta_{j-1}}/g_{\zeta_j} - 1)(2\gamma_j - g_{\zeta_j}) \cdot a_{\zeta_j} \equiv_n -a_{\zeta_1} \equiv_n n - a_{\zeta_1}.$$

Let  $v = n - a_{\zeta_1}$ . As  $d[v, 0] = d(a_{\zeta_1})$  holds,  $\alpha_D$ , and  $v$  verifies the hypotheses of *Theorem 18*. The claim thus follows.  $\square$

### 7. 2-striped symmetric circulant matrices

Let  $D$  be a matrix in  $SC(\mathbb{N}^{n \times n})$ , let  $\alpha_D = (a_i)_{i=1}^{\lfloor n/2 \rfloor}$  be a presentation for it, and let  $\tau$  be a fixed integer in  $[\lfloor n/2 \rfloor]$ . Any Hamiltonian tour  $T: \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  such that  $\{i, T(i)\} \in D(a_i)$ , for some  $i \in \mathbb{Z}_n$ , and some  $t \geq \tau$ , verifies  $sum_D(T) \geq SHP(\alpha_D) + d(a_\tau)$ . Indeed, if  $P$  denotes the Hamiltonian path obtained by deleting the arc  $[i, T(i)]$  from the Hamiltonian cycle for  $G_{\lfloor n/2 \rfloor}(\alpha_D)$  induced by  $T$ , then  $c_D(P) \geq SHP(\alpha_D)$ , and  $sum_D(T) \geq c_D(P) + d(a_\tau)$ . Any such tour is not optimal if  $SHP(\alpha_D) + d(a_\tau) > UB(D)$  holds, since a Hamiltonian tour for  $D$  of cost  $UB(D)$  always exists (see §4). Thus, we may ignore the  $a_i$ -stripe, for any  $t \geq \tau$ , if  $d(a_\tau) > UB(D) - SHP(\alpha_D)$  holds.

Note that any other  $a$ -stripe cannot be a priori ignored, even if no presentation for  $D$  contains  $a$  in its s.s.. Thus, a first step for solving SCTSP is analyzing the case in which each presentation for  $D$  has the same s.s., and any stripe not belonging to the s.s. can be ignored.

**Definition 20** A matrix  $D \in SC(\mathbb{N}^{n \times n})$  is an  **$s$ -striped** matrix, for some  $s \geq 1$ , if a presentation

$\alpha_D = (a_i)_{i=1}^{\lfloor n/2 \rfloor}$  for it verifies the following properties:

- (i)  $(a_s, a_{s-1}, \dots, a_1)$  is the s.s. of  $\alpha_D$ , and  $d(a_i) < d(a_{i+1})$ , for any  $t \in [s]$ ;
- (ii)  $d(a_{s+1}) > UB(D) - SHP(\alpha_D)$ .

Definition 20 does not depend on the presentation. Indeed, let  $\beta_D = (b_i)_{i=1}^{\lfloor n/2 \rfloor}$  be a presentation for  $D$ , possibly different from  $\alpha_D$ . As both  $\alpha_D$ , and  $\beta_D$  sort in non decreasing order the multi-set containing the stripe costs of  $D$ , then  $d(a_i) = d(b_i)$  holds, for any  $t \in [\lfloor n/2 \rfloor]$ . In particular,  $d(b_{s+1}) = d(a_{s+1})$ , and, thus,  $d(b_{s+1})$  verifies property (ii). As a consequence of property (i), no other stripe different from  $a_i$  costs  $d(a_i)$ , for any  $t \in [s]$ . Hence,  $a_t = b_t$ , and  $g_t(\alpha_D) = g_t(\beta_D)$  hold, for any  $t \in [s]$ , and, thus,  $(a_s, a_{s-1}, \dots, a_1)$  is also the s.s. of  $\beta_D$ .

The case  $s = 1$  is trivial: condition (b) in Proposition 19 holds, and thus  $opt(D) = LB(D)$ . In this section we deal with the case  $s = 2$ .

By  $D(n; d_1, d_2; a_1, a_2)$  we denote the 2-striped matrix in  $SC(\mathbb{N}^{n \times n})$  verifying  $d(a_1) = d_1$ , and  $d(a_2) = d_2$ , for some presentation  $\alpha_D = (a_i)_{i=1}^{\lfloor n/2 \rfloor}$ . As any two presentations have  $(a_2, a_1)$  as s.s., we denote by  $g_1$  the integer  $g_1(\alpha_D) = \gcd(n, a_1)$ , and by  $G_1$ , and  $G_2$  the weighted undirected graphs  $G_1(\alpha_D)$ , and  $G_2(\alpha_D)$ . Note that  $g_1 > 1$ , and that  $\gcd(g_1, a_2) = 1$ , as a consequence of Definition 20, applied for  $s = 2$ .

The weighted adjacency matrix of  $G_2$  is a symmetric circulant matrix with two stripes, according to the definition given in Gerace, and Greco (2008a). Aim of this section is restating for the 2-striped matrices in  $SC(\mathbb{N}^{n \times n})$  the results obtained in Gerace, and Greco (2008a). Let  $D$  be the matrix  $D(n; d_1, d_2; a_1, a_2)$ . As a consequence of Theorem 9, of Theorem 17, and of (11) (respectively, of Theorem 9, and of (12)), the integer  $UB(D)$  (respectively,  $LB(D)$ ) verifies:

$$\begin{aligned}
 UB(D) &= (n - 2(g_1 - 1)) \cdot d_1 + 2(g_1 - 1) \cdot d_2; \\
 LB(D) &= (n - g_1) \cdot d_1 + g_1 \cdot d_2.
 \end{aligned}
 \tag{13}$$

If  $g_1 = 2$ , condition (c) of Proposition 19 implies that  $opt(D) = LB(D)$ .

**Definition 21** Let  $D$  be the matrix  $D(n; d_1, d_2; a_1, a_2)$ , and let  $T : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  be an Hamiltonian tour for  $D$ .  $T$  is **feasible** if  $\{i, T(i)\} \in D(a_1) \cup D(a_2)$ , for any  $i \in \mathbb{Z}_n$ .

Any stripe of  $D$  different from  $a_1$ , and  $a_2$  can be ignored. Thus, an optimal Hamiltonian tour for  $D$  is also a feasible one. As a consequence of Definition 6, Hamiltonian cycles for  $G_2$ , and feasible Hamiltonian tours for  $D$  are in correspondence.

Let  $T : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  be a feasible Hamiltonian tour for  $D$ , and let  $H_T = [v_0, v_1, \dots, v_{n-1}, v_0]$  be the Hamiltonian cycle for  $G_2$  associated to  $T$ .  $[v_\lambda, v_{\langle \lambda+1 \rangle_n}]$  is a  **$(+a_1)$ -arc**, for some  $\lambda \in \mathbb{Z}_n$ , if

$(v_{\langle i+1 \rangle_n} - v_i) \equiv_n +a_1$  holds. In a similar way,  $(-a_1)$ -arcs,  $(+a_2)$ -arcs, and  $(-a_2)$ -arcs are defined.  $\pi_{2,T}^+$  (respectively,  $\pi_{2,T}^-$ ) denotes the number of  $(+a_2)$ -arcs (respectively, of  $(-a_2)$ -arcs). If  $g_1 \geq 3$ ,  $(\pi_{2,T}^+ + \pi_{2,T}^-)$  corresponds the number of arcs of  $H_T$  belonging to  $D(a_2)$ , as the next remark shows.

**Remark.** An arc is at the same time a  $(+a_2)$ -arc, and a  $(-a_2)$ -arc if and only if  $a_2 \equiv_n -a_2$ , that is, if and only if  $n$  is even, and  $a_2 = n/2$ . As already observed,  $g_1 = \gcd(n, a_1) > 1$ , and  $1 = \gcd(n, a_1, a_2) = \gcd(g_1, n/2)$  hold. Thus,  $g_1 = 2$  holds if  $n$  is even, and  $a_2 = n/2$ .

**Theorem 22** Let  $D$  be the matrix  $D(n; d_1, d_2; a_1, a_2)$ . If  $g_1 \geq 3$ , there exists an optimal Hamiltonian tour  $T$  for  $D$  such that  $(\pi_{2,T}^+ - \pi_{2,T}^-) \in \{0, g_1\}$ . In particular, if  $(\pi_{2,T}^+ - \pi_{2,T}^-) = 0$ , then,  $\text{opt}(D) = \text{UB}(D)$  holds.

**PROOF.** (Sketch) Let  $S : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  be an optimal Hamiltonian tour for  $D$ . As  $g_1 \geq 3$  holds, the number of arcs in  $D(a_2)$  is  $(\pi_{2,S}^+ + \pi_{2,S}^-)$ . Since either  $\{i, S(i)\} \in D(a_1)$ , or  $\{i, S(i)\} \in D(a_2)$  holds, for any  $i \in \mathbb{Z}_n$ , then

$$\text{opt}(D) = \text{sum}_D(S) = (n - (\pi_{2,S}^+ + \pi_{2,S}^-)) \cdot d_1 + (\pi_{2,S}^+ + \pi_{2,S}^-) \cdot d_2.$$

Clearly,  $\text{LB}(D) \leq \text{sum}_D(S) \leq \text{UB}(D)$  holds. Hence, it follows from (13), and from  $d_1 < d_2$  that  $g_1 \leq (\pi_{2,S}^+ + \pi_{2,S}^-) \leq 2(g_1 - 1)$ . On the other hand,  $(\pi_{2,S}^+ - \pi_{2,S}^-) \equiv_{g_1} 0$ , since any arc in  $D(a_2)$  links two different connected components of  $G_1$ , and the starting one coincides with the ending one.

Hence,  $(\pi_{2,S}^+ - \pi_{2,S}^-) \in \{-g_1, 0, g_1\}$ . If  $(\pi_{2,S}^+ - \pi_{2,S}^-) \in \{0, g_1\}$ , it suffices to take  $T = S$ . If  $(\pi_{2,S}^+ - \pi_{2,S}^-) = -g_1$ , it suffices to take  $T = S^-$ .

Suppose that  $(\pi_{2,T}^+ - \pi_{2,T}^-) = 0$ . Since  $(\pi_{2,T}^+ + \pi_{2,T}^-) \leq 2(g_1 - 1)$  also holds, it follows that  $0 \leq \pi_{2,T}^+ = \pi_{2,T}^- \leq (g_1 - 1)$ .

For any  $i \in \mathbb{Z}_n$ , the nodes  $i$ , and  $T(i)$  belong to different connected components of  $G_1$  if and only if  $\{i, T(i)\} \in D(a_2)$ .  $G_1$  has  $g_1$  connected components, and the Hamiltonian cycle  $H_T$  induced by  $T$  starts, and ends at the same connected components, after having passed through each other connected component. It follows that  $\pi_{2,T}^+ = \pi_{2,T}^- \geq (g_1 - 1)$  also holds. The claim, thus, follows.  $\square$

**Theorem 23** Let  $D$  be the matrix  $D(n; d_1, d_2; a_1, a_2)$ . Assume that  $g_1 \geq 3$  holds. Let  $A_D = \{y \in \mathbb{Z} : 0 \leq y < n/g_1, (n/g_1 - 1)(g_1 - 2y)a_1 + g_1a_2 \equiv_n 0\}$ . If  $A_D$  is not empty, let  $y_1$ , and  $y_2$  be, respectively, the minimum, and the maximum of  $A_D$ , and let  $m = \min\{y_1 - g_1, n/g_1 - y_2\}$ .

The following statements hold.

- (i) If  $A_D$  is empty, then  $\text{opt}(D) = \text{UB}(D)$ .
- (ii)  $A_D$  is not empty, and  $m \leq 0$  if and only if  $\text{opt}(D) = \text{LB}(D)$ .
- (iii) If  $A_D$  is not empty, and  $m > 0$ , there exists a Hamiltonian tour for  $D$  of cost  $\text{LB}(D) + 2m \cdot (d_2 - d_1)$ .

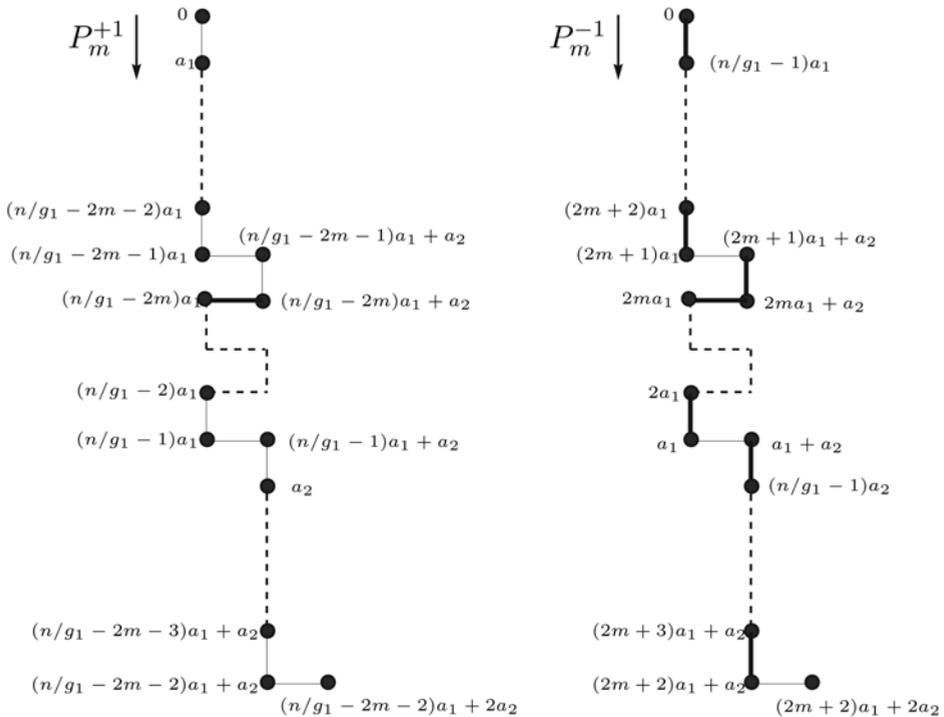


Fig. 6.  $P_m^{+1}$ , and  $P_m^{-1}$ , for a fixed  $m > 0$

**PROOF.** (Sketch) If  $A_D$  is empty, it can be shown that no Hamiltonian tour  $T$  for  $D$  verifies  $(\pi_{2,T}^+ - \pi_{2,T}^-) = g_1$ . Claim (i), thus, follows by Theorem 22.

Suppose that  $A_D$  is not empty, and that  $m \leq 0$  holds. As  $(n/g_1 - y) > 0$  holds, for any  $y \in A_D$ , we have that  $m = (y_1 - g_1)$ . It follows from  $m \leq 0$  that  $y_1$  verifies  $0 \leq y_1 \leq g_1$ , and from  $y_1 \in A_D$  that  $(n/g_1 - 1)(g_1 - 2y_1)a_1 + g_1 a_2 \equiv_n 0$ . As  $(a_2, a_1)$  is the s.s. of any presentation for  $D$ , condition (d) of Proposition 19 is verified. Thus,  $\text{opt}(D) = \text{LB}(D)$  follows.

By arguing as in the proof of the second claim of Theorem 18, it can be shown that  $\text{opt}(D) = \text{LB}(D)$  implies that there exists  $y \in A_D$  such that  $0 \leq y \leq g_1$ . Clearly,  $m \leq 0$ , in this case. Claim (ii) is thus proved.

Suppose that  $A_D$  is not empty, and that  $m > 0$  holds. Then  $m$  is a positive integer less than  $n/2g_1$ . Let us denote by  $\Delta_\lambda$ , for any  $\lambda \in \mathbb{Z}_{g_1}$ , the connected component of  $G_1$  having as node set  $\{v \in \mathbb{Z}_n : v \equiv_{g_1} \lambda a_2\}$ . Let  $P_m^{+1}$ , and  $P_m^{-1}$  be the path in  $G_2$  described in Figure 6<sup>1</sup>. They pass through any node in  $\Delta_0$ , and in  $\Delta_1$ , and cost  $(2n/g_1 - 2m) \cdot d_1 + (2 + 2m) \cdot d_2$ . For any  $\lambda \in \mathbb{Z}_{g_1}$ , let

<sup>1</sup> In the figures of this section, thin vertical lines represent  $(+a_1)$ -arcs, bold vertical lines represent  $(-a_1)$ -arcs, any other thin line represents a  $(+a_2)$ -arc, and, finally, any other bold line represents a  $(-a_2)$ -arcs.

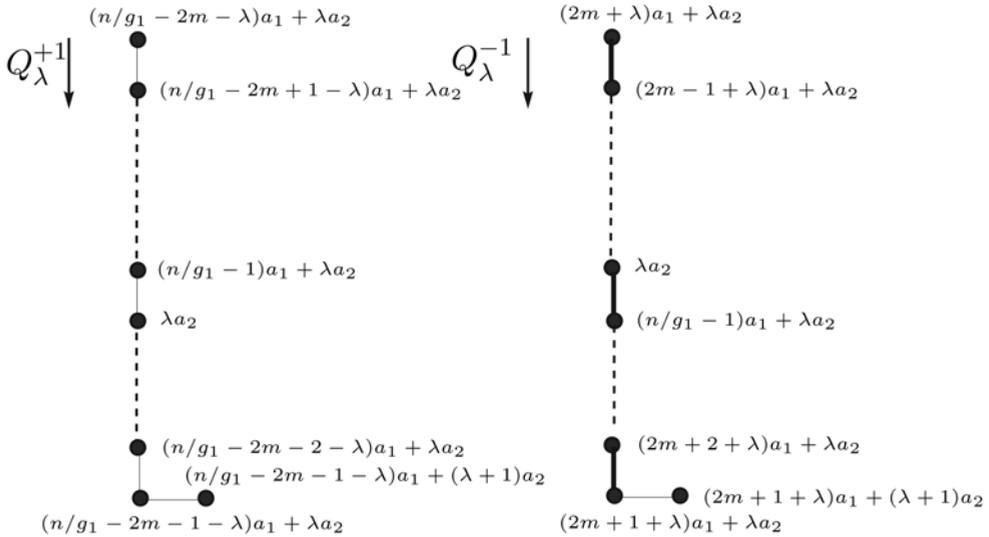


Fig. 7.  $Q_\lambda^{+1}$ , and  $Q_\lambda^{-1}$ , for a fixed  $\lambda \in \mathbb{Z}_{g_1}$

$Q_\lambda^{+1}$ , and  $Q_\lambda^{-1}$  be the path in  $G_2$  described in Figure 7. They pass through any node in  $\Delta_\lambda$ , and  $\text{cost}_{c_D}(Q_\lambda^\varepsilon) = (n/g_1 - 1) \cdot d_1 + d_2$ . For  $\varepsilon = +1, -1$ , let  $H_m^\varepsilon$  be the path obtained by composing  $P_m^\varepsilon, Q_2^\varepsilon, \dots, Q_{g_1-1}^\varepsilon$ .  $H_m^\varepsilon$  starts at the node 0, and passes through any node in  $G_2$ . Its cost verifies

$$\begin{aligned} c_D(H_m^\varepsilon) &= c_D(P_m^\varepsilon) + (g_1 - 2)c_D(Q_m^\varepsilon) = \\ &= (n - g_1 - 2m) \cdot d_1 + (g_1 + 2m) \cdot d_2 = LB(D) + 2m \cdot (d_2 - d_1). \end{aligned}$$

If  $m = y_1 - g_1$ ,  $H_m^{+1}$  is a Hamiltonian cycle for  $G_2$ , as its ending point is

$$v \equiv_n (2m + g_1)a_1 + g_1a_2 \equiv_n (2y_1 - g_1)a_1 + g_1a_2 \equiv_n 0.$$

If  $m = n/g_1 - y_2$ ,  $H_m^{-1}$  is a Hamiltonian cycle for  $G_2$  as its ending point is

$$v \equiv_n (2m + g_1)a_1 - g_1a_2 \equiv_n (2y_2 - g_1)a_1 + g_1a_2 \equiv_n 0.$$

The second part of claim (ii) thus follows, since either  $H_m^{+1}$ , or  $H_m^{-1}$  induced a Hamiltonian tour for  $D$  of the required cost.  $\square$

**Example 24** Let  $D_1$  be the matrix  $D(32; 1, 2; 8, 1)$ . It is easy to verify that  $g_1 = \text{gcd}(32, 8) = 8$ , and that  $n/g_1 = 4$ . The equation  $3(8 - 2y)8 + 8 \equiv_{32} 0$  has no integer solutions. Thus,  $A_{D_1}$  is empty. It follows from Theorem 23, and from (13) that  $\text{opt}(D_1) = UB(D_1) = 46$ .



## 8. Conclusions

In this chapter the attention has been focused on the Symmetric Circulant Traveling Salesman Problem (SCTSP), a subcase of the Traveling Salesman Problem explicitly introduced for the first time in 1992. The most remarkable results obtained in the last 16 years are reported: In the general case, there are given an upper bound, a lower bound, and a polynomial time 2-approximation algorithm; In the so-called 2-striped case, there are given an algebraic characterization for those matrices having the optimal cost equal either to the upper bound, or to the lower bound, and a new Hamiltonian tour construction for the remaining matrices.

At the moment the main research direction is that of generalizing to the  $s$ -striped case the results obtained in the 2-striped case. It seems the first necessary step in the direction of solving SCTSP.

To sum up, the problem of finding a polynomial time solution for SCTSP seems harder, and more interesting than it was expected. In general, it is less easy than it was expected dealing with circulant graphs, and with their algebraic structure. As a matter of fact, also showing that *Graph Isomorphism* is polynomial time solvable in the circulant graph case has required a forty year research.

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## **Traveling Salesman Problem**

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The idea behind TSP was conceived by Austrian mathematician Karl Menger in mid 1930s who invited the research community to consider a problem from the everyday life from a mathematical point of view. A traveling salesman has to visit exactly once each one of a list of  $m$  cities and then return to the home city. He knows the cost of traveling from any city  $i$  to any other city  $j$ . Thus, which is the tour of least possible cost the salesman can take? In this book the problem of finding algorithmic technique leading to good/optimal solutions for TSP (or for some other strictly related problems) is considered. TSP is a very attractive problem for the research community because it arises as a natural subproblem in many applications concerning the every day life. Indeed, each application, in which an optimal ordering of a number of items has to be chosen in a way that the total cost of a solution is determined by adding up the costs arising from two successively items, can be modelled as a TSP instance. Thus, studying TSP can never be considered as an abstract research with no real importance.

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