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# Unruh Radiation via WKB Method

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## 1. Introduction

Quantum mechanics has many features which are distinct from classical physics. Perhaps none more so than tunneling – the ability of a quantum particle to pass through some potential barrier even when, classically, it would not have enough energy to do so. The examples of tunneling phenomenon range from the nuclear (e.g. alpha decay of nuclei) to the molecular (oscillations of the ammonia molecule). Every text book on quantum mechanics devotes a good fraction of of page space to tunneling (usually introduced via the tunneling through a one dimensional step potential) and its applications.

In general, tunneling problems can not be solved, easily or at all, in closed, analytic form and so one must resort to various approximation techniques. One of the first and mostly useful approximations techniques is the WKB method [1] names after its co-discoverers Wentzel, Krammers and Brillouin. For a particle with an energy  $E$  and rest mass  $m$  moving in a one-dimensional potential  $V(x)$  (where  $E < V(x)$  for some range of  $x$ , say  $a \leq x \leq b$ , which is the region through which the particle tunnels) the tunneling amplitude is given by

$$\exp \left[ -\frac{1}{\hbar} \int_a^b [2m(V(x) - E)]^{1/2} dx \right] = \exp \left[ -\frac{1}{\hbar} \int_a^b p(x) dx \right], \quad (1)$$

where  $p(x)$  is the canonical momentum of the particle. Taking the square of (1) gives the probability for the particle to tunnel through the barrier.

In this chapter we show how the essentially quantum field theory phenomenon of Unruh radiation [4] can be seen as a tunneling phenomenon and how one can calculate some details of Unruh radiation using the WKB method. Unruh radiation is the radiation seen by an observer who accelerates through Minkowski space-time. Via the equivalence principle (i.e. the local equivalence between observations in a gravitational field versus in an accelerating

frame) Unruh radiation is closely related to Hawking radiation [2] – the radiation seen by an observer in the space-time background of a Schwarzschild black hole.

In the WKB derivation of Unruh radiation presented here we do not recover all the details of the radiation that the full quantum field theory calculation of the Unruh effect yields. The most obvious gap is that from the quantum field theory calculations it is known that Unruh radiation as well as Hawking radiation have a thermal/Planckian spectrum. In the simple treatment given here we do not obtain the thermal character of the spectrum of Unruh radiation but rather one must assume the spectrum is thermal (however as shown in [5] one can use the density matrix formalism to obtain the thermal nature of Unruh radiation as well as Hawking radiation in the WKB tunneling approach). The advantage of the present approach (in contrast to the full quantum field theory calculation) is that it is easy to apply to a wide range of observer and space-times. For example, an observer in de Sitter space-time (the space-time with a positive cosmological constant) will see Hawking–Gibbons radiation [3]; an observer in the Friedmann–Robertson–Walker metric of standard Big Bang cosmology will see Hawking-like radiation [6]. One can easily calculate the basic thermal features of many space-times (e.g. Reissner–Nordstrom [9], de Sitter [14], Kerr and Kerr–Newmann [15, 16], Unruh [17]) using the WKB tunneling method. Additionally, one can easily incorporate the Hawking radiation of particles with different spins [18] and one can begin to take into account back reaction effects on the metric [9, 10, 19] i.e. the effect that due to the emission of Hawking radiation the space-time will change which in turn will modify the nature of subsequent Hawking radiation.

The WKB tunneling method of calculating the Unruh and Hawking effects also corresponds to the heuristic picture of Hawking radiation given in the original work by Hawking (see pg. 202 of [2]). In this paper Hawking describes the effect as a tunneling outward of positive energy modes from behind the black hole event horizon and a tunneling inward of negative energy modes. However only after a span of about twenty five years were mathematical details given to this heuristic tunneling picture with the works [7–10]. These works showed that the action for a particle which crosses the horizon of some space-time picked up an imaginary contribution on crossing the horizon. This imaginary contribution was then interpreted as the tunneling probability.

One additional advantage of the WKB tunneling method for calculating some of the features of Hawking and Unruh radiation is that this method does not rely on quantum field theory techniques. Thus this approach should make some aspects of Unruh radiation accessible to beginning graduate students or even advanced undergraduate students.

Because of the strong equivalence principle (i.e., locally, a constant acceleration and a gravitational field are observationally equivalent), the Unruh radiation from Rindler space-time is the prototype of this type of effect. Also, of all these effects – Hawking radiation, Hawking–Gibbons radiation – Unruh radiation has the best prospects for being observed experimentally [20–23]. This WKB approach to Unruh radiation draws together many different areas of study: (i) classical mechanics via the Hamilton–Jacobi equations; (ii) relativity via the use of the Rindler metric; (iii) relativistic field theory through the Klein–Gordon equation in curved backgrounds; (iv) quantum mechanics via the use of the WKB-like method applied to gravitational backgrounds; (v) thermodynamics via the use of the Boltzmann distribution to extract the temperature of the radiation; (vi) mathematical methods in physics via the use of contour integrations to evaluate the imaginary part of

the action of the particle that crosses the horizon. Thus this single problem serves to show students how the different areas of physics are interconnected.

Finally, through this discussion of Unruh radiation we will highlight some subtle features of the Rindler space-time and the WKB method which are usually overlooked. In particular, we show that the gravitational WKB amplitude has a contribution coming from a change of the time coordinate from crossing the horizon [14]. This temporal contribution is never encountered in ordinary quantum mechanics, where time acts as a parameter rather than a coordinate. Additionally we show that the invariance under canonical transformations of the tunneling amplitude for Unruh radiation is crucially important to obtaining the correct results in the case of tunneling in space-time with a horizon.

## 2. Some details of Rindler space-time

We now introduce and discuss some relevant features of Rindler space-time. This is the space-time seen by an observer moving with constant proper acceleration through Minkowski space-time. Thus in some sense this is distinct from the case of a gravitational field since here we are dealing with flat, Minkowski space-time but now seen by an accelerating observer. However, because of the equivalence principle this discussion is connected to situations where one does have gravitational fields such as Hawking radiation in the vicinity of a black hole.

The Rindler metric can be obtained by starting with the Minkowski metric, i.e.,  $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ , where we have set  $c = 1$ , and transforming to the coordinates of the accelerating observer. We take the acceleration to be along the  $x$ -direction, thus we only need to consider a 1+1 dimensional Minkowski space-time

$$ds^2 = -dt^2 + dx^2 . \tag{2}$$

Using the Lorentz transformations (LT) of special relativity, the worldlines of an accelerated observer moving along the  $x$ -axis in empty spacetime can be related to Minkowski coordinates  $t, x$  according to the following transformations

$$\begin{aligned} t &= (a^{-1} + x_R) \sinh(at_R) \\ x &= (a^{-1} + x_R) \cosh(at_R) , \end{aligned} \tag{3}$$

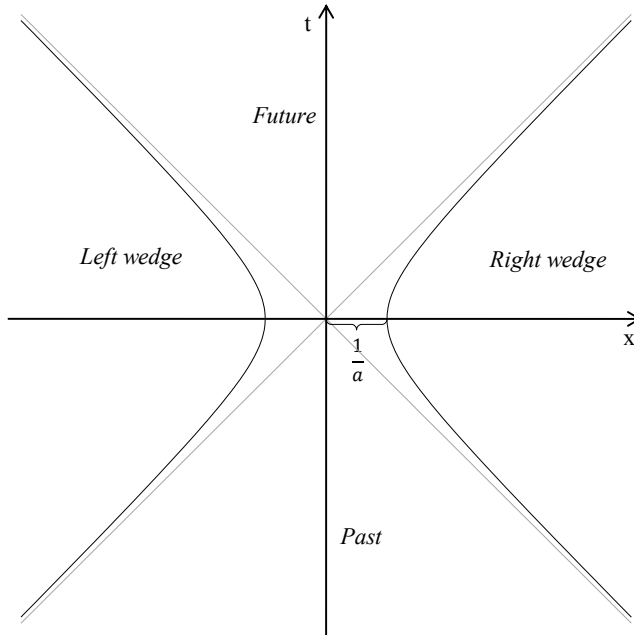
where  $a$  is the constant, proper acceleration of the Rindler observer measured in his instantaneous rest frame. One can show that the acceleration associated with the trajectory of (3) is constant since  $a_\mu a^\mu = (d^2x_\mu/dt_R^2)^2 = a^2$  with  $x_R = 0$ . The trajectory of (3) can be obtained using the definitions of four-velocity and four-acceleration of the accelerated observer in his instantaneous inertial rest frame [24]. Another derivation of (3) uses a LT to relate the proper acceleration of the non-inertial observer to the acceleration of the inertial observer [25]. The text by Taylor and Wheeler [26] also provides a discussion of the Rindler observer.

The coordinates  $x_R$  and  $t_R$ , when parametrized and plotted in a spacetime diagram whose axes are the Minkowski coordinates  $x$  and  $t$ , result in the familiar hyperbolic trajectories (i.e.,  $x^2 - t^2 = a^{-2}$ ) that represent the worldlines of the Rindler observer.

Differentiating each coordinate in (3) and substituting the result into (2) yields the standard Rindler metric

$$ds^2 = -(1 + ax_R)^2 dt_R^2 + dx_R^2 . \tag{4}$$

When  $x_R = -\frac{1}{a}$ , the determinant of the metric given by (4),  $\det(g_{ab}) \equiv g = -(1 + ax_R)^2$ , vanishes. This indicates the presence of a coordinate singularity at  $x_R = -\frac{1}{a}$ , which can not be a real singularity since (4) is the result of a global coordinate transformation from Minkowski spacetime. The horizon of the Rindler space-time is given by  $x_R = -\frac{1}{a}$ .



**Figure 1.** Trajectory of the Rindler observer as seen by the observer at rest.

In the spacetime diagram shown above, the horizon for this metric is represented by the null asymptotes,  $x = \pm t$ , that the hyperbola given by (3) approaches as  $x$  and  $t$  tend to infinity [27]. Note that this horizon is a particle horizon, since the Rindler observer is not influenced by the whole space-time, and the horizon’s location is observer dependent [28].

One can also see that the transformations (3) that lead to the Rindler metric in (4) only cover a quarter of the full Minkowski space-time, given by  $x - t > 0$  and  $x + t > 0$ . This portion of Minkowski is usually labeled *Right wedge*. To recover the *Left wedge*, one can modify the second equation of (3) with a minus sign in front of the transformation of the  $x$  coordinate, thus recovering the trajectory of an observer moving with a negative acceleration. In fact, we will show below that the coordinates  $x_R$  and  $t_R$  double cover the region in front of the horizon,  $x_R = -\frac{1}{a}$ . In this sense, the metric in (4) is similar to the Schwarzschild metric written in isotropic coordinates. For further details, see reference [28].

There is an alternative form of the Rindler metric that can be obtained from (4) by the following transformation:

$$(1 + a x_R) = \sqrt{|1 + 2 a x_{R'}|} . \tag{5}$$

Using the coordinate transformation given by (5) in (4), we get the following Schwarzschild-like form of the Rindler metric

$$ds^2 = -(1 + 2 a x_{R'}) dt_{R'}^2 + (1 + 2 a x_{R'})^{-1} dx_{R'}^2 . \tag{6}$$

If one makes the substitution  $a \rightarrow GM/x_{R'}^2$ , one can see the similarity to the usual Schwarzschild metric. The horizon is now at  $x_{R'} = -1/2a$  and the time coordinate,  $t_{R'}$ , does change sign as one crosses  $x_{R'} = -1/2a$ . In addition, from (5) one can see explicitly that as  $x_{R'}$  ranges from  $+\infty$  to  $-\infty$  the standard Rindler coordinate will go from  $+\infty$  down to  $x_R = -1/a$  and then back out to  $+\infty$ .

The Schwarzschild-like form of the Rindler metric given by (6) can also be obtained directly from the 2-dimensional Minkowski metric (2) via the transformations

$$\begin{aligned} t &= \frac{\sqrt{|1 + 2ax_{R'}|}}{a} \sinh(at_{R'}) \\ x &= \frac{\sqrt{|1 + 2ax_{R'}|}}{a} \cosh(at_{R'}) \end{aligned} \tag{7}$$

for  $x_{R'} \geq -\frac{1}{2a}$ , and

$$\begin{aligned} t &= \frac{\sqrt{|1 + 2ax_{R'}|}}{a} \cosh(at_{R'}) \\ x &= \frac{\sqrt{|1 + 2ax_{R'}|}}{a} \sinh(at_{R'}) \end{aligned} \tag{8}$$

for  $x_{R'} \leq -\frac{1}{2a}$ . Note that imposing the above conditions on the coordinate  $x_{R'}$  fixes the signature of the metric, since for  $x_{R'} \leq -\frac{1}{2a}$  or  $1 + 2ax_{R'} \leq 0$  the metric signature changes to  $(+, -)$ , while for  $1 + 2ax_{R'} \geq 0$  the metric has signature  $(-, +)$ . Thus one sees that the crossing of the horizon is achieved by the crossing of the coordinate singularity, which is precisely the tunneling barrier that causes the radiation in this formalism. As a final comment, we note that the determinant of the metric for (4) is zero at the horizon  $x_R = -1/a$ , while the determinant of the metric given by (6) is 1 everywhere.

### 3. The WKB/Tunneling method applied to Rindler space-time

In this section we study a scalar field placed in a background metric. Physically, these fields come from the quantum fields, i.e., vacuum fluctuations, that permeate the space-time given by the metric. By applying the WKB method to this scalar field, we find that the phase of the scalar field develops imaginary contributions upon crossing the horizon. The exponential of

these imaginary contributions is interpreted as a tunneling amplitude through the horizon. By assuming a Boltzmann distribution and associating it with the tunneling amplitude, we obtain the temperature of the radiation.

To begin we derive the Hamilton–Jacobi equations for a scalar field,  $\phi$ , in a given background metric. In using a scalar field, we are following the original works [2, 4]. The derivation with spinor or vector particles/fields would only add the complication of having to carry around spinor or Lorentz indices without adding to the basic understanding of the phenomenon. Using the WKB approach presented here it is straightforward to do the calculation using spinor[18] or vector particles. The scalar field in some background metric,  $g^{\mu\nu}$  is taken to satisfy the Klein-Gordon (KG) equation

$$\left( \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) - \frac{m^2 c^2}{\hbar^2} \right) \phi = 0, \tag{9}$$

where  $c$  is the speed of light,  $\hbar$  is Planck’s constant,  $m$  is the mass of the scalar field and  $g_{\mu\nu}$  is the background metric. For Minkowski space-time, the (9) reduces to the free Klein–Gordon equation, i.e.,  $(\square - m^2 c^2 / \hbar^2) \phi = (-\partial^2 / c^2 \partial t^2 + \nabla^2 - m^2 c^2 / \hbar^2) \phi = 0$ . This equation is nothing other than the fundamental relativistic equation  $E^2 - p^2 c^2 = m^2 c^4$  with  $E \rightarrow i\hbar \partial_t$  and  $p \rightarrow -i\hbar \nabla$ .

Setting the speed of light  $c = 1$ , multiplying (9) by  $-\hbar$  and using the product rule, (9) becomes

$$\frac{-\hbar^2}{\sqrt{-g}} \left[ (\partial_\mu \sqrt{-g}) g^{\mu\nu} \partial_\nu \phi + \sqrt{-g} (\partial_\mu g^{\mu\nu}) \partial_\nu \phi + \sqrt{-g} g^{\mu\nu} \partial_\mu \partial_\nu \phi \right] + m^2 \phi = 0. \tag{10}$$

The above equation can be simplified using the fact that the covariant derivative of any metric  $g$  vanishes

$$\nabla_\alpha g^{\mu\nu} = \partial_\alpha g^{\mu\nu} + \Gamma_{\alpha\beta}^\mu g^{\beta\nu} + \Gamma_{\alpha\beta}^\nu g^{\mu\beta} = 0, \tag{11}$$

where  $\Gamma_{\alpha\beta}^\mu$  is the Christoffel connection. All the metrics that we consider here are diagonal so  $\Gamma_{\alpha\beta}^\mu = 0$ , for  $\mu \neq \alpha \neq \beta$ . It can also be shown that

$$\Gamma_{\mu\gamma}^\mu = \partial_\gamma (\ln \sqrt{-g}) = \frac{\partial_\gamma \sqrt{-g}}{\sqrt{-g}}. \tag{12}$$

Using (11) and (12), the term  $\partial_\mu g^{\mu\nu}$  in (10) can be rewritten as

$$\partial_\mu g^{\mu\nu} = -\Gamma_{\mu\gamma}^\mu g^{\gamma\nu} - \Gamma_{\mu\rho}^\nu g^{\mu\rho} = -\frac{\partial_\gamma \sqrt{-g}}{\sqrt{-g}} g^{\gamma\nu}, \tag{13}$$

since the harmonic condition is imposed on the metric  $g^{\mu\nu}$ , i.e.,  $\Gamma_{\mu\rho}^{\nu}g^{\mu\rho} = 0$ . Thus (10) becomes

$$-\hbar^2 g^{\mu\nu} \partial_{\mu} \partial_{\nu} \phi + m^2 \phi = 0. \tag{14}$$

We now express the scalar field  $\phi$  in terms of its action  $S = S(t, \vec{x})$

$$\phi = \phi_0 e^{\frac{i}{\hbar} S(t, \vec{x})}, \tag{15}$$

where  $\phi_0$  is an amplitude [29] not relevant for calculating the tunneling rate. Plugging this expression for  $\phi$  into (14), we get

$$-\hbar g^{\mu\nu} (\partial_{\mu} (\partial_{\nu} (iS))) + g^{\mu\nu} \partial_{\nu} (S) \partial_{\mu} (S) + m^2 = 0. \tag{16}$$

Taking the classical limit, i.e., letting  $\hbar \rightarrow 0$ , we obtain the Hamilton–Jacobi equations for the action  $S$  of the field  $\phi$  in the gravitational background given by the metric  $g_{\mu\nu}$ ,

$$g^{\mu\nu} \partial_{\nu} (S) \partial_{\mu} (S) + m^2 = 0. \tag{17}$$

For stationary space-times (technically space-times for which one can define a time-like Killing vector that yields a conserved energy,  $E$ ) the action  $S$  can be split into a time and space part, i.e.,  $S(t, \vec{x}) = Et + S_0(\vec{x})$ .

If  $S_0$  has an imaginary part, this then gives the tunneling rate,  $\Gamma_{QM}$ , via the standard WKB formula. The WKB approximation tells us how to find the transmission probability in terms of the incident wave and transmitted wave amplitudes. The transition probability is in turn given by the exponentially decaying part of the wave function over the non-classical (*tunneling*) region [30]

$$\Gamma_{QM} \propto e^{-\text{Im} \frac{1}{\hbar} \oint p_x dx}. \tag{18}$$

The tunneling rate given by (18) is just the lowest order, quasi-classical approximation to the full non-perturbative Schwinger [31] rate.<sup>1</sup>

In most cases (with an important exception of Painlevé–Gulstrand form of the Schwarzschild metric which we discuss below),  $p^{out}$  and  $p^{in}$  have the same magnitude but opposite signs. Thus  $\Gamma_{QM}$  will receive equal contributions from the ingoing and outgoing particles, since the sign difference between  $p^{out}$  and  $p^{in}$  will be compensated for by the minus sign that is picked up in the  $p^{in}$  integration due to the fact that the path is being traversed in the

<sup>1</sup> The Schwinger rate is found by taking the Trace-Log of the operator  $(\square_g - m^2 c^2 / \hbar^2)$ , where  $\square_g$  is the d’Alembertian in the background metric  $g_{\mu\nu}$ , i.e., the first term in (9). As a side comment, the Schwinger rate was initially calculated for the case of a uniform electric field. In this case, the Schwinger rate corresponded to the probability of creating particle–antiparticle pairs from the vacuum field at the expense of the electric field’s energy. This electric field must have a critical strength in order for the Schwinger effect to occur. A good discussion of the calculation of the Schwinger rate for the usual case of a uniform electric field and the connection of the Schwinger effect to Unruh and Hawking radiation can be found in reference [32].

backward  $x$ -direction. In all quantum mechanical tunneling problems that we are aware of this is the case: the tunneling rate across a barrier is the same for particles going right to left or left to right. For this reason, the tunneling rate (18) is usually written as [30]

$$\Gamma_{QM} \propto e^{-2\text{Im} \frac{1}{\hbar} \int p_x^{\text{out},\text{in}} dx}, \tag{19}$$

In (19) the  $-$  sign goes with  $p_x^{\text{out}}$  and the  $+$  sign with  $p_x^{\text{in}}$ .

There is a technical reason to prefer (18) over (19). As was remarked in references [33–35], equation (18) is invariant under canonical transformations, whereas the form given by (19) is not. Thus the form given by (19) is not a proper observable.

Moreover, we now show that the two formulas, (18) and (19), *are not* even numerically equivalent when one applies the WKB method to the Schwarzschild space-time in Painlevé-Gulstrand coordinates. The Painlevé-Gulstrand form of the Schwarzschild space-time is obtained by transforming the Schwarzschild time  $t$  to the Painlevé-Gulstrand time  $t'$  using the transformation

$$dt = dt' - \frac{\sqrt{\frac{2M}{r}}}{1 - \frac{2M}{r}} dr. \tag{20}$$

Applying the above transformation to the Schwarzschild metric gives us the Painlevé-Gulstrand form of the Schwarzschild space-time

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt'^2 + 2\sqrt{\frac{2M}{r}} dr dt' + dr^2. \tag{21}$$

The time is transformed, but all the other coordinates  $(r, \theta, \phi)$  are the same as the Schwarzschild coordinates. If we use the metric in (21) to calculate the spatial part of the action as in (35) and (29), we obtain

$$S_0 = - \int_{-\infty}^{\infty} \frac{dr}{1 - \frac{2M}{r}} \sqrt{\frac{2M}{r}} E \tag{22}$$

$$\pm \int_{-\infty}^{\infty} \frac{dr}{1 - \frac{2M}{r}} \sqrt{E^2 - m^2 \left(1 - \frac{2M}{r}\right)}. \tag{23}$$

Each of these two integrals has an imaginary contribution of equal magnitude, as can be seen by performing a contour integration. Thus one finds that for the ingoing particle (the  $+$  sign in the second integral) one has a zero net imaginary contribution, while for the outgoing particle (the  $-$  sign in the second integral) there is a non-zero net imaginary contribution. Also as anticipated above the ingoing momentum (i.e, the integrand in (22) with the  $+$  sign in the second term) is not equal to the outgoing momentum (i.e, the integrand in (22) with the  $-$  sign in the second term) In these coordinates there is a difference by a



factor of two between using (18) and (19) which comes exactly because the tunneling rates from the spatial contributions in this case do depend upon the direction in which the barrier (i.e., the horizon) is crossed. The Schwarzschild metric has a similar temporal contribution as for the Rindler metric [36]. The Painlevé–Gulstrand form of the Schwarzschild metric actually has *two* temporal contributions: (i) one coming from the jump in the Schwarzschild time coordinate similar to what occurs with the Rindler metric in (7) and (8); (ii) the second temporal contribution coming from the transformation between the Schwarzschild and Painlevé–Gulstrand time coordinates in (20). If one integrates equation (20), one can see that there is a pole coming from the second term. One needs to take into account both of these time contributions in addition to the spatial contribution, to recover the Hawking temperature. Only by adding the temporal contribution to the spatial part from (18), does one recover the Hawking temperature [36]  $T = \frac{\hbar}{8\pi M}$ . Thus for both reasons – canonical invariance and to recover the temperature – it is (18) which should be used over (19), when calculating  $\Gamma_{QM}$ . In ordinary quantum mechanics, there is never a case – as far as we know – where it makes a difference whether one uses (18) or (19). This feature – dependence of the tunneling rate on the direction in which the barrier is traverse – appears to be a unique feature of the gravitational WKB problem. So in terms of the WKB method as applied to the gravitational field, we have found that there are situations (e.g. Schwarzschild space-time in Painlevé–Gulstrand coordinates) where the tunneling rate depends on the direction in which barrier is traversed so that (18) over (19) are not equivalent and will thus yield different tunneling rates,  $\Gamma$ .

For the case of the gravitational WKB problem, equation (19) only gives the imaginary contribution to the total action coming from the spatial part of the action. In addition, there is a temporal piece,  $E\Delta t$ , that must be added to the total imaginary part of the action to obtain the tunneling rate. This temporal piece originates from an imaginary change of the time coordinate as the horizon is crossed. We will explicitly show how to account for this temporal piece in the next section, where we apply the WKB method to the Rindler space-time. This imaginary part of the total action coming from the time piece is a unique feature of the gravitational WKB problem. Therefore, for the case of the gravitational WKB problem, the tunneling rate is given by

$$\Gamma \propto e^{-\frac{1}{\hbar}[\text{Im}(\oint p_x dx) - E\text{Im}(\Delta t)]} \tag{24}$$

In order to obtain the temperature of the radiation, we assume a Boltzmann distribution for the emitted particles

$$\Gamma \propto e^{-\frac{E}{T}} \tag{25}$$

where  $E$  is the energy of the emitted particle,  $T$  is the temperature associated with the radiation, and we have set Boltzmann’s constant,  $k_B$ , equal to 1. Equation (25) gives the probability that a system at temperature  $T$  occupies a quantum state with energy  $E$ . One weak point of this derivation is that we had to assume a Boltzmann distribution for the radiation while the original derivations [2, 4] obtain the thermal spectrum without any assumptions. Recently, this shortcoming of the tunneling method has been addressed in reference [5], where the thermal spectrum was obtained within the tunneling method using density matrix techniques of quantum mechanics.

By equating (25) and (24), we obtain the following formula for the temperature  $T$

$$T = \frac{E\hbar}{\text{Im}(\oint p_x dx) - E\text{Im}(\Delta t)}. \quad (26)$$

#### 4. Unruh radiation via WKB/tunneling

We now apply the above method to the alternative Rindler metric previously introduced. For the 1 + 1 Rindler space-times, the Hamilton–Jacobi equations (H–J) reduce to

$$g^{tt}\partial_t S \partial_t S + g^{xx}\partial_x S \partial_x S + m^2 = 0.$$

For the Schwarzschild–like form of Rindler given in (6) the H–J equations are

$$-\frac{1}{(1+2ax_{R'})}(\partial_t S)^2 + (1+2ax_{R'})^2(\partial_x S)^2 + m^2 = 0. \quad (27)$$

Now splitting up the action  $S$  as  $S(t, \vec{x}) = Et + S_0(\vec{x})$  in (27) gives

$$-\frac{E}{(1+2ax_{R'})^2} + (\partial_x S_0(x_{R'}))^2 + \frac{m^2}{1+2ax_{R'}} = 0. \quad (28)$$

From (28),  $S_0$  is found to be

$$S_0^\pm = \pm \int_{-\infty}^{\infty} \frac{\sqrt{E^2 - m^2(1+2ax_{R'})}}{1+2ax_{R'}} dx_{R'}. \quad (29)$$

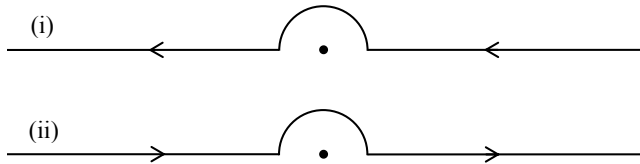
In (29), the + sign corresponds to the ingoing particles (i.e., particles that move from right to left) and the – sign to the outgoing particles (i.e., particles that move left to right). Note also that (29) is of the form  $S_0 = \int p_x dx$ , where  $p_x$  is the canonical momentum of the field in the Rindler background. The Minkowski space-time expression for the momentum is easily recovered by setting  $a = 0$ , in which case one sees that  $p_x = \sqrt{E^2 - m^2}$ .

From (29), one can see that this integral has a pole along the path of integration at  $x_{R'} = -\frac{1}{2a}$ . Using a contour integration gives an imaginary contribution to the action. We will give explicit details of the contour integration since this will be important when we try to apply this method to the standard form of the Rindler metric (4) (see Appendix I for the details of this calculation). We go around the pole at  $x_{R'} = -\frac{1}{2a}$  using a semi-circular contour which we parameterize as  $x_{R'} = -\frac{1}{2a} + \epsilon e^{i\theta}$ , where  $\epsilon \ll 1$  and  $\theta$  goes from 0 to  $\pi$  for the ingoing path and  $\pi$  to 0 for the outgoing path. These contours are illustrated in the figure below. With this parameterization of the path, and taking the limit  $\epsilon \rightarrow 0$ , we find that the imaginary part of (29) for ingoing (+) particles is

$$S_0^+ = \int_0^\pi \frac{\sqrt{E^2 - m^2\epsilon} e^{i\theta}}{2a\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta = \frac{i\pi E}{2a}, \tag{30}$$

and for outgoing (-) particles, we get

$$S_0^- = - \int_\pi^0 \frac{\sqrt{E^2 - m^2\epsilon} e^{i\theta}}{2a\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta = \frac{i\pi E}{2a}. \tag{31}$$



**Figure 2.** Contours of integration for (i) the ingoing and (ii) the outgoing particles.

In order to recover the Unruh temperature, we need to take into account the contribution from the time piece of the total action  $S(t, \vec{x}) = Et + S_0(\vec{x})$ , as indicated by the formula of the temperature, (26), found in the previous section. The transformation of (7) into (8) indicates that the time coordinate has a discrete imaginary jump as one crosses the horizon at  $x_{R'} = -\frac{1}{2a}$ , since the two time coordinate transformations are connected across the horizon by the change  $t_{R'} \rightarrow t_{R'} - \frac{i\pi}{2a}$ , that is,

$$\sinh(at_{R'}) \rightarrow \sinh\left(at_{R'} - \frac{i\pi}{2}\right) = -i \cosh(at_{R'}).$$

Note that as the horizon is crossed, a factor of  $i$  comes from the term in front of the hyperbolic function in (7), i.e.,

$$\sqrt{1 + 2ax_{R'}} \rightarrow i\sqrt{|1 + 2ax_{R'}|},$$

so that (8) is recovered.

Therefore every time the horizon is crossed, the total action  $S(t, \vec{x}) = S_0(\vec{x}) + Et$  picks up a factor of  $E\Delta t = -\frac{i\pi E}{2a}$ . For the temporal contribution, the direction in which the horizon is crossed does not affect the sign. This is different from the situation for the spatial contribution. When the horizon is crossed once, the total action  $S(t, \vec{x})$  gets a contribution of  $E\Delta t = -\frac{iE\pi}{2a}$ , and for a round trip, as implied by the spatial part  $\oint p_x dx$ , the total contribution is  $E\Delta t_{total} = -\frac{iE\pi}{a}$ . So using the equation for the temperature (26) developed in the previous section, we obtain

$$T_{Unruh} = \frac{E\hbar}{\frac{\pi E}{a} + \frac{\pi E}{a}} = \frac{\hbar a}{2\pi}, \quad (32)$$

which is the Unruh temperature. The interesting feature of this result is that the gravitational WKB problem has contributions from both spatial and temporal parts of the wave function, whereas the ordinary quantum mechanical WKB problem has only a spatial contribution. This is natural since time in quantum mechanics is treated as a distinct parameter, separate in character from the spatial coordinates. However, in relativity time is on equal footing with the spatial coordinates.

## 5. Conclusions and summary

We have given a derivation of Unruh radiation in terms of the original heuristic explanation as tunneling of virtual particles tunneling through the horizon [2]. This tunneling method can easily be applied to different space-times and to different types of virtual particles. We chose the Rindler metric and Unruh radiation since, because of the local equivalence of acceleration and gravitational fields, it represents the prototype of all similar effects (e.g. Hawking radiation, Hawking–Gibbons radiation).

Since this derivation touches on many different areas – classical mechanics (through the H–J equations), relativity (via the Rindler metric), relativistic field theory (through the Klein–Gordon equation in curved backgrounds), quantum mechanics (via the WKB method for gravitational fields), thermodynamics (via the Boltzmann distribution to extract the temperature), and mathematical methods (via the contour integration to obtain the imaginary part of the action) – this single problem serves as a reminder of the connections between the different areas of physics.

This derivation also highlights several subtle points regarding the Rindler metric and the WKB tunneling method. In terms of the Rindler metric, we found that the different forms of the metric (4) and (6) do not cover the same parts of the full spacetime diagram. Also, as one crosses the horizon, there is an imaginary jump of the Rindler time coordinate as given by comparing (7) and (8).

In addition, for the gravitational WKB problem,  $\Gamma$  has contributions from both the spatial and temporal parts of the action. Both these features are not found in the ordinary quantum mechanical WKB problem.

As a final comment, note that one can define an absorption probability (i.e.,  $P_{abs} \propto |\phi_{in}|^2$ ) and an emission probability (i.e.,  $P_{emit} \propto |\phi_{out}|^2$ ). These probabilities can also be used to obtain the temperature of the radiation via the “detailed balance method” [8]

$$\frac{P_{emit}}{P_{abs}} = e^{-E/T}.$$

Using the expression of the field  $\phi = \phi_0 e^{i\hbar S(t, \vec{x})}$ , the Schwarzschild–like form of the Rindler metric given in (6), and taking into account the spatial *and* temporal contributions gives an

an absorption probability of

$$P_{abs} \propto e^{\frac{\pi E}{a} - \frac{\pi E}{a}} = 1$$

and an emission probability of

$$P_{emit} \propto e^{-\frac{\pi E}{a} - \frac{\pi E}{a}} = e^{-\frac{2\pi E}{a}}.$$

The first term in the exponents of the above probabilities corresponds to the spatial contribution of the action  $S$ , while the second term is the time piece. When using this method, we are not dealing with a directed line integral as in (18), so the spatial parts of the absorption and emission probability have opposite signs. In addition, the absorption probability is 1, which physically makes sense – particles should be able to fall into the horizon with unit probability. If the time part were not included in  $P_{abs}$ , then for some given  $E$  and  $a$  one would have  $P_{abs} \propto e^{\frac{\pi E}{a}} > 1$ , i.e., the probability of absorption would exceed 1 for some energy. Thus for the detailed balance method the temporal piece is crucial to ensure that one has a physically reasonable absorption probability.

### Appendix I: Unruh radiation from the standard Rindler metric

For the standard form of the Rindler metric given by (4), the Hamilton–Jacobi equations become

$$-\frac{1}{(1 + a x_R)^2} (\partial_t S)^2 + (\partial_x S)^2 + m^2 = 0. \tag{33}$$

After splitting up the action as  $S(t, \vec{x}) = Et + S_0(\vec{x})$ , we get

$$-\frac{E}{(1 + a x_R)^2} + (\partial_x S_0(x_R))^2 + m^2 = 0. \tag{34}$$

The above yields the following solution for  $S_0$

$$S_0^\pm = \pm \int_{-\infty}^{\infty} \frac{\sqrt{E^2 - m^2 (1 + a x_R)^2}}{1 + a x_R} dx_R, \tag{35}$$

where the  $+(-)$  sign corresponds to the ingoing (outgoing) particles.

Looking at (35), we see that the pole is now at  $x_R = -1/a$  and a naive application of contour integration appears to give the results  $\pm \frac{i\pi E}{a}$ . However, this cannot be justified since the two forms of the Rindler metric – (4) and (6) – are related by the simple coordinate transformation (5), and one should not change the value of an integral by a change of variables. The resolution to this puzzle is that one needs to transform not only the integrand but the path of integration, so applying the transformation (5) to the semi-circular contour  $x_{R'} = -\frac{1}{2a} + \epsilon e^{i\theta}$  gives  $x_R = -\frac{1}{a} + \frac{\sqrt{\epsilon}}{a} e^{i\theta/2}$ . Because  $e^{i\theta}$  is replaced by  $e^{i\theta/2}$  due to the square root in the transformation (5), the semi-circular contour of (30) is replaced by a quarter-circle, which then leads to a contour integral of  $i\frac{\pi}{2} \times \text{Residue}$  instead of  $i\pi \times \text{Residue}$ . Thus both forms of Rindler yield the same spatial contribution to the total imaginary part of the action.

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