

Nonlinear Observer-Based Control Allocation

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1. Introduction

Control allocation is the process of mapping virtual control inputs (such as torque and force) into actual actuator deflections in the design of control systems (Benosman et al., 2009; Bodson, 2002; Buffington et al., 1998; Liao et al., 2007; 2010). Essentially, it is considered as a constrained optimization problem as one usually wants to fully utilize all actuators in order to minimize power consumption, drag and other costs related to the use of control, subject to constraints such as actuator position and rate limits. In the design of control allocation, full state information is required. However, in practice, states may not be measurable. Hence, estimation of these unmeasurable states becomes inevitable.

The unmeasurable states are generally estimated based on available measurements and the knowledge of the physical system. For linear systems, the property of observability guarantees the existence of an observer. Luenberger or Kalman observers are known to give a systematic solution (Luenberger, 1964). In the case of nonlinear systems, observability in general depends on the input of the system. In other words, observability of a nonlinear system does not exclude the existence of inputs for which two distinct initial states generate identical measured outputs. Hence, in general, observer gains can be expected to depend on the applied input (Nijmeijer & Fossen, 1999). This makes the design of a nonlinear observer for a general nonlinear system a challenging problem. Although various results have been proposed over the past decades (Ahmed-Ali & Lamnabhi-Lagarrigue, 1999; Alami, 1999; Besancon, 2007; Besancon & Ticlea, 2007; Bestle & Zeitz, 1983; Bornard & Hammouri, 1991; Gauthier & Kupka, 1994; Krener & Isidori, 1983; Krener & Respondek, 1985; Michalska & Mayne, 1995; Nijmeijer & Fossen, 1999; Teel & Praly, 1994; Tsinias, 1989; 1990; Zimmer, 1994), none of them can claim to provide a general solution with the same convergence properties as in the linear case.

Over the past decades, a variety of methods have been developed for constructing nonlinear observers for nonlinear systems (Ahmed-Ali & Lamnabhi-Lagarrigue, 1999; Alami, 1999; Besancon, 2007; Besancon & Ticlea, 2007; Bestle & Zeitz, 1983; Bornard & Hammouri, 1991; Gauthier & Kupka, 1994; Krener & Isidori, 1983; Krener & Respondek, 1985; Michalska & Mayne, 1995; Nijmeijer & Fossen, 1999; Teel & Praly, 1994; Tsinias, 1989; 1990; Zimmer, 1994). They may be classified into optimization-based methods (Alami, 1999; Michalska

& Mayne, 1995; Zimmer, 1994) and feedback-based methods (Bestle & Zeitz, 1983; Bornard & Hammouri, 1991; Gauthier & Kupka, 1994; Krener & Isidori, 1983; Krener & Respondek, 1985; Teel & Praly, 1994; Tsiniias, 1989; 1990). Optimization-based methods obtain an estimate $\hat{x}(t)$ of the state $x(t)$ by searching for the best estimate $\hat{x}(0)$ of $x(0)$ (which can explain the evolution $y(\tau)$ over $[0, t]$) and integrating the deterministic nonlinear system from $\hat{x}(0)$ and under $u(\tau)$. These methods take advantage of their systematic formulation, but suffer from usual drawbacks of nonlinear optimization (like computation burden, local minima, and so on). Feedback-based methods can correct on-line the estimation $\hat{x}(t)$ from the error between the measurement output and the estimated output. These methods include linearization methods (Bestle & Zeitz, 1983; Krener & Isidori, 1983; Krener & Respondek, 1985), Lyapunov-based approaches (Tsiniias, 1989; 1990), sliding mode observer approaches (Ahmed-Ali & Lamnabhi-Lagarrigue, 1999) and high gain observer approaches (Bornard & Hammouri, 1991; Gauthier & Kupka, 1994; Teel & Praly, 1994), and so on. Among them, linearization methods (Krener & Isidori, 1983) transform nonlinear systems into linear systems by change of state variables and output injection. It is applicable to a special class of nonlinear systems. Sliding mode observer approaches (Ahmed-Ali & Lamnabhi-Lagarrigue, 1999) is to force the estimation error to join a stabilizing variety. The difficulty is to find a variety attainable and having this property. High gain observer approaches (Besancon, 2007) use the uniform observability and weight a gain based on the linear part so as to make the linear dynamics of the observer error to dominate the nonlinear one. Due to the requirement of the uniform observability, these approaches can only be applied to a class of nonlinear systems with special structure. Interestingly, Lyapunov-based approaches (Tsiniias, 1989; 1990) provide a general sufficient Lyapunov condition for the observer design of a general class of nonlinear systems and the proposed observer is a direct extension of Luenberger observer in linear case.

In this chapter, we extend the control allocation approach developed in (Benosman et al., 2009; Liao et al., 2007; 2010) from state feedback to output feedback and adopt the Lyapunov-type observer for a general class of nonlinear systems in (Tsiniias, 1989; 1990) to estimate the unmeasured states. Sufficient Lyapunov-like conditions in the form of the dynamic update law are proposed for the control allocation design via output feedback. The proposed approach ensures that the estimation error and its rate converge exponentially to zero as $t \rightarrow +\infty$ and the closed-loop system exponentially converges to the stable reference model as $t \rightarrow +\infty$. The advantage of the proposed approach is that it is applicable to a wide class of nonlinear systems with unmeasurable states, and it is computational efficiency as it is not necessary to optimize the control allocation problem exactly at each time instant.

This chapter is organized as follows. In Section 2, the observer-based control allocation problem is formulated where the control allocation design is based on the estimated states which exponentially converge to the true states as $t \rightarrow +\infty$. In Section 3, the main result of the observer-based control allocation design is presented in the form of dynamic update law. An illustrative example is given in Section 4, followed by some conclusions in Section 5.

Throughout this chapter, given a real map $f(\mathbf{v}, \mathbf{w})$, $(\mathbf{v}, \mathbf{w}) \in \mathbb{R}^n \times \mathbb{R}^m$, $D_{\mathbf{v}}f(\mathbf{v}_0, \mathbf{w}_0)$ denotes its derivative with respect to \mathbf{v} at the point $(\mathbf{v}_0, \mathbf{w}_0)$. For given real map $h(\mathbf{v})$ with $\mathbf{v} \in \mathbb{R}^n$, $Dh(\mathbf{v}_0)$ denotes its derivative with respect to \mathbf{v} at the point \mathbf{v}_0 . In addition, $\|\cdot\|$ represent the induced 2-norm.

2. Problem formulation

Consider the following nonlinear system:

$$\begin{cases} \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) \\ \mathbf{y} = h(\mathbf{x}) \end{cases} \tag{1}$$

where $\mathbf{x} \in \mathcal{X} \subset \mathbb{R}^n$ is the state vector with \mathcal{X} a open subset of \mathbb{R}^n , $\mathbf{y} \in \mathbb{R}^l$ is the measurement output vector, and $\mathbf{u} \in \mathbb{R}^m$ is the control input vector satisfying the constraints

$$\mathbf{u} \in \Omega \triangleq \left\{ \mathbf{u} = [u_1 \ u_2 \ \dots \ u_m]^T \mid \underline{u}_i \leq u_i \leq \bar{u}_i, \ i = 1, 2, \dots, m \right\} \tag{2}$$

with $\underline{\mathbf{u}} = [\underline{u}_1 \ \underline{u}_2 \ \dots \ \underline{u}_m]^T$ and $\bar{\mathbf{u}} = [\bar{u}_1 \ \bar{u}_2 \ \dots \ \bar{u}_m]^T$ being vectors of lower and upper control limits, respectively.

We assume that the system (1) satisfies the following assumption:

Assumption 1. *The function $f(\mathbf{x}, \mathbf{u})$ is smooth and the output function $h(\mathbf{x})$ is continuously differentiable.*

Since control allocation need full state information, the state estimation for the system (1) is required.

Consider a dynamic observer of the following form

$$\dot{\hat{\mathbf{x}}} = f(\hat{\mathbf{x}}, \mathbf{u}) - \Phi(\hat{\mathbf{x}}, \mathbf{u})[\mathbf{y} - h(\hat{\mathbf{x}})] \tag{3}$$

Define the error \mathbf{e} as

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}} \tag{4}$$

To estimate the state \mathbf{x} , we wish to design the mapping $\Phi(\hat{\mathbf{x}}, \mathbf{u})$ such that the trajectory of \mathbf{e} with the dynamics

$$\dot{\mathbf{e}} = f(\mathbf{x}, \mathbf{u}) - f(\hat{\mathbf{x}}, \mathbf{u}) + \Phi(\hat{\mathbf{x}}, \mathbf{u})[\mathbf{y} - h(\hat{\mathbf{x}})] \tag{5}$$

exponentially converges to zero as $t \rightarrow +\infty$, uniformly on $\mathbf{u} \in \Omega$, for every $\mathbf{x}(0)$ subject to $\mathbf{e}(0) = \mathbf{x}(0) - \hat{\mathbf{x}}(0)$ near zero.

The aim is to design a nonlinear control allocation law based on the state observer (3) such that a reference model that represents a predefined dynamics of the closed-loop system is tracked subject to the control constraint $\mathbf{u} \in \Omega$.

Given that the predefined dynamics of the closed-loop system is described by the following asymptotically stable reference model

$$\dot{\mathbf{x}} = \mathbf{A}_d \mathbf{x} + \mathbf{B}_d \mathbf{r} \tag{6}$$

where $\mathbf{A}_d \in \mathbb{R}^{n \times n}$, $\mathbf{B}_d \in \mathbb{R}^{n \times n_r}$ and the reference $\mathbf{r} \in \mathbb{R}^{n_r}$ satisfy the following assumption.

Assumption 2. *\mathbf{A}_d is Hurwitz, and $\mathbf{r} \in \Sigma \subset \mathbb{R}^{n_r}$ is continuously differentiable where Σ is an open subset defined by: for each $\mathbf{r} \in \Sigma$, there exist $\mathbf{x} \in \mathcal{X}$ and $\mathbf{u} \in \Omega$ such that the system (1) matches the reference system (6).*

Since the state \mathbf{x} is unmeasurable, the control allocation design is then based on its estimate $\hat{\mathbf{x}}$. In other words, we have to first choose the mapping $\Phi(\hat{\mathbf{x}}, \mathbf{u})$ in (3) such that the estimation error \mathbf{e} exponentially converges to zero as $t \rightarrow +\infty$, uniformly on $\mathbf{u} \in \Omega$, for every $\mathbf{x}(0) \in \mathcal{X}$ subject to $\mathbf{e}(0)$ near zero; then minimize the cost function

$$J(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{u}) = \frac{1}{2} \mathbf{u}^T \mathbf{H}_1 \mathbf{u} + \frac{1}{2} \tau^T(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{u}) \mathbf{H}_2 \tau(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{u}) \quad (7)$$

where $\mathbf{H}_1 \in \mathbb{R}^{m \times m}$ and $\mathbf{H}_2 \in \mathbb{R}^{n \times n}$ are positive definite weighting matrices, and

$$\tau(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{u}) \triangleq f(\hat{\mathbf{x}}, \mathbf{u}) - \mathbf{A}_d \hat{\mathbf{x}} - \mathbf{B}_d \mathbf{r} \quad (8)$$

is the matching error between the actual dynamics and desired dynamics. Since power consumption minimization introduced by the term $\frac{1}{2} \mathbf{u}^T \mathbf{H}_1 \mathbf{u}$ is a secondary objective, we choose $\|\mathbf{H}_1\| \ll \|\mathbf{H}_2\|$.

Now the control allocation problem is formulated in terms of solving the following nonlinear static minimization problem:

$$\begin{aligned} & \min_{\mathbf{u}} J(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{u}) \quad \text{subject to} \\ & \mathbf{u} \in \Omega \quad \text{and } \hat{\mathbf{x}} \text{ converges to } \mathbf{x} \text{ exponentially} \end{aligned} \quad (9)$$

Define

$$\Delta(\mathbf{u}) = [S(u_1) \ S(u_2) \ \cdots \ S(u_m)] \quad (10)$$

with

$$S(u_i) = \min((u_i - \underline{u}_i)^3, (\bar{u}_i - u_i)^3, 0), i = 1, 2, \dots, m \quad (11)$$

Then the constraint condition $\mathbf{u} \in \Omega$ is equivalent to

$$\Delta(\mathbf{u}) = 0 \quad (12)$$

Introduce the Lagrangian

$$L(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{u}, \lambda) = J(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{u}) + \Delta(\mathbf{u}) \lambda \quad (13)$$

where $\lambda \in \mathbb{R}^m$ is a Lagrange multiplier. And assume that

Assumption 3. *There exists a constant $\gamma_1 > 0$ such that $\frac{\partial^2 L}{\partial \mathbf{u}^2} \geq \gamma_1 \mathbf{I}_m$.*

The following lemma is immediate ((Wismer & Chattergy, 1978), p. 42).

Lemma 1. *If Assumptions 1 and 3 hold, the Lagrangian (13) achieves a local minimum if and only if $\frac{\partial L}{\partial \lambda} = 0$ and $\frac{\partial L}{\partial \mathbf{u}} = 0$.*

Proof. Necessity: The necessary condition is obvious. *Sufficiency:* Since $\frac{\partial L}{\partial \lambda} = 0$, we have $\Delta(\mathbf{u}) = 0$. In this case, the Lagrangian (13) is independent of the Lagrange multiplier λ , which achieves a local minimum if $\frac{\partial L}{\partial \mathbf{u}} = 0$ and $\frac{\partial^2 L}{\partial \mathbf{u}^2} > 0$. As $\frac{\partial^2 L}{\partial \mathbf{u}^2} > 0$ is guaranteed by Assumption 3, thus, $\frac{\partial L}{\partial \lambda} = 0$ and $\frac{\partial L}{\partial \mathbf{u}} = 0$ implies the local minimum. The proof is completed. \square

Remark 1. *It should be noted that Assumption 3 is satisfied if all control inputs are within their limits (i.e., $\frac{\partial L}{\partial \lambda} = \Delta^T(\mathbf{u}) = 0$) and the nonlinear system (1) is affine in control (i.e., $f(\mathbf{x}, \mathbf{u}) = f_1(\mathbf{x}) + g(\mathbf{x})\mathbf{u}$). It is because, in this case, $\frac{\partial^2 L}{\partial \mathbf{u}^2} = \mathbf{H}_1 + g^T(\mathbf{x})\mathbf{H}_2g(\mathbf{x})$ is positive definite matrix for $\mathbf{H}_1 > 0$ and $\mathbf{H}_2 > 0$. Furthermore, since the Lagrangian (13) is convex in this case, Lemma 1 holds for a global minimum.*

To solve the control allocation problem (9) with the state estimate $\hat{\mathbf{x}}$ from the observer (3), we consider the following control Lyapunov-like function

$$V(\hat{\mathbf{x}}, \mathbf{e}, \mathbf{r}, \mathbf{u}, \lambda) = V_m(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{u}, \lambda) + \frac{1}{2} \mathbf{e}^T \mathbf{P} \mathbf{e} \tag{14}$$

where $\mathbf{P} > 0$ is a known positive-definite matrix and

$$V_m(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{u}, \lambda) = \frac{1}{2} \left[\left(\frac{\partial L}{\partial \mathbf{u}} \right)^T \frac{\partial L}{\partial \mathbf{u}} + \left(\frac{\partial L}{\partial \lambda} \right)^T \frac{\partial L}{\partial \lambda} \right] \tag{15}$$

Here the function V_m is designed to attract (\mathbf{u}, λ) so as to minimize the Lagrangian (13). The term $\frac{1}{2} \mathbf{e}^T \mathbf{P} \mathbf{e}$ forms a standard Lyapunov-like function for observer estimation error \mathbf{e} which is required to exponentially converge to zero as $t \rightarrow +\infty$.

Following the observer design in (Tsinias, 1989), we define a neighborhood Q of zero with $Q \subset \mathcal{X}$, a neighborhood W of \mathcal{X} with $\{\mathbf{x} - \mathbf{e} : \mathbf{x} \in \mathcal{X}, \mathbf{e} \in Q\} \subset W$, and a closed ball S of radius $r > 0$, centered at zero, such that $S \subset Q$. Then define the boundary of S as ∂S . Figure 1 illustrates the geometrical relationship of these defined sets.

Let \mathcal{H} denote the set of the continuously differentiable output mappings $h(\mathbf{x}) : \mathcal{X} \rightarrow \mathbb{R}^l$ such that for every $\mathbf{m}_0 \in Q$ and $\hat{\mathbf{x}} \in W$,

$$R(\hat{\mathbf{x}}, \mathbf{m}_0) \geq 0 \tag{16}$$

and

$$\ker R(\hat{\mathbf{x}}, \mathbf{m}_0) \subset \ker Dh(\hat{\mathbf{x}}) \tag{17}$$

where

$$R(\hat{\mathbf{x}}, \mathbf{m}_0) \triangleq [Dh(\hat{\mathbf{x}})]^T Dh(\hat{\mathbf{x}} + \mathbf{m}_0) + [Dh(\hat{\mathbf{x}} + \mathbf{m}_0)]^T Dh(\hat{\mathbf{x}}) \tag{18}$$

Remark 2. *Obviously, every linear map $\mathbf{y} = \mathbf{H}\mathbf{x}$ belongs to \mathcal{H} . Furthermore, \mathcal{H} contains a wide family of nonlinear mappings.*

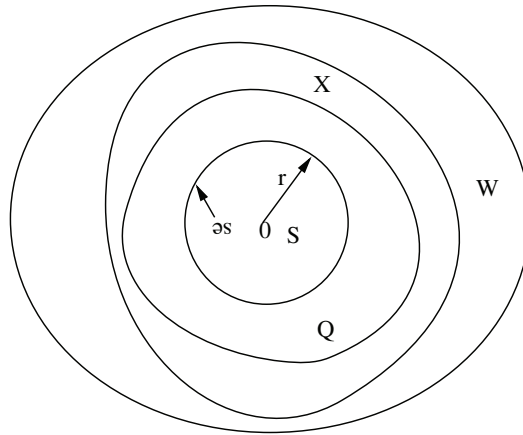


Fig. 1. Geometrical representation of sets

We assume that

Assumption 4. $h(\mathbf{x})$ in the system (1) belongs to the set \mathcal{H} , namely, $h(\mathbf{x}) \in \mathcal{H}$.

Further, we define

$$N \triangleq \left\{ \mathbf{e} \in \mathbb{R}^{n_x} \mid \mathbf{e}^T \mathbf{P} D_{\mathbf{x}} f(\hat{\mathbf{x}} + \mathbf{m}_1, \mathbf{u}) \mathbf{e} \leq -k_0 \|\mathbf{e}\|^2 \right\} \tag{19}$$

and assume that

Assumption 5. There exist a positive definite matrix $\mathbf{P} \in \mathbb{R}^{n_x \times n_x}$ and a positive constant k_0 such that $\ker Dh(\hat{\mathbf{x}}) \subset N$ holds for any $(\hat{\mathbf{x}}, \mathbf{m}_1, \mathbf{u}) \in W \times Q \times \Omega$.

Remark 3. Assumption 5 ensures that the estimation error system (5) is stable in the case of $h(\mathbf{x}) = h(\hat{\mathbf{x}})$ and $\mathbf{x} \neq \hat{\mathbf{x}}$. In particular, for linear systems, the condition in Assumption 5 is equivalent to detectability.

3. Main results

Denote

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 L}{\partial \mathbf{u}^2} & \frac{\partial^2 L}{\partial \lambda \partial \mathbf{u}} \\ \frac{\partial^2 L}{\partial \mathbf{u} \partial \lambda} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \frac{\partial L}{\partial \mathbf{u}} \\ \frac{\partial L}{\partial \lambda} \end{bmatrix} \tag{20}$$

and define

$$M \triangleq \left\{ \nu \in \mathbb{R}^{n_x} \mid \nu = r \|\mathbf{e}\|^{-1} \mathbf{e}, \mathbf{e} \in N \cap S \right\} \tag{21}$$

Let

$$\gamma_1(\hat{\mathbf{x}}, \mathbf{u}) = \max \left\{ r^2 (\|\mathbf{P}\| \|D_{\mathbf{x}} f(\hat{\mathbf{x}} + \mathbf{m}_1, \mathbf{u})\| + k_0), \mathbf{m}_1 \in S, (\hat{\mathbf{x}}, \mathbf{u}) \in W \times \Omega \right\} \tag{22}$$

$$\gamma_2(\hat{\mathbf{x}}) = \min \left\{ \frac{1}{2} \nu^T R(\hat{\mathbf{x}}, \mathbf{m}_0) \nu, \mathbf{m}_0 \in S, \nu \in \partial S - M, \hat{\mathbf{x}} \in W \right\} \tag{23}$$

Theorem 1. Consider the system (1) with $\mathbf{x} \in \mathcal{X}$ and $\mathbf{u} \in \Omega$. Suppose that Assumptions 1-5 are satisfied. For a given asymptotically stable matrix \mathbf{A}_d and a matrix \mathbf{B}_d , given symmetric positive-definite matrices Γ_1 and Γ_2 , and a given positive constants ω , for $\mathbf{e}(0)$ near zero, $\left(\frac{\partial L}{\partial \lambda}, \frac{\partial L}{\partial \mathbf{u}}, \mathbf{e}\right)$ exponentially converges to zero as $t \rightarrow +\infty$, and the dynamics of the nonlinear system (1) exponentially converges to that of the stable system (6) if the following dynamic update law

$$\begin{cases} \dot{\mathbf{u}} = -\Gamma_1 \alpha + \zeta_1 \\ \dot{\lambda} = -\Gamma_2 \beta + \zeta_2 \end{cases} \tag{24}$$

and the observer system

$$\dot{\hat{\mathbf{x}}} = f(\hat{\mathbf{x}}, \mathbf{u}) - \Phi(\hat{\mathbf{x}}, \mathbf{u}) [\mathbf{y} - h(\hat{\mathbf{x}})] \tag{25}$$

are adopted. Here $\alpha, \beta \in \mathbb{R}^m$ are as in (20), and $\zeta_1, \zeta_2 \in \mathbb{R}^m$ satisfy

$$\alpha^T \zeta_1 + \beta^T \zeta_2 + \delta + \omega V_m = 0 \tag{26}$$

with V_m as in (15) and

$$\delta = \left(\frac{\partial L}{\partial \mathbf{u}}\right)^T \frac{\partial^2 L}{\partial \mathbf{r} \partial \mathbf{u}} \dot{\mathbf{r}} + \left(\frac{\partial L}{\partial \mathbf{u}}\right)^T \frac{\partial^2 L}{\partial \hat{\mathbf{x}} \partial \mathbf{u}} \dot{\hat{\mathbf{x}}} \tag{27}$$

and the mapping

$$\Phi(\hat{\mathbf{x}}, \mathbf{u}) = -\theta(\hat{\mathbf{x}}, \mathbf{u}) \mathbf{P}^{-1} [Dh(\hat{\mathbf{x}})]^T \tag{28}$$

where

$$\theta(\hat{\mathbf{x}}, \mathbf{u}) \geq \frac{\gamma_1(\hat{\mathbf{x}}, \mathbf{u})}{\gamma_2(\hat{\mathbf{x}})} > 0 \tag{29}$$

with $\gamma_1(\hat{\mathbf{x}}, \mathbf{u}) > 0$ and $\gamma_2(\hat{\mathbf{x}}) > 0$ defined as in (22) and (23).

Proof. From the Lyapunov-like function (14), we obtain its time derivative as

$$\begin{aligned} \dot{V} = & \left[\left(\frac{\partial L}{\partial \mathbf{u}}\right)^T \frac{\partial^2 L}{\partial \mathbf{u}^2} + \left(\frac{\partial L}{\partial \lambda}\right)^T \frac{\partial^2 L}{\partial \mathbf{u} \partial \lambda} \right] \dot{\mathbf{u}} + \left(\frac{\partial L}{\partial \mathbf{u}}\right)^T \frac{\partial^2 L}{\partial \lambda \partial \mathbf{u}} \dot{\lambda} \\ & + \left(\frac{\partial L}{\partial \mathbf{u}}\right)^T \frac{\partial^2 L}{\partial \mathbf{r} \partial \mathbf{u}} \dot{\mathbf{r}} + \left(\frac{\partial L}{\partial \mathbf{u}}\right)^T \frac{\partial^2 L}{\partial \hat{\mathbf{x}} \partial \mathbf{u}} \dot{\hat{\mathbf{x}}} + \mathbf{e}^T \mathbf{P} \dot{\mathbf{e}} \end{aligned} \tag{30}$$

Substituting $\dot{\mathbf{e}}$ in (5), α and β as in (20) and δ as in (27) into (30), we have

$$\dot{V} = \alpha^T \dot{\mathbf{u}} + \beta^T \dot{\lambda} + \delta + \mathbf{e}^T \mathbf{P} \{f(\mathbf{x}, \mathbf{u}) - f(\hat{\mathbf{x}}, \mathbf{u}) + \Phi(\hat{\mathbf{x}}, \mathbf{u})[\mathbf{y} - h(\hat{\mathbf{x}})]\} \tag{31}$$

Consider $\mathbf{e} \in S$. Since S is convex, according to Mean Value Theorem, there exists $\mathbf{m}_0, \mathbf{m}_1 \in S$ satisfying

$$f(\mathbf{x}, \mathbf{u}) - f(\hat{\mathbf{x}}, \mathbf{u}) = D_x f(\hat{\mathbf{x}} + \mathbf{m}_1, \mathbf{u}) \mathbf{e} \tag{32}$$

$$\mathbf{y} - h(\hat{\mathbf{x}}) = Dh(\hat{\mathbf{x}} + \mathbf{m}_0) \mathbf{e} \tag{33}$$

Then substituting (24), (26), (32) and (33) into (31), we obtain

$$\dot{V} = -\alpha^T \Gamma_1 \alpha - \beta^T \Gamma_2 \beta - \omega V_m + \mathbf{e}^T \mathbf{P} [D_{\mathbf{x}} f(\hat{\mathbf{x}} + \mathbf{m}_1, \mathbf{u}) + \Phi(\hat{\mathbf{x}}, \mathbf{u}) Dh(\hat{\mathbf{x}} + \mathbf{m}_0)] \mathbf{e} \quad (34)$$

After substituting $\Phi(\hat{\mathbf{x}}, \mathbf{u})$ as in (28) and $R(\hat{\mathbf{x}}, \mathbf{m}_0)$ as in (18), (34) can be rewritten as

$$\dot{V} = -\alpha^T \Gamma_1 \alpha - \beta^T \Gamma_2 \beta - \omega V_m + \mathbf{e}^T \mathbf{P} D_{\mathbf{x}} f(\hat{\mathbf{x}} + \mathbf{m}_1, \mathbf{u}) \mathbf{e} - \frac{\theta(\hat{\mathbf{x}}, \mathbf{u})}{2} \mathbf{e}^T R(\hat{\mathbf{x}}, \mathbf{m}_0) \mathbf{e} \quad (35)$$

Since the matrices $\Gamma_1 > 0$ and $\Gamma_2 > 0$, we have

$$\dot{V} \leq -\omega V_m + \mathbf{e}^T \mathbf{P} D_{\mathbf{x}} f(\hat{\mathbf{x}} + \mathbf{m}_1, \mathbf{u}) \mathbf{e} - \frac{\theta(\hat{\mathbf{x}}, \mathbf{u})}{2} \mathbf{e}^T R(\hat{\mathbf{x}}, \mathbf{m}_0) \mathbf{e} \quad (36)$$

For $\mathbf{e} = 0$ where \mathbf{x} is determined by the observer accurately, we have

$$\dot{V} \leq -\omega V_m = -\omega V \quad (37)$$

Since $\omega > 0$, V exponentially converges to zero as $t \rightarrow +\infty$. Hence, $\left(\frac{\partial L}{\partial \lambda}, \frac{\partial L}{\partial \mathbf{u}}\right)$ exponentially converges to zero.

For any nonzero $\mathbf{e} \in S$, let $v = r \|\mathbf{e}\|^{-1} \mathbf{e}$. Obviously, $v \in \partial S$. Then we have

$$\dot{V} \leq -\omega V_m + \frac{1}{r^2} \|\mathbf{e}\|^2 v^T \mathbf{P} D_{\mathbf{x}} f(\hat{\mathbf{x}} + \mathbf{m}_1, \mathbf{u}) v - \frac{\theta(\hat{\mathbf{x}}, \mathbf{u})}{2r^2} \|\mathbf{e}\|^2 v^T R(\hat{\mathbf{x}}, \mathbf{m}_0) v \quad (38)$$

In the following, we shall show that V converges exponentially to zero for all $\mathbf{m}_0, \mathbf{m}_1 \in S$, $\hat{\mathbf{x}} \in W$, $\mathbf{u} \in \Omega$, $\mathbf{e} \in S$, $\mathbf{e} \neq 0$ and $v \in \partial S$.

First let us consider nonzero $\mathbf{e} \in N \cap S$. From $v = r \|\mathbf{e}\|^{-1} \mathbf{e}$, we have $v \in M$. Since $\mathbf{m}_0, \mathbf{m}_1 \in S \subset Q$, $\hat{\mathbf{x}} \in W$ and $\mathbf{u} \in \Omega$, according to Assumptions 1-5, it follows that

$$v^T R(\hat{\mathbf{x}}, \mathbf{m}_0) v = 0 \quad (39)$$

and

$$v^T \mathbf{P} D_{\mathbf{x}} f(\hat{\mathbf{x}} + \mathbf{m}_1, \mathbf{u}) v \leq -k_0 \|v\|^2 \quad (40)$$

with the constant $k_0 > 0$. From (38), we have

$$\dot{V} \leq -\omega V_m - k_0 \|\mathbf{e}\|^2 \leq -\sigma V \quad (41)$$

with the constant $\sigma > 0$. Hence, $\left(\frac{\partial L}{\partial \lambda}, \frac{\partial L}{\partial \mathbf{u}}, \mathbf{e}\right)$ exponentially converges to zero as $t \rightarrow +\infty$.

Then we consider nonzero $\mathbf{e} \in S - N \cap S$, namely, $v \in \partial S - M$. From (38), taking into account (22)-(23), we obtain

$$\dot{V} \leq -\omega V_m + \frac{1}{r^2} \|\mathbf{e}\|^2 \left[\gamma_1(\hat{\mathbf{x}}, \mathbf{u}) - k_0 r^2 - \theta(\hat{\mathbf{x}}, \mathbf{u}) \gamma_2(\hat{\mathbf{x}}) \right] \quad (42)$$

Since $\theta(\hat{\mathbf{x}}, \mathbf{u})$ satisfy the condition (29), we obtain (41) again. Hence, in this case, $\left(\frac{\partial L}{\partial \lambda}, \frac{\partial L}{\partial \mathbf{u}}, \mathbf{e}\right)$ also exponentially converges to zero as $t \rightarrow +\infty$.

Since $\left(\frac{\partial L}{\partial \lambda}, \frac{\partial L}{\partial \mathbf{u}}, \mathbf{e}\right)$ exponentially converges to zero as $t \rightarrow +\infty$, the closed-loop system exponentially converges to

$$\begin{cases} \dot{\hat{\mathbf{x}}} = \mathbf{A}_d \hat{\mathbf{x}} + \mathbf{B}_d \mathbf{r} \\ \dot{\mathbf{e}} = \{D_x f(\hat{\mathbf{x}} + \mathbf{m}_1, \mathbf{u}) - \theta(\hat{\mathbf{x}}, \mathbf{u}) \mathbf{P}^{-1} [Dh(\hat{\mathbf{x}})]^T Dh(\hat{\mathbf{x}} + \mathbf{m}_0)\} \mathbf{e} \end{cases} \tag{43}$$

Since \mathbf{A}_d is a asymptotically stable matrix, we know that $\hat{\mathbf{x}} \in W$ is bounded. According to Assumptions 1 and 4, $D_x f(\hat{\mathbf{x}} + \mathbf{m}_1, \mathbf{u})$, $Dh(\hat{\mathbf{x}})$ and $Dh(\hat{\mathbf{x}} + \mathbf{m}_0)$ are all bounded for $\mathbf{m}_0, \mathbf{m}_1 \in S$ and $\mathbf{u} \in \Omega$. From $k_0 > 0$, we have $0 < \gamma_1(\hat{\mathbf{x}}, \mathbf{u}) < +\infty$. According to Assumption 4, we have $\ker R(\hat{\mathbf{x}}, \mathbf{m}_0) \subset \ker Dh(\hat{\mathbf{x}})$ which ensures that $0 < v^T R(\hat{\mathbf{x}}, \mathbf{m}_0) v < +\infty$ for every $v \in \partial S - M$, $\mathbf{m}_0 \in S$ and $\hat{\mathbf{x}} \in W$. Thus, we have $0 < \gamma_2(\hat{\mathbf{x}}) < +\infty$. As a result, $0 < \theta(\hat{\mathbf{x}}, \mathbf{u}) < +\infty$. From (43), we know that $\dot{\mathbf{e}}$ exponentially converges to zero as \mathbf{e} exponentially converges to zero. Moreover, we have

$$\dot{\mathbf{x}} - \dot{\mathbf{e}} = \mathbf{A}_d \mathbf{x} - \mathbf{A}_d \mathbf{e} + \mathbf{B}_d \mathbf{r} \tag{44}$$

Since $\dot{\mathbf{e}}$ and \mathbf{e} exponentially converges to zero, we have the system (1) exponentially converges to $\dot{\mathbf{x}} = \mathbf{A}_d \mathbf{x} + \mathbf{B}_d \mathbf{r}$. This completes the proof. \square

Consider now the issue of solving (26) with respect to ξ_1 and ξ_2 . One method to achieve a well-defined unique solution to the under-determined algebraic equation is to solve a least-square problem subject to (26). This leads to the Lagrangian

$$l(\xi_1, \xi_2, \rho) = \frac{1}{2}(\tilde{\xi}_1^T \tilde{\xi}_1 + \tilde{\xi}_2^T \tilde{\xi}_2) + \rho(\alpha^T \tilde{\xi}_1 + \beta^T \tilde{\xi}_2 + \delta + \omega V_m) \tag{45}$$

where $\rho \in \mathbb{R}$ is a Lagrange multiplier. The first order optimality conditions

$$\frac{\partial l}{\partial \tilde{\xi}_1} = 0, \quad \frac{\partial l}{\partial \tilde{\xi}_2} = 0, \quad \frac{\partial l}{\partial \rho} = 0 \tag{46}$$

leads to the following system of linear equations

$$\begin{bmatrix} I_m & 0 & \alpha \\ 0 & I_m & \beta \\ \alpha^T & \beta^T & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \rho \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\delta - \omega V_m \end{bmatrix} \tag{47}$$

Remark 4. It is noted that Equation (47) always has a unique solution for ξ_1 and ξ_2 if any one of α and β is nonzero.

4. Example

Consider the pendulum system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin x_1 + u_1 \cos x_1 + u_2 \sin x_1 \end{bmatrix} \tag{48}$$

$$y = x_1 + x_2 \tag{49}$$

with $\mathbf{x} = [x_1 \ x_2]^T \in \mathbb{R}^2$, $\mathbf{u} = [u_1 \ u_2]^T \in \Omega$ and

$$\Omega \triangleq \left\{ \mathbf{u} = [u_1 \ u_2]^T \mid -1 \leq u_1 \leq 1, \ -0.5 \leq u_2 \leq 0.5 \right\} \quad (50)$$

As the system is affine in control and its measurement output y is a linear map of its state \mathbf{x} , Assumptions 1, 3 and 4 are satisfied automatically.

Choose

$$\mathbf{P} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

For $\mathbf{e} \neq 0$ and $\mathbf{e} \in \ker[1 \ 1]$, we have $e_1 = -e_2$ and

$$\begin{aligned} & \mathbf{e}^T \mathbf{P} D_{\mathbf{x}} f(\mathbf{x}, u) \mathbf{e} |_{e_1 = -e_2} \\ &= [e_1 \ e_2] \begin{bmatrix} 0 & 3 \\ -\cos x_1 - u_1 \sin x_1 + u_2 \cos x_1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} |_{e_1 = -e_2} \\ &= (-\cos x_1 - u_1 \sin x_1 + u_2 \cos x_1 + 3) e_1 e_2 |_{e_1 = -e_2} \\ &\leq [-1.5 \cos(\arctan \frac{2}{3}) - \sin(\arctan \frac{2}{3}) + 3] e_1 e_2 |_{e_1 = -e_2} \\ &= (-1.8028 + 3) e_1 e_2 |_{e_1 = -e_2} \\ &= -0.5986 \|\mathbf{e}\|^2 |_{e_1 = -e_2} \\ &< -k_0 \|\mathbf{e}\|^2 |_{e_1 = -e_2} \end{aligned}$$

with $0 < k_0 < 0.5986$. Hence, Assumption 5 is satisfied. Let S be the ball of radius $r = 1$, centered at zero and ∂S is the boundary of S . Define $M \subset \partial S$ and

$$M = \left\{ v = [v_1 \ v_2]^T \in \mathbb{R}^2 : \|v\| = 1, \ 3v_1 v_2 + 1.8028|v_1 v_2| < -k_0 \right\}$$

Obviously,

$$\partial S - M = \left\{ v = [v_1 \ v_2]^T \in \mathbb{R}^2 : \|v\| = 1, \ 3v_1 v_2 + 1.8028|v_1 v_2| \geq -k_0 \right\}$$

As $\gamma_1(\hat{\mathbf{x}}, u) = 3 \times 1.8028 + k_0$ and

$$\gamma_2(\hat{\mathbf{x}}) = \min \left\{ (v_1 + v_2)^2, \ v \in \partial S - M \right\} = 1 - \frac{2k_0}{3 - 1.8028}$$

choosing $k_0 = 0.5$, we have $\frac{\gamma_1(\hat{\mathbf{x}}, \mathbf{u})}{\gamma_2(\hat{\mathbf{x}})} = 35.8699$. Let $\theta(\hat{\mathbf{x}}, \mathbf{u}) = 36 > 35.8699$ and we have $\Phi(\hat{\mathbf{x}}, \mathbf{u}) = -[12 \ 36]^T$.

Now the nonlinear observer becomes

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} \hat{x}_2 \\ -\sin \hat{x}_1 + u_1 \cos \hat{x}_1 + u_2 \sin \hat{x}_1 \end{bmatrix} + \begin{bmatrix} 12 \\ 36 \end{bmatrix} (y - \hat{x}_1 - \hat{x}_2)$$

Choose the reference model (6) where

$$\mathbf{A}_d = \begin{bmatrix} 0 & 1 \\ -25 & -10 \end{bmatrix}, \quad \mathbf{B}_d = \begin{bmatrix} 0 \\ 25 \end{bmatrix}$$

and the reference is given by

$$r = \begin{cases} r_f \left[6 \left(\frac{t}{t_1} \right)^5 - 15 \left(\frac{t}{t_1} \right)^4 + 10 \left(\frac{t}{t_1} \right)^3 \right], & 0 \leq t < t_1 \\ r_f, & t_1 \leq t < t_2 \\ -r_f \left[6 \left(\frac{t-t_2}{t_f-t_2} \right)^5 - 15 \left(\frac{t-t_2}{t_f-t_2} \right)^4 + 10 \left(\frac{t-t_2}{t_f-t_2} \right)^3 \right] + r_f, & t_2 \leq t < t_f \\ 0, & t \geq t_f \end{cases}$$

with $t_1 = 10s, t_2 = 20s, t_f = 30s$ and $r_f = 0.5$. Obviously, Assumption 2 is satisfied.

Set $H_1 = 0, H_2 = 10^{-4}I_2, \omega = 1, \Gamma_1 = \Gamma_2 = 2I_2$, and $x_1(0) = 0.3$ and $x_2(0) = 0.5$. Using the proposed approach, we have the simulation result of the pendulum system (48)-(50) shown in Figures 2-5 where the control u_2 is stuck at -0.5 from $t = 12s$ onward.

From Figure 2, it is observed that the estimated states \hat{x}_1 and \hat{x}_2 converge to the actual states x_1 and x_2 and match the desired states x_{1d} and x_{2d} well, respectively, even when u_2 is stuck at -0.5 . This observation is further verified by Figure 3 where both the state estimation errors $e_1(= x_1 - \hat{x}_1)$ and $e_2(= x_2 - \hat{x}_2)$ of the nonlinear observer as in (4) and the matching errors $\tau_1(= 0)$ and $\tau_2(= -\sin \hat{x}_1 + u_1 \cos \hat{x}_1 + u_2 \sin \hat{x}_1 + 25\hat{x}_1 + 10\hat{x}_2 - 25r)$ as in (8) exponentially converge to zero. Moreover, Figure 4 shows that the control u_1 roughly satisfies the control constraint $u_1 \in [-1, 1]$ while the control u_2 strictly satisfies the control constraint $u_2 \in [-0.5, 0.5]$. This is because, in this example, the Lagrange multiplier λ_1 is first activated by the control $u_1 < -1$ at $t = 0$ (see Figure 5 where λ_1 is no longer zero from $t = 0$),

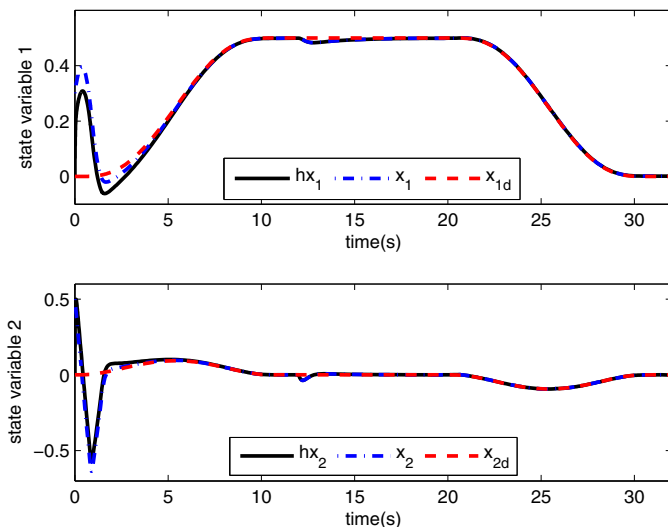


Fig. 2. Responses of the desired, estimated and actual states

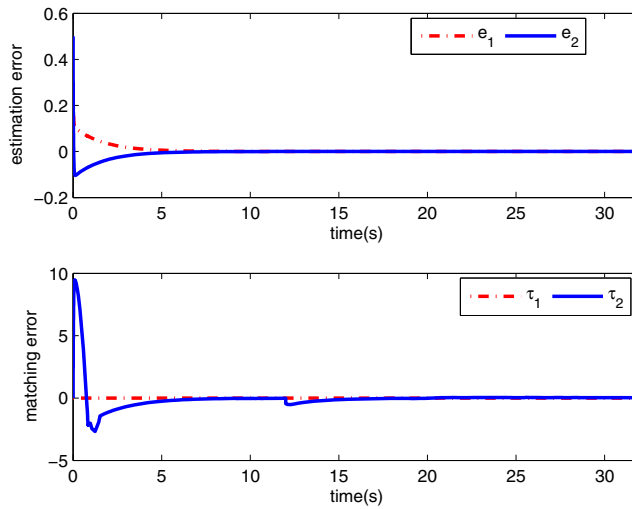


Fig. 3. Responses of estimation error and matching error

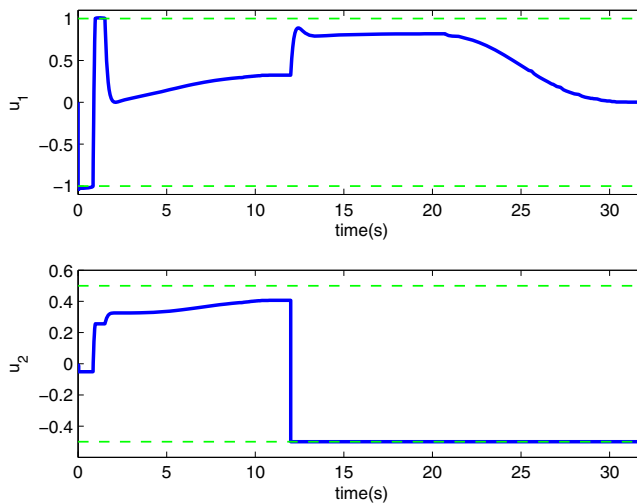


Fig. 4. Responses of control u

and then the proposed dynamic update law forces the control u_1 to satisfy the constraint $u_1 \in [-1, 1]$. It is also noted from Figure 5 that the Lagrange multiplier λ_2 is not activated in this example as the control u_2 is never beyond the range $[-0.5, 0.5]$. In addition, the output y and the Lyapunov-like function V_m are shown in Figure 6. From Figure 6, it is observed that the Lyapunov-like function V_m exponentially converges to zero.

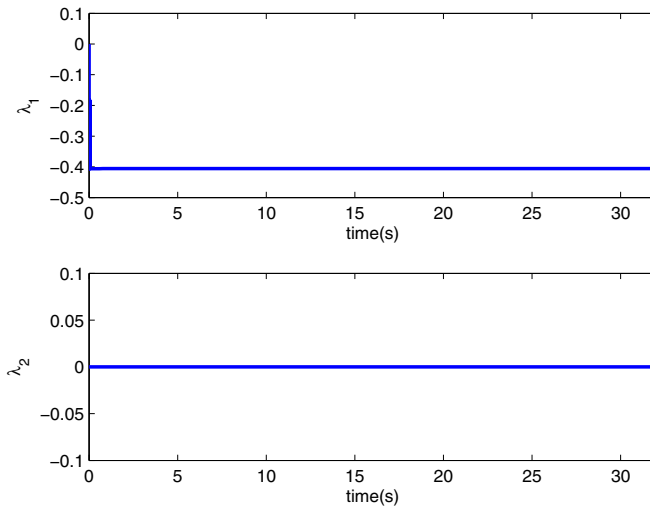


Fig. 5. Responses of Lagrangian multiplier λ

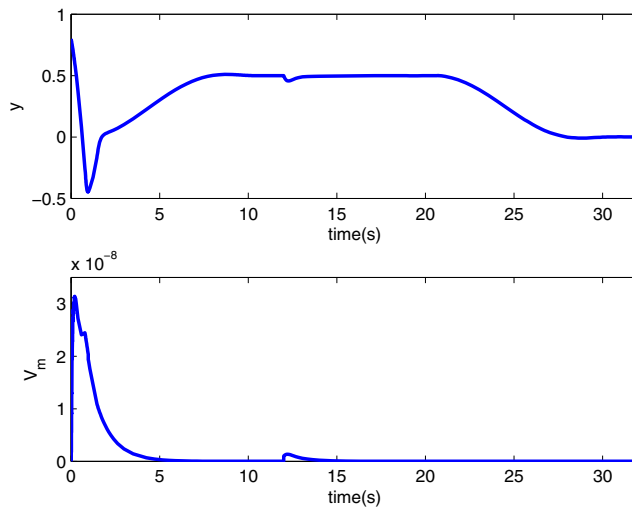


Fig. 6. Responses of output y and Lyapunov-like function V_m

5. Conclusions

Sufficient Lyapunov-like conditions have been proposed for the control allocation design via output feedback. The proposed approach is applicable to a wide class of nonlinear systems. As the initial estimation error $e(0)$ need be near zero and the predefined dynamics of the

closed-loop is described by a linear stable reference model, the proposed approach will present a local nature.

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