Chapter 5

Extended Theories of Gravitation and the Curvature of the Universe – Do We Really Need Dark Matter?

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Additional information is available at the end of the chapter

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1. Introduction

Cosmology is a young science. Less than a century ago cosmology stopped to be a branch of philosophy and it crossbred with General Relativity to become a science. Until very recently cosmological observations were quite rough and qualitative. Until the 80s one was quite satisfied with data with error bars of few percent.

Since then a number of extremely precise surveys have been carried over producing a massive amount of very precise data. The current picture that emerged from those data is quite awkward. In order to fit observations and maintain standard GR as general framework for gravity one is forced to introduce dark sources, at least in a large amount different from the matter that can be seen in the universe and which has somehow odd behavior; see [1], [2], [3], [4].

Actually, following this direction one is led to assume that about 70% of gravitational sources in the universe is constituted by some strange kind of dark energy, closely resembling a (small and positive) cosmological constant, about 25% of gravitational sources is constituted by some kind of dark matter (for which different models have been proposed and discussed), while visible matter amounts to few percents (about 4-5% depending on the model) of the total amount of matter. It is important to notice that we do not have any direct evidence or data about dark energy and dark matter other than their supposed gravitational effects on visible matter. Moreover, the best models for dark energy and dark matter are often definitely unsatisfactory from a fundamental viewpoint; see [5].

On the other hand, it has been suggested that the description of the gravitational field given by standard GR may fail at cosmological scale and we missed something, so that a good agreement with data can be obtained by modifying the description of gravity more than adding exotic sources. In any event it is now clear that something has to be changed in our standard framework in order to understand the universe out there.

Besides these obvious considerations let us add quite a trivial remark. Our understanding of the meaning of observations is generally weak and often depending on the model. Standard GR has a good set of protocols which allow one to make predictions and tests. The theory is extremely well tested at Solar system scales, while it is known to require corrections (by adding dark sources or by modifying dynamics) at galactic, astrophysical and cosmological scale (oddly enough whenever non-vacuum solutions are considered).

However, what we observe when we measure the distance of a supernova is not clear at all. GR is a relativistic theory with a huge symmetry group, namely all spacetime diffeomorphisms. The observable quantities should be then invariant with respect to spacetime diffeomorphisms, i.e. gauge invariant. Unfortunately, due to the particular nature of diffeomorphisms and their action on the geometry of spacetime, we do not know any non-trivial quantity which is diff-invariant. Also scalars are not (unless they are constant) since the Lie derivative of a scalar with respect to a generic spacetime vector field (i.e. a generic generator of spacetime diffeomorphisms) is

$$\pounds_\xi f = \xi^\mu \partial_\mu f \tag{1}$$

which is in general not zero, showing that in fact the quantity is not gauge invariant. This is known since the very beginning and it is the starting point of the celebrated hole argument; see [6]. Since we do measure quantities that are not gauge invariant, the only possible explanation is that we set observational protocols which as a matter of fact break gauge invariance on a conventional basis (possibly using matter references, as suggested in [6]).

That would not be too bad, if we clearly understood the details of such conventions and gauge fixing, that we do not. Instead, standard GR mixes from its very beginning physical quantities (i.e. the gravitational field) and the observational protocols (e.g. for measuring distances and times) in the same object (namely, the metric tensor). Originally, Einstein had not many options, since at that time the only way to describe curvature was through a metric structure and general (linear) connections were still to be fully described. As a consequence it becomes very difficult to keep the two things separated as they should.

In the 70s Ehlers, Pirani and Schild (EPS) gave a fundamental contribution to the understanding of the foundations of any reasonable theory of spacetime and gravity. They proposed an axiomatic approach to gravitational theories which, instead of assuming a metric or a connection on spacetime, assumed as fundamental potentially observable quantities (namely the worldlines of particles and light rays) and derived from them the geometry of spacetime; see [7], [8]. The original project was to obtain standard GR. However, the proposal finally turned out to give us a fundamental insight about what is to be considered observable and which geometrical structures are really essential for gravity.

In particular EPS framework allows a more general geometric structure on spacetime in which standard GR comes out to be just one of many possible theories of gravitation. Moreover, a more general framework potentially allows to test which geometric structure on spacetime is actually physically realized. As a side effect, EPS has an impact on observational protocols (not all standard protocols can be trivially extended to a general extended theory).

We shall hereafter review the EPS framework, define extended theories of gravitation and attempt a rough classification of possible extended theories. Finally we shall discuss some simple application to cosmology and observational protocols.

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2. EPS structures on spacetime

As already mentioned, in the early 70s Ehlers, Pirani and Schild (EPS) proposed an axiomatic framework for relativistic theories in which they showed how one can derive the geometric structure of spacetime from potentially observable quantities, i.e. worldlines of particles and light rays; see [7]. Accordingly, in the EPS framework the geometry of spacetime is not assumed but derived by more fundamental objects. We shall first briefly review EPS formalism; in the next Sections we shall discuss its consequences in gravitational theories and cosmology.

Let M be an (orientable, time orientable, connected, paracompact, smooth) m-dimensional manifold. Points in M are called *events* and M is called accordingly a *spacetime*. Let us stress that although M is chosen so that it allows global Lorentzian metrics, we do not fix any metric structure on M.

On M we consider two congruences of trajectories. Let $\mathcal{P}$ be the congruence of all possible motions of massive particles and $\mathcal{L}$ be the congruence of all possible light rays. Of course there are reasonable physical requirement to be asked about $\mathcal{P}$ and $\mathcal{L}$ since we expect they cannot be chosen to be completely generic or unrelated since we expect photons to feel the gravitational field as well as we expect matter to interact with the electromagnetic field.

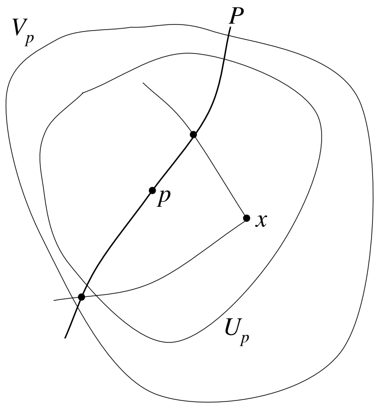
If we restrict ourselves to particles and light rays passing through an event $x \in M$, we *know* that the directions of light rays form a *cone* (the *light cone*). We can express this experimental fact by asking that the directions of light rays divide spacetime directions (i.e. the projective space of T_xM) into two connected components (i.e. the directions *inside* and *outside* the light cone).

We also know that the set of vectors *inside* the light cone is topologically different from the set of vectors *outside* the light cone. If one removes the zero vector then the set of vectors *inside* the light cone disconnects into two connected components (namely, *future* and *past* directed timelike vectors), while the set of vectors *outside* the light cone keeps being connected (there is nothing like future directed spacelike vectors!).

Moreover, we know that one has two kinds of vectors tangent to light rays: the ones pointing to the future and the ones pointing to the past. Thus we assume that (once the zero vector is removed) the set of vectors tangent to light rays also splits into two connected components (namely, *future* and *past* directed vectors). Let us stress that past and future are defined *at a point* x and it does not really matter which one of them is called future or past. These three requirements are physically well founded and in the end they constrain the light cones to be *cones* without resorting to a metric structure we did not define yet.

Then we have a number of regularity conditions. We need axioms to certify that one has enough light rays to account for physical standard messaging. Let us thus assume that for any particle $P \in \mathcal{P}$ and for any event $p \in P \subset M$ there exists a neighbourhood V_p and a neighbourhood $U_p \subset V_p$ such that for any event $x \in U_p$ there are two light rays through x hitting P within V_p.

Let us remark that in Minkowski spacetime one can set $U_p = V_p = M$ and there are always two such light rays (as one can check by direct calculation remembering that particles and light rays are given as straight lines in Minkowski spacetime).



Given two particles $P, Q \in \mathcal{P}$ we can consider the family of light rays $\lambda \in \mathcal{L}$ intersecting P and Q. By the above assumption, when P and Q are close enough such family is not empty. This family of light rays does define a local one-to-one map between P and Q which is called a *message* which is denoted by $\mu : P \to Q$. If one takes the composition of a message from P to Q and a message from Q to P the resulting map $\epsilon : P \to P$ is called and *echo* of P on Q. Both messages and echoes are assumed to be smooth maps.

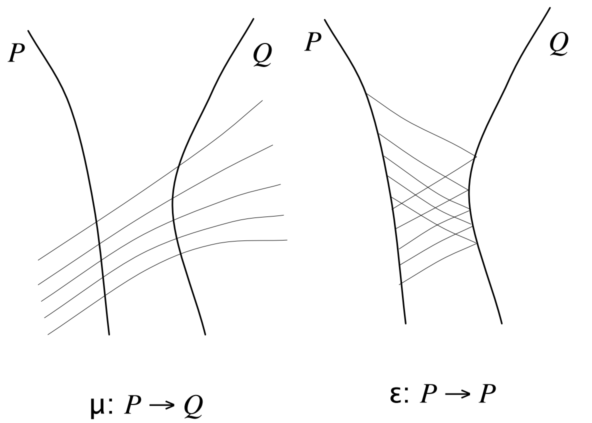


Figure 1. Messages and Echoes

Finally, we have to guarantee that there exist enough particles in $\mathcal{P}$ and light rays in $\mathcal{L}$ (which until now could be empty, as far as we know). We assume that there is a particle for each vector inside the light cone and a light ray for each vector on the light cone.

Let us now define a *clock* to be a parametrized particle, the parametrization accounting for the time maintained by the clock; [9]. For any clock $P \in \mathcal{P}$, for any event $p \in P$ one can set the parameter to be $s = 0$ at p. Using echoes one can use a number of clocks to define a special class of local coordinates, *called radar coordinates* or *parallax coodinates*. If $\dim(M) = m$ one can always choose m clocks P_i near an event $p \in M$ so that there exists a neibourhood

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U_p such that for any $x \in U_p$ there is a (future directed) light ray through x then hitting the clock P_i at its parameter value s_i. The values of the parameters s_i do form a good coordinate system in U_p. We assume that the spacetime differential structure on M is the one compatible with these charts. Let us remark that parallax coordinates mimick how astronomers define positions of objects.

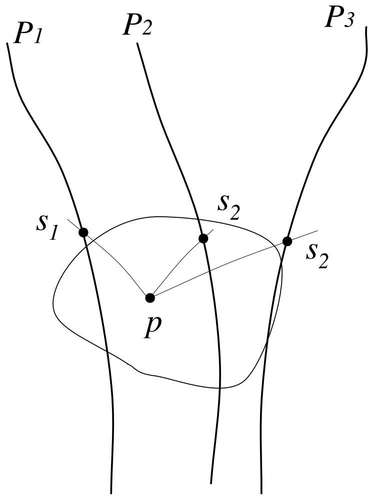

Figure 2. Parallax coordinates in dimension $m = 3$

One can show that as a consequence of these assumptions a class of Lorentzian metrics g is defined on M. Let us then fix a clock P though an event p. For any event $x \in U_p$ one has two light rays through x intersecting P, say at events $p_\pm$ which correspond to the parameter values $s_\pm$. Then we can define a local function $\Phi : U_p \to \mathbb{R} : x \mapsto -s_+ \cdot s_-$. As one can easily show, if there exists a light ray through x and p then $\Phi(x) = 0$. According to the topological assumptions made on the light cones then one can show that $\Phi(p) = 0$ and $d\Phi(p) = 0$. Then one can consider the Hessian $\partial_{\mu\nu}\Phi(p)$ as the first non-zero term in the Taylor expansion of Φ around the event p. In this case it defines a tensor field (a bilinear form)

$$g_p = g_{\mu\nu}(p)\, dx^\mu \otimes dx^\nu = \partial_{\mu\nu}\Phi(p)\, dx^\mu \otimes dx^\nu \tag{2}$$

For any light ray direction v at p one has $g(v, v) = 0$. One can also easily show that for u tangent to the clock P one has $g(u, u) < 0$.

Accordingly, g cannot be definite positive. In order not to contradict again assumptions about light cones, one can show that g is necessarily non-degenerate and Lorentzian (see [10]). Of course the the tensor g depends on the conventional choice of the clock. If one changes clock one defines a different tensor $\tilde{g}$ which is related to the previous one by a conformal transformation, namely $\tilde{g} = \varphi(x) \cdot g$ for some positive scalar field φ.

Let us now consider the set $\mathrm{Lor}(M)$ of all (global) Lorentzian metrics on M. Let us say that two metrics $g_{(1)}, g_{(2)} \in \mathrm{Lor}(M)$ are *conformally equivalent* iff there exists a positive scalar field

φ such that $\tilde{g}_x = \varphi(x) \cdot g_x$. The construction above shows that one can define out of light rays (i.e. out of the electromagnetic field) a *conformal class* of metrics $\mathfrak{C} = [g]$. Let us remark that the choice of a representative $\tilde{g} \in \mathfrak{C}$ is conventional and in fact part of the specification of the observer; conformal transformations are gauge transformations.

Notice that light cones are invariant with respect to conformal transformations; given a conformal structure $\mathfrak{C}$ on M one can define *lightlike* (*timelike* and *spacelike*, respectively) vectors being $g(v, v) = 0$ ($g(u, u) < 0$ and $g(w, w) > 0$, respectively) recovering standard notations used in GR.

Finally, we have to focus on particles. Let us first assume that we have one particle through $p \in M$ for any timelike direction and a light ray for any lightlike direction. Some constraint on particles must be set. Originally, EPS resorted to the equivalence principle and special relativity (SR) assuming that particle worldlines are geodesics of some connection $\tilde{\Gamma}$.

We cannot, for various reasons, be totally satisfied with this assumption, even if we accept of course the result. First of all relativistic theories are more fundamental than SR, which should hence be obtained in some limit from GR rather than being used to define it. Then the equivalence principle is an experimental fact and we would like to keep the possibility to test it rather than assuming it as a must. Luckily enough, one can obtain geodesic equations (together with a better insight on the nature of gravitational field) also without resorting to SR and equivalence principle. In fact, if one assumes that free fall must be described by differential equations of the second order, deterministic, covariant with respect to spacetime diffeomorphisms and with respect to arbitrary reparametrizations of worldlines those candidate equations are strongly constrained; see [11], [12]. If one then defines gravitational interaction to be the one which cannot be cancelled in a way independent of the coordinates and parametrizations then the equation uniquely determined are geodesic equations

$$\ddot{q}^\lambda + \tilde{\Gamma}^\lambda_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta = \lambda \dot{q}^\lambda \tag{3}$$

for some (global torsionless) connection $\tilde{\Gamma}(x)$ and some function $\lambda(s)$. In this way free fall is naturally associated to a connection $\tilde{\Gamma}$ and one is considering the Einstein's lift experiment as showing that *there are* observers who see a gravitational field rather than a *gedanken* experiment showing that there is a class of observers who do not (approximately) observe it.

As is well known, different connections can define the same autoparallel trajectories. In fact the connection

$$\tilde{\Gamma}'^\alpha_{\beta\mu} = \tilde{\Gamma}^\alpha_{\beta\mu} + \delta^\alpha_{(\beta} V_{\mu)} \tag{4}$$

defines the same geodesic trajectories as $\tilde{\Gamma}^\alpha_{\beta\mu}$ for any covector V_μ. In this case we say that $\tilde{\Gamma}$ and $\tilde{\Gamma}'$ are *projectively equivalent*. Accordingly, free fall corresponds to a projective class $\mathfrak{P} = [\tilde{\Gamma}]$; see [13].

Finally, we need (as we said above) a compatibility condition between the conformal class $\mathfrak{C}$ associated to light cones and the projective class $\mathfrak{P}$ associated to free fall. This is due by the simple fact that we know that light rays (and hence light cones) feel the gravitational fields as mass particles. Noticing that g-lightlike g-geodesics are conformally invariant (unlike general g-geodesics), we have then to assume that g-lightlike g-geodesics are a subset of

$\tilde{\Gamma}$-autoparallel trajectories. According to EPS-compatibility condition one can show that a representative $\tilde{\Gamma} \in \mathfrak{P}$ of the projective structure can be always (and uniquely) chosen so that there exists a covector $A = A_\mu \, dx^\mu$ such that

$$\tilde{\nabla} g = 2A \otimes g \tag{5}$$

where g is a representative of the conformal structure $g \in \mathfrak{C}$ and the covariant derivative $\tilde{\nabla}$ is the one associated to $\tilde{\Gamma}$; see [14]. Equivalently one has

$$\tilde{\Gamma}^\alpha_{\beta\mu} = \{g\}^\alpha_{\beta\mu} + (g^{\alpha\epsilon} g_{\beta\mu} - 2\delta^\alpha_{(\beta}\delta^\epsilon_{\mu)})A_\epsilon \tag{6}$$

To summarize, by assuming particles and light rays one can define on spacetime a *EPS structure*, i.e. a triple $(M, \mathfrak{C}, \mathfrak{P})$. The conformal structure $\mathfrak{P}$ describes light cones and it is associated to light rays. Notice that having just a conformal structure one cannot yet define distances (that are not conformally invariant) and this not being a gauge covariant must resort to a *convention* which corresponds to the choice of a representative $g \in \mathfrak{C}$. On the other hand, the projective structure $\mathfrak{P}$ is associated to free fall so that one can make a canonical gauge fixing by choosing the only representative in the form (6) or, equivalently, the 1-form A.

The triple $(M, \mathfrak{C}, \tilde{\Gamma})$ (or, equivalently, the triple $(M, \mathfrak{C}, A)$) is called a *Weyl geometry* on spacetime. This setting is more general than the setting for standard GR where one has just a Lorentzian metric g determining both the conformal structure $g \in \mathfrak{C}$ and the free fall $\tilde{\Gamma} = \{g\}$ (i.e. the Levi-Civita connection uniquely associated to g). Hence standard GR is a very peculiar case of EPS framework, where there is a gauge fixing of the conformal gauge. Such a fixing is possible iff the covector $A = A_\mu dx^\mu$ is exact, i.e. $A = d\varphi$. In this case, there exists a Lorentzian metric $\tilde{g} \in \mathfrak{C}$ also determining free fall by $\tilde{\Gamma} = \{\tilde{g}\}$. When this happens the Weyl geometry $(M, [\tilde{g}], \{\tilde{g}\})$ is called a *metric Weyl geometry*. Notice that this is still more general than standard GR in the sense that the metric determining free fall and light cones is not the original g chosen to describe dynamics, but a conformal one $\tilde{g} \in [g]$. Reverting to standard GR in a sense amounts to choose φ to be a constant (so that A vanishes identically).

At this point the reader could argue that in a metric Weyl geometry one could fix the conformal metric $\tilde{g}$ at the beginning and use it to describe dynamics, thus obtaining a framework which *exactly* reproduces standard GR. We shall discuss this issue below in greater details. Now we simply notice that this is not the case. The choice of a representative of the conformal structure is, in fact, what allows us to define distances.

In fact, since astronomers do measure distances, we do have a protocol (or better a number of protocols) to measure distances. As a matter of fact, such a protocol selects (in a rather obscure way) a precise representative g' of the conformal structure which is the one that corresponds to the distances that we measure. If one metric g geometrically accounts for a given physical distance measured between two events then obviously no other conformal metric $\tilde{g}$ (i.e. no other representative of the same conformal factor) can geometrically account for the same measure (modulo constant conformal factors which can be treated as a definition of units).

In standard GR one assumes that such a representative g' *also* determines light cones and free fall. In metric Weyl geometries there is nothing ensuring that the canonical representative $\tilde{g}$ also gives us the measured distances, that as far as we can see could as well be related to any other conformally equivalent metric g. Fixing the metric that we use to calculate distances is, in the end, a choice that we can do only *a posteriori*, on the basis of observations.

At a fundamental level one can either decide to be strict on the interpretation of conformal gauge symmetry (and accordingly quantities that are not gauge invariant, such as distances, cannot be really observable) or one accepts conventional gauge fixing to define such quantities as observable, thus restricting symmetries of the system to the conformal transformations which preserves these gauge fixing. In the first case standard GR is equivalent to metric Weyl geometry (though we cannot measure distances) or, in the second case, we define distances but standard GR is not necessarily equivalent to metric Weyl geometries. Again, deciding which is the metric that really enters observational protocols is something that should not be imposed *a priori* but rather something to be tested locally.

3. Extended theories of gravitation

EPS analysis sets a number of constraints to any theoretical framework that can be called a *reasonable theory of gravitation*. Such constraints are much weaker than the strong metricity assumptions done in standard GR.

Before explicitly analyzing these constraints, let us first discuss about the interpretation of a relativistic theory. One usually chooses fundamental fields in kinematics and then considers dynamics. In gauge theories these two levels are usually quite disconnected since one is free to change fundamental field variables; this induces a change of dynamics (which is in fact assumed to be gauge covariant) and it does not affect observable quantities (which are also assumed to be gauge invariant).

However, the situation in gravitational theories is quite different. In relativistic theories, as we already discussed in the introduction, there are no known non-trivial gauge-invariant quantities. If we want to retain some connection with astrophysics and cosmology we are forced to assume as a fact that matter allows some conventional (partial) gauge fixing. Strictly speaking observables are not gauge invariant and accordingly they are not preserved under changes of fundamental field coordinates. When discussing the equivalence between different formalisms one must additionally declare how observational and measuring protocols are modified by the transformations allowed and/or chosen. For example, let us consider a metric Weyl theory in which the dynamics is described in terms of a metric g and a connection $\tilde{\Gamma}$. When a solution of field equations is found then one can determine both light cones and free fall by a single conformal representative $\tilde{g}$. Of course one can rewrite the dynamics in terms of $\tilde{g}$ only. Is this metric theory fully equivalent to GR, especially in presence of matter?

To better understand this apparently trivial question, let us recall that changing a metric g to a conformally equivalent one $\tilde{g} = \varphi \cdot g$ will not change electromagnetism but will certainly change the coupling with non-electromegnetic matter (e.g., a cosmological fluid).

We cannot answer this simple question, before considering which metric is used to define distances. In standard GR, one makes the *a priori* (unjustified) ansatz that distances would be defined by the same metric which defines free fall, i.e. in this case $\tilde{g}$. If in the original model distances were defined using g (which by the way is the only way to select a conformal gauge to write a non conformally invariant dynamics) the new model is only *similar* to standard GR but inequivalent as far as distances are concerned. If the original theory is recognized to be inequivalent to standard GR based on the Hilbert-Einstein-Palatini Lagrangian $L = \sqrt{g} g^{\mu\nu} R_{\mu\nu}(\Gamma)\, ds$, let us remark that as a matter of fact dark sources are precisely related to mismatches in observed distances. . . !

Now let us suppose for the sake of argument that standard GR is still a perfect theory to describe the actual universe (something that we know to be strongly questioned by actual observations). Still we believe that analyzing it within a wider framework as the one of EPS structures and Weyl geometries is in any case useful if not even necessary. If we can understand observations in this wider framework, in fact, we can better test gravity and maybe eventually show that standard GR is compatible with observations. If we assume standard GR setting and we build observational protocols for it then it may become difficult to understand which data come from assumptions and which data come instead from real physical facts, especially in a theory of gravitation in which we clearly made exceptions about gauge invariant observables.

Having said that, we see now that EPS formalism points to a Weyl geometry on spacetime in which one has a conformal structure $\mathfrak{C}$ defining light cones and a compatible (torsionless) connection $\tilde{\Gamma}$ defining free fall. Of course, whenever interaction with matter of half-integer spin is considered nothing prevents from relaxing the symmetry requirements on connections. However, until only test particles are considered matter is unaffected by torsion and one can drop it from the beginning. Then our protocols for measuring distances select a representative $g \in \mathfrak{C}$ for the conformal structure. In particular there is no reason why one should assume *a priori* that the connection $\tilde{\Gamma}$ is metric or, if such, that it is metric for the same metric one happens to have selected for distances.

Accordingly, one can use the kinematic and interpretation suggested by EPS to constrain dynamics. In a Palatini or *metric-affine* formalism the metric and connection are completely unrelated *a priori*, so that only dynamics may give their reciprocal relations. Then field equations may force a relation between the metric and the connection. That is exactly what happens in vacuum standard Palatini GR: field equations force the connection to be the Levi-Civita connection of the given metric. The same happens with some specific kind of matter, but for general matter such a feature is generally lost, and in general the connection cannot be the one associated to the original metric.

However, EPS analysis shows that the connection and the metric cannot be completely arbitrary if one wants a theory that fits fundamental principles; in fact there must be a covector A_μ for which (6) holds true. If the compatibility condition is not already imposed at the kinematical level —for example writing the theory for the fundamental fields $(g_{\mu\nu}, A_\mu)$ instead of $(g_{\mu\nu}, \tilde{\Gamma}^\alpha_{\beta\mu})$— then the only option is that field equations impose the compatibility conditions *a posteriori* as a consequence of field equations.

We shall thence call *extended theory of gravitation* any field theory for independent variables $(g_{\mu\nu}, \tilde{\Gamma}^\alpha_{\beta\mu})$ in which field equations imply the compatibility condition (6) as a consequence. In these models the geometry of spacetime is described by a Weyl geometry.

Let us call *extended metric theory of gravitation* any extended theory of gravitation in which field equations imply dynamically that the connection is a metric connection, so that in that case the geometry of spacetime is described by a *metric* Weyl geometry.

We know a class of dynamics which are in fact extended metric theories of gravitation. As is well known, any Palatini $f(\mathcal{R})$-theory is in fact an extended metric theory of gravitation; see [8], [15], [19], [20], [21]. Standard GR is a specific extended metric theory of gravitation in which field equations imply that $A_\mu = 0$ (and then $\Gamma = \{g\}$).

Of course it is well known that general Weyl geometries may have unpleasant holonomy problems in the definition of length (namely, the length of a ruler depends on the path). However, metric Weyl geometries are not affected by these problems and they are still more general than standard GR as we shall discuss hereafter for $f(\mathcal{R})$-models.

With such theories rulers cannot change length when parallel transported, although one has to be careful to notice that the metric scales can change point by point because of conformal rescaling.

3.1. Palatini $f(\mathcal{R})$-theories

In order to fix notation let us briefly review a generic $f(\mathcal{R})$-theory with matter.

Let us consider a Lorentzian metric $g_{\mu\nu}$ and a torsionless connection $\tilde{\Gamma}^\alpha_{\beta\mu}$ on spacetime M of dimension $m > 2$. The conformal class $[g] = \mathfrak{C}$ of metrics defines light cones. The representative $g \in \mathfrak{C}$ is chosen to define distances. The connection $\tilde{\Gamma}$ is associated to free fall and it is chosen to be torsionless since geodesic equations is insensitive to torsion.

Let us remark that in this context conformal transformations are defined to be $\tilde{g}(x) = \varphi(x) \cdot g(x)$ and they leave the connection unchanged. One is *forced* to leave the connection unchanged (as it is possible in Palatini formalism) since our connection $\tilde{\Gamma}$ is uniquely selected to describe free fall (and by the projective gauge fixing $\nabla g = 2A \otimes g$). One could say that this definition of conformal transformations preserves the interpretation of fields.

Let us restrict our analysis to dynamics induced by a Lagrangian in the form

$$L = \sqrt{g} f(\mathcal{R}) + L_m(\phi, g) \tag{7}$$

where f is a generic (analytic or *sufficiently regular*) function, ϕ is a collection of matter fields and we set $\mathcal{R} := g^{\mu\nu}\tilde{R}_{\mu\nu}$.

With this choice we are implicitly assuming that matter fields ψ minimally couple with the metric g which in turn encodes electromagnetic properties (photons and light cones). Since gravity, according to EPS formalism, is mostly inherent with the equivalence principle and free fall, that is encoded in the projective structure $\mathfrak{P}$, one should better assume that matter couples *also* with $\tilde{\Gamma}$ and investigate the more general case in which the matter Lagrangian has the form $L_m(\psi, g, \tilde{\Gamma})$. However, this case in much harder to be investigated since it entails that a second stress tensor is generated by the variational derivative

$$\sqrt{g} T^{\mu\nu}_\alpha = \frac{\delta L_m}{\delta \tilde{\Gamma}^\alpha_{\mu\nu}} \tag{8}$$

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No relevant progress in this direction is still at hands, although it corresponds to an even more physically reasonable situation; only few concrete examples have been worked out insofar; see [15], [22], [23].

Let us remark that *a priori* the Ricci tensor $\tilde{R}_{\mu\nu}$ of the connection $\tilde{\Gamma}$ is not necessarily symmetric since the connection is not necessarily metric. As we said the matter Lagrangian L_m is here assumed to depend only on matter and metric (together with their derivatives up to order 1). Thus if one needs covariant derivatives of matter fields they are explicitly defined with respect to the metric field. Requiring that the matter Lagrangian does not depend on the connection $\tilde{\Gamma}$ is a standard requirement to simplify the analysis of field equation below although (as we said above) it would correspond to more reasonable physical situations. Let us here notice that what follows can be in fact extended to a more general framework; there are in fact matter Lagrangians depending on the connection $\tilde{\Gamma}$ in which field equations still imply the EPS-compatibility condition (6); see [15], [16], [17].

Field equations of (7) are

$$\begin{cases} f'(\mathcal{R})\tilde{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu} \\ \tilde{\nabla}_\alpha\left(\sqrt{g}f'(\mathcal{R})g^{\beta\mu}\right) = T^{\beta\mu}_\alpha = 0 \end{cases} \tag{9}$$

We do not write the matter field equations (which will be considered as matter equations of state). The constant $\kappa = 8\pi G/c^4$ is the coupling constant between matter and gravity. The second stress tensor $T^{\beta\mu}_\alpha$ vanishes since the matter Lagrangian is assumed to be independent of the connection $\tilde{\Gamma}$. The first stress tensor $T_{\mu\nu}$ arises since the matter Lagrangian is a function of the metric

$$\sqrt{g}T_{\mu\nu} = \frac{\delta L_m}{\delta g^{\mu\nu}} \tag{10}$$

Notice that $T_{\mu\nu}$ depends both on the matter fields and on the metric g.

Under these simplifying assumptions the second field equations can be solved explicitly. Let us consider in fact a conformal transformation $\tilde{g}_{\mu\nu} = (f'(\mathcal{R}))^{\frac{2}{m-2}} \cdot g_{\mu\nu}$ (with $m > 2$). One has

$$\tilde{g}^{\mu\nu} = (f'(\mathcal{R}))^{\frac{2}{2-m}} \cdot g^{\mu\nu} \qquad\qquad \sqrt{\tilde{g}} = (f'(\mathcal{R}))^{\frac{m}{m-2}}\sqrt{g} \tag{11}$$

and then

$$\sqrt{\tilde{g}}\tilde{g}^{\beta\mu} = \sqrt{g}f'(\mathcal{R}) \cdot g^{\beta\mu} \tag{12}$$

Thus the second field equation in (9) can be recast as

$$\tilde{\nabla}_\alpha\left(\sqrt{g}f'(\mathcal{R})g^{\beta\mu}\right) = \tilde{\nabla}_\alpha\left(\sqrt{\tilde{g}}\tilde{g}^{\beta\mu}\right) = 0 \tag{13}$$

which by the Levi-Civita theorem implies

$$\tilde{\Gamma}^\alpha_{\beta\mu} = \{\tilde{g}\}^\alpha_{\beta\mu} \tag{14}$$

i.e. the connection $\tilde{\Gamma}$ is the Levi-Civita connection of the conformal metric $\tilde{g}$. Thus in these theories the connection is *a posteriori* metric and the geometry of spacetime is described by a metric Weyl geometry. As a consequence the Ricci tensor $\tilde{R}_{\mu\nu}$ is symmetric being the Ricci tensor of the metric $\tilde{g}$.

The first field equation now reads as

$$f'(\mathcal{R})\tilde{R}_{\mu\nu} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu} \tag{15}$$

The trace of this equation (with respect to $g^{\mu\nu}$) is so important in the analysis of these models that it is called the *master equation*. It reads

$$f'(\mathcal{R})\mathcal{R} - \frac{m}{2}f(\mathcal{R}) = \kappa T := \kappa g^{\mu\nu}T_{\mu\nu} \tag{16}$$

For any given (analytic) function f, the master equation is an *algebraic* (i.e. not differential) equation between $\mathcal{R}$ and T. Assuming that $m \neq 2$ and excluding the degenerate case in which the following holds

$$f''(\mathcal{R})\mathcal{R} + \frac{2-m}{2}f'(\mathcal{R}) = 0 \quad \Rightarrow f'(\mathcal{R}) = \frac{m}{2}C_1\mathcal{R}^{\frac{m-2}{2}} \quad \Rightarrow f(\mathcal{R}) = C_1\mathcal{R}^{\frac{m}{2}} + C_2 \tag{17}$$

we see that the function $F(\mathcal{R}, T) := f'(\mathcal{R})\mathcal{R} - \frac{m}{2}f(\mathcal{R}) - \kappa T$ is also analytic and can be generically (i.e. except a discrete set of values for $\mathcal{R}$) solved for $\mathcal{R} = r(T) = \kappa\hat{r}(T)$.

In vacuum or for purely electromagnetic matter obeying Maxwell equations, one has $T = 0$, i.e. the trace T of $T_{\mu\nu}$ is zero and $\mathcal{R}$ takes a constant value from a discrete set $\mathcal{R} \equiv \rho \in \{\rho_0, \rho_1, \dots\}$ that of course depends on f. In this vacuum (as well as in purely electromagnetic) case the field equations simplify to

$$\tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} = \left(\frac{2-m}{2m}(f'(\rho))^{\frac{2}{2-m}}\rho\right)\tilde{g}_{\mu\nu} = \Lambda(\rho)\tilde{g}_{\mu\nu} \tag{18}$$

Accordingly, vacuum (or purely electromagnetic) Palatini $f(\mathcal{R})$-theories are generically equivalent to Einstein models with cosmological constant and the possible value of the cosmological constant is chosen in a discrete set which depends on the function f. This is known as *universality theorem* for Einstein equations (see [18]). The meaning of this result is not to be overestimated; the equivalence is important but one has a huge freedom in choosing the function f (which depends on countable infinite parameters) so that any value for the cosmological constant can be in principle attained. Let us stress once more that this includes all cases in which matter is present but the trace $T = 0$ as it happens for the electromagnetic field.

Accordingly, the physics described by Palatini $f(\mathcal{R})$-theories in vacuum is not richer than standard GR physics with cosmological constant. Still one should notice that in these vacuum $f(\mathcal{R})$-theories free fall is given by $\tilde{g}$ while in standard GR it is given by g (while distances

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are defined by g in both cases); however, the conformal factor $\varphi = (f'(\rho))^{\frac{2}{2-m}}$ is constant and it does not affect geodesics and it can be compensated by a change of units.

However, when *real matter* is present the situation is completely different. In this more general case, we have that $\mathcal{R} = r(T)$ depends on $x \in M$. The first field equation becomes then

$$\tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} = \kappa\left(\frac{1}{f'(r(T))}\left(T_{\mu\nu} - \frac{1}{m}Tg_{\mu\nu}\right) + \frac{2-m}{2m}\hat{r}(T)g_{\mu\nu}\right) = \kappa\tilde{T}_{\mu\nu} \qquad (19)$$

so that a Palatini $f(\mathcal{R})$-theory with *real* matter behaves like standard GR with a strongly modified source stress tensor. Naively speaking, one can reasonably hope that the modifications dictated by the choice of the function f can be chosen to fit observational data.

In a sense, whenever $T \neq 0$ the presence of standard visible matter ψ (assumed to generate, through the matter Lagrangian $L_m(g,\psi)$, an energy momentum stress tensor $T_{\mu\nu}$) would produce by gravitational interaction with $\tilde{\Gamma}$ (i.e. with the Levi-Civita connection of the conformal metric $\tilde{g} = f'(T)\cdot g$) a kind *effective* energy momentum stress tensor $\tilde{T}_{\mu\nu}$ in which standard matter ψ is seen to exist together with *dark (virtual)* matter generated by the gauging of the rulers imposed by the T-dependent conformal transformations on g. In a sense, the *dark side* of Einstein equations can be mimicked by suitably choosing f and L_m, as a curvature effect induced by $T = g^{\mu\nu}T_{\mu\nu} \neq 0$.

3.2. Equivalence with Brans-Dicke theories

Let us hereafter briefly review the mathematical equivalence between Palatini $f(\mathcal{R})$-theories and Brans-Dicke theories and discuss about how physical is such an equivalence. Let us hereafter restrict to the case in dimension $m = 4$.

A *Brans-Dicke* theory is a theory for a metric $g_{\mu\nu}$ and a scalar field φ. The dynamics is described by a Lagrangian in the following form

$$L_{BD} = \sqrt{g}\left[\varphi R - \frac{\omega}{\varphi}\nabla_\mu\varphi\nabla^\mu\varphi + U(\varphi)\right] + L_m(g,\psi) \qquad (20)$$

where ω is a real parameter and $U(\varphi)$ is a potential function.

Field equations for such a theory are

$$\begin{cases} \varphi(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = \kappa T_{\mu\nu} + \nabla_{\mu\nu}\varphi + \Box\varphi g_{\mu\nu} + \frac{\omega}{\varphi}\left(\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}\nabla_\alpha\varphi\nabla^\alpha\varphi g_{\mu\nu}\right) + \frac{1}{2}Ug_{\mu\nu} \\ R = \frac{\omega}{\varphi^2}\nabla_\alpha\varphi\nabla^\alpha\varphi - 2\frac{\omega}{\varphi}\Box\varphi - U'(\varphi) \end{cases} \qquad (21)$$

If one considers now the field equation (19) for a Palatini $f(\mathcal{R})$-theory and writes them for the original metric $g_{\mu\nu} = \varphi^{-1}\cdot\tilde{g}_{\mu\nu}$ and the conformal factor $\varphi = f'(\mathcal{R})$ field equation reads as

$$\varphi R_{\mu\nu} = \nabla_{\mu\nu}\varphi + \frac{1}{2}\Box\varphi g_{\mu\nu} - \frac{3}{2\varphi}\nabla_\mu\varphi\nabla_\nu\varphi + \frac{1}{4}\varphi\hat{r}(T)g_{\mu\nu} + \kappa\left(T_{\mu\nu} - \frac{1}{4}Tg_{\mu\nu}\right) \qquad (22)$$

while the master equation reads as

$$\varphi R = 3\Box\varphi - \frac{3}{2\varphi}\nabla_\alpha\varphi\nabla^\alpha\varphi + \kappa T + 2f \tag{23}$$

Within the framework for $f(\mathcal{R})$-theory, one can generically invert the definition of the conformal factor

$$\varphi = f'(\mathcal{R}) \quad \Rightarrow \mathcal{R} = \sigma(\varphi) \tag{24}$$

and define a potential function

$$U(\varphi) = -\varphi\sigma(\varphi) + f(\sigma(\varphi)) \qquad (U'(\varphi) = -\sigma'\varphi - \sigma + f'\sigma' = -\sigma) \tag{25}$$

Then one has a manifest correspondence between a Palatini $f(\mathcal{R})$-theory and a Brans-Dicke theory with the potential $U(\varphi) = -\varphi\sigma(\varphi) + f(\sigma(\varphi))$ and $\omega = -\frac{3}{2}$. This correspondence holds at the level of field equations (and solutions) but it can be shown at the level of action principles as well; see [20].

This equivalence is sometimes used against $f(\mathcal{R})$-theories since Brans-Dicke theories go to standard GR for $\omega \to \infty$ and the value $\omega = -\frac{3}{2}$ is ruled out by standard tests in the solar system, e.g. by precession of perihelia of Mercury. In view of the correspondence shown above the same tests would rule out $f(\mathcal{R})$-theories as well.

Letting aside the fact that tests rule out Brans-Dicke theories *without potential*, there is a further aspect that we believe is worth discussing here. In Brans-Dicke theory the gravitational interaction is mediated by a scalar field as well as the metric field. That means that g determines light cones, free fall and distances while the scalar field φ just participates to the dynamics.

In the corresponding Palatini $f(\mathcal{R})$-theory the metric g defines distances, it defines light cones (as well as $\tilde{g}$ does), but free fall is described by $\tilde{g}$ *not* by g!

The standard tests (as the precession of perihelia of Mercury) which rule out Brans-Dicke theories (see e.g. [24]) simply do not apply to the corresponding $f(\mathcal{R})$-theory since, in the two different models, Mercury moves along the geodesics of two different metrics. In Brans-Dicke theories it moves along the geodesics of a metric g which can be expanded in series of ω^{-1} around the standard Schwarzschild solution of standard GR; in the corresponding $f(\mathcal{R})$-theory it moves along geodesics of a different metric $\tilde{g}$ which, being in vacuum and in view of universality theorem, is a Schwarzschild-AdS solution; see also [25]. Since it is reasonable to assume a value for the cosmological constant which has no measurable effect at solar system scales, Mercury can be assumed move with good approximation along geodesics of the standard Schwarzschild metric and, despite the mathematical equivalence, $f(\mathcal{R})$-theories pass the tests while Brans-Dicke does not.

This is a pretty neat example in which a mathematical equivalence between two field theories is broken by the interpretation of the theories since the physical assumptions are not preserved by the transformation mapping one framework into the other; see [26], [27].

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4. Extended cosmologies

Let us now apply to a cosmological situation the discussion above for a general Palatini $f(\mathcal{R})$-model.

In a cosmological setting let us assume that the matter stress tensor $T_{\mu\nu}$ is the energy momentum tensor of a (perfect) fluid

$$T_{\mu\nu} := p g_{\mu\nu} + (p+\rho) u_\mu u_\nu \tag{26}$$

where $u^\alpha u^\beta g_{\alpha\beta} = -1$ and we set ρ for the fluid density and p for its pressure. Matter field equations are assumed to provide a relation between pressure and density under the form $p = w\rho$ for some (constant) w. Then one has

$$\begin{aligned} \tilde{T}_{\mu\nu} =& \frac{1}{f'}\left(T_{\mu\nu} - \frac{1}{m} T g_{\mu\nu}\right) + \frac{2-m}{2m}\hat{r}(T) g_{\mu\nu} = \\ =& (\rho+p)(f')^{\frac{m}{2-m}} \tilde{u}_\mu \tilde{u}_\nu + \left(\frac{p+\rho}{m}(f')^{\frac{m}{2-m}} - \frac{m-2}{2m}\hat{r}(f')^{\frac{2}{2-m}}\right)\tilde{g}_{\mu\nu} = \\ =& (\tilde{\rho}+\tilde{p})\tilde{u}_\mu \tilde{u}_\nu + \tilde{p}\tilde{g}_{\mu\nu} \end{aligned} \tag{27}$$

where we set $\tilde{u}_\mu = (f')^{\frac{1}{m-2}} u_\mu$ and

$$\begin{cases} \tilde{\rho} = \frac{m-1}{m}(p+\rho)(f')^{\frac{m}{2-m}} + \frac{m-2}{2m}\hat{r}(f')^{\frac{2}{2-m}} \\ \tilde{p} = \frac{p+\rho}{m}(f')^{\frac{m}{2-m}} - \frac{m-2}{2m}\hat{r}(f')^{\frac{2}{2-m}} \end{cases} \tag{28}$$

Thus the effect of a Palatini $f(\mathcal{R})$-dynamics is to modify the fluid tensor representing sources into another stress tensor which is again in the form of a (perfect) fluid, with modified pressure and density. This can be split quite naturally (though of course non-uniquely) into three fluids with

$$\begin{cases} \tilde{\rho}_1 = \rho \\ \tilde{p}_1 = p \end{cases} \qquad \begin{cases} \tilde{\rho}_2 = \frac{m-1}{m}(p+\rho)(f')^{\frac{m}{2-m}} - \rho \\ \tilde{p}_2 = \frac{p+\rho}{m}(f')^{\frac{m}{2-m}} - p \end{cases} \qquad \begin{cases} \tilde{\rho}_3 = \frac{m-2}{2m}\hat{r}(f')^{\frac{2}{2-m}} \\ \tilde{p}_3 = -\frac{m-2}{2m}\hat{r}(f')^{\frac{2}{2-m}} = -\tilde{\rho}_3 \end{cases} \tag{29}$$

The first fluid accounts for what we see as visible matter and it has standard equation of states $p_1 = w_1\rho_1$ with $w_1 = w$, i.e. the same state equation chosen for the visible matter. The third fluid has equation of states in the form $p_3 = w_3\rho_3$ with $w_3 = -1$, i.e. it is a quintessence field.

For the second fluid, taking into account the equation of state $p = w\rho$ of visible matter, one can set

$$\begin{cases} \tilde{\rho}_2 = \frac{m-1}{m}(f')^{\frac{m}{2-m}} p + \left(\frac{m-1}{m}(f')^{\frac{m}{2-m}} - 1\right)\rho = \left(\frac{m-1}{m}(f')^{\frac{m}{2-m}}(w+1) - 1\right)\rho \\ \tilde{p}_2 = \left(\frac{1}{m}(f')^{\frac{m}{2-m}} - 1\right) p + \frac{1}{m}(f')^{\frac{m}{2-m}}\rho = \left(\frac{1}{m}(f')^{\frac{m}{2-m}}(w+1) - w\right)\rho \end{cases} \tag{30}$$

which corresponds to an equation of state of the form $p_2 = w_2\rho_2$ with

$$w_2 = \frac{\frac{1}{m}(f')^{\frac{m}{2-m}}(w+1) - w}{\frac{m-1}{m}(f')^{\frac{m}{2-m}}(w+1) - 1} \tag{31}$$

Within the standard viewpoint this kind of matter is quite puzzling. Its equation of state is changing in time (since in cosmology $f'(r((m-1)p(t) - \rho(t)))$ is a function of time).

It is reasonable to assume that at present time visible matter is dominated by dust ($w = 0$) and $m = 4$, in which case we have

$$w_2^{\text{dust}} = \frac{1}{3 - 4(f')^2} \tag{32}$$

Of course, the splitting of the fluid is not canonical or unique. In particular the second fluid can be further split in different components (for example in order to isolate components which are dominant in various regimes).

This is probably the main reason to consider Palatini $f(\mathcal{R})$-theories as good as models also for cosmology: although we assumed only dust at fundamental level, from the gravitational viewpoint that behaves effectively as a more general fluid the characteristics of which depend on the extended gravitational theories chosen, i.e. on f.

Morever, let us also remark that this simple toy model can be easily tested and falsified by current data and it makes predictions about near future surveys. In the standard ΛCDM one assumes a cosmological constant Λ which is here modeled by the third fluid. Thus in order to fit data one has to fix the current value for f', which in turn fixes the current equation of state for the CDM dark matter which is also observed. Of course one can consider other reasonable models considering more realistic and finer descriptions of visible matter. Near future surveys will provide data about the evolution of the cosmological constant in time allowing in principle to observe $f'(t)$ directly.

Let us now set $m = 4$, $w = 0$ and impose the cosmological principle ansazt, i.e. homogeneity and isotropy. Again should we impose it for g or $\tilde{g}$? It is fortunate that this does not matter at all! If one does that for $\tilde{g}$ assuming the form

$$\tilde{g} = -d\tilde{t}^2 - \tilde{a}^2(\tilde{t})\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right)$$

then also g is homogeneous and isotropic, i.e. in the form

$$g = -dt^2 - a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right)$$

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provided one rescales the cosmological time with the conformal factor (which depends only on time)

$$d\tilde{t} = \sqrt{f'}\, dt \qquad \Rightarrow \tilde{t}(t) = \int \sqrt{f'}\, dt$$

and rescales the Friedmann-Lemaître-Robertson-Walker (FLRW) scale factor accordingly

$$\tilde{a}(\tilde{t}) = \sqrt{f'}\, a(t)$$

The equations for the scale factor are the celebrated Friedmann equations

$$\begin{cases} \frac{\dot{\tilde{a}}^2+K}{\tilde{a}^2} = \frac{\kappa}{3}\tilde{\rho} = \frac{\kappa}{12f'}\left(\hat{r}(\rho) + \frac{3\rho}{f'}\right) \\ \frac{\ddot{\tilde{a}}}{\tilde{a}} = -\frac{\kappa}{6}\left(\tilde{\rho} + 3\tilde{p}\right) = \frac{\kappa}{12f'}\left(\hat{r}(\rho) - \frac{3\rho}{f'}\right) \end{cases}$$

For a given f (and the associated $\hat{r}(\rho)$) these are two equations for the two unknowns $\tilde{a}(\tilde{t})$ and $\rho(t)$ which in principle should be determined as functions of t.

There is no much one can say in general without specifying f. Nevertheless, one can still notice that the worldlines $\gamma : s \mapsto (t_0 + s, r_0, \theta_0, \phi_0)$ are always geodesics (something that depends on the cosmological principle, not on Friedmann equations). Also the curves $\bar{\gamma} : s \mapsto (t_0, r_0 s, \theta_0, \phi_0)$ are geodesic trajectories and their length is thence related to spacial distances at time t_0.

Let us thus consider a point $(t_0, r_0, \theta_0, \phi_0)$ representing for example a galaxy, and let us suppose we want to compute its distance from us. If we defined distances by $\tilde{g}$ (as one would probably do in scalar tensor theories) such a distance would be given by

$$\tilde{d} = \tilde{a}(\tilde{t}_0) r_0 \int_0^1 \frac{ds}{\sqrt{1 - Ks^2 r_0^2}}$$

However, we defined distances by using g. Accordingly, one has

$$d = a(t_0) r_0 \int_0^1 \frac{ds}{\sqrt{1 - Ks^2 r_0^2}} = \frac{1}{\sqrt{f'}}\, \tilde{d}$$

Then these $f(\mathcal{R})$-theories have an extra time-dependent mismatch in measuring distances. Being the conformal factor dependent on time, it would affect non-trivially the measured acceleration of faraway galaxies.

To the best of our knowledge such a possible effect has not only been totally ignored in interpreting raw data, but it has not been discussed or proved to vanish.

For example, in this context the *measured* acceleration of the universe (which is defined to be the acceleration of galaxies per unit of distance) would be

$$\frac{\ddot{a}}{a} = \Phi^2 \frac{\ddot{\tilde{a}}}{\tilde{a}} - \frac{\dot{\Phi}}{\Phi}\frac{\dot{a}}{a} - \frac{\ddot{\Phi}}{\Phi} + \left(\frac{\dot{\Phi}}{\Phi}\right)^2$$

where we set $\Phi^2(t) = f'$ for the conformal factor.

It is therefore not difficult to find whole classes of functions f for which a solution in the FLRW form is allowed $\ddot{\tilde{a}}/\tilde{a}$ is negative (corresponding to an ever slower expansion) while $\ddot{a}/a$ is positive (corresponding to an accelerating expansion). When this were the case part of the effect of dark energy would be explained as a simple aberration of distance measurement.

Whether for some f this can fit experimental data better than the acceleration $\ddot{\tilde{a}}/\tilde{a}$ is something to be discussed on the observational ground. We just remark on a fundamental ground that extrapolating *our terrestrial current* rulers to 10 billion years ago and 10 billion light years away (in a theory in which geometry is dynamical and measurement protocols depends on all sorts of physical assumptions on the behavior of electromagnetic and gravitational fields) could be slightly hasty.

Of course we are not claiming these effect to be real. However, they are plausible and hence they should be considered in data analysis (and possibly eventually shown to be null). They were not introduced by *ad hoc* argument. On the contrary they are quite natural in metric extended theories of gravitation.

5. Conclusions and perspectives

The astrophysical and cosmological observations of the last decade clearly point to a deep reconsideration of standard scenarios based on standard GR, either on the source side or on the gravitational dynamics; or both. Basically, all observations about gravity in non-vacuum situations need to be somehow corrected.

If one decides to keep stuck to standard gravitational dynamics, then observations force us to modify the matter energy momentum tensor by adding dark sources. If one decides to modify gravity dynamics, the family of different available (covariant, variational, ...) theories is huge. Moreover, one variational model for gravity usually may support (for example when the model contains more than one metric) many inequivalent definitions of observational protocols. It is quite natural that in such a huge family one can find (many) models which fit the observations.

Thus usually one has to choose which of these two ways is preferable. In any event, one should reconsider foundations of gravitational theories from a more general perspective. EPS formalism provides us with such a reconsideration. It clearly shows that on spacetime coexist a conformal structure (associated to light rays and defining light cones and causality), a (torsionless) projective structure (associated to particles and free fall), and a metric structure (associated to our definition of clocks and rulers). These three structures can be assumed to be *a priori* independent, provided that dynamics forces *a posteriori* some compatibility conditions. The metric structure should also define the conformal structure and the projective structure can be represented by an affine (torsionless) connection so that lightlike geodesics of the metric structure are *also* autoparallel curves of the connection. This framework strongly

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constrains possible dynamics and it leads to *extended theories of gravitation*. Standard GR is a model within this extended family of *reasonable* gravitational models.

In extended gravitational theories one can also recast the fields so that an extended gravitational theory looks like standard GR with additional effective sources. This scenario thoroughly agrees with observations as long as dark matter and dark energy will be detected only through their gravitational effects. This scenario is reasonable also in view of cosmological observations which clearly suggest that the spacetime geometrical structure at large scale might be substantially different from the simple standard GR that we observe at Solar System scale.

Even if in the end standard GR were the correct theory and dark energy and matter will be understood at a fundamental level, this wider framework would be fundamental. It provides an extended framework in which one could test *directly* the assumptions of standard GR on an experimental basis without resorting to uncertain approximations.

In this paper we reviewed EPS formalism and defined *extended theories of gravitation* and *extended metric theories of gravitation*. Then we showed that Palatini $f(\mathcal{R})$-theories provide a family of such metric extended theories of gravitation.

If we restrict and apply $f(\mathcal{R})$-theories to cosmology we showed that matter naturally induces effective sources which can naturally modelled by fluid energy momentum source tensors which at least qualitatively present the main features of dark source models used phenomenologically to fit data. A (running) cosmological constant naturally emerges as well as a fluid with a running equation of states which depends explicitly of the $f(\mathcal{R})$ dynamics chosen. We also briefly discussed how one should define distances (as well as velocities and acceleration parameteres) in this extended framework.

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