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# Free Vibration Analysis of Centrifugally Stiffened Non Uniform Timoshenko Beams

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## 1. Introduction

Rotating beams – like structures are widely used in many engineering fields and are of great interest as they can be used to model blades of wind turbines, helicopter rotors, robotic manipulators, turbo-machinery and aircraft propellers. The governing differential equations of motion in free vibration of a non-uniform rotating Timoshenko beam, with general elastic restraints at the ends are solved using the differential quadrature method, (Bellman & Roth, 1986; Felix et al., 2008, 2009). The equations of motion are derived to include the effects of shear deformation, rotary inertia, hub radius, ends elastically restrained and non-uniform variation of the cross-sectional area of the beam. The presence of a centrifugal force due to the rotational motion is considered as Banerjee has developed, using Hamilton's principle to capture the centrifugal stiffening arising in fast rotating structures, (Banerjee, 2001). With the proposed model, a great number of different situations are admitted to be solved. Particular cases with classical restraints can be deduced for limiting values of the rigidities. Also step changes in cross-section are considered (Naguleswaran, 2004).

The natural vibration frequencies and mode shapes of rotating beams have been a topic of interest and have received considerable attention. A large number of researchers have studied the dynamic behavior of rotating uniform or tapered Euler-Bernoulli beams. (Yang el al., 2004; Özdemir & Kaya, 2006; Lin & Hsiao, 2001). Banerjee derived the dynamic stiffness matrix of a rotating Bernoulli-Euler beam using the Frobenius method of solution in power series and he includes the presence of an axial force at the outboard end of the beam in addition to the existence of the usual centrifugal (Banerjee, 2000).

Not so many studies have tackled the problem of rotating beams taking into account rotary inertia, shear deformation and their combined effects, hub radius and ends elastically restrained, (Bambill et al., 2010). In applications where the rotary inertia and the shear deformation effects are not significant, an analysis based on the Euler–Bernoulli beam theory can be used. However, Timoshenko theory allows describing the vibration of short beams, sandwich composite beams or high modes of a slender beam, (Rossi et al., 1991; Seon et al., 1999). (Banerjee et al., 2006) investigated the free bending vibration of rotating tapered Timoshenko beams by the dynamic stiffness method. (Ozgumus & Kaya, 2010) used the Differential Transform Method for free vibration analysis of a rotating, tapered Timoshenko beam.

The finite element method was used by (Hodges & Rutkowski, 1981). (Vinod et al., 2007) presented a study about spectral finite element formulation for a rotating beam subjected to small duration impact. (Gunda & Ganguli, 2008) developed a new beam finite element whose basis functions were obtained by the exact solution of the governing static homogenous differential equation of a stiff string, which resulted from an approximation in the rotating beam equation. (Singh et al., 2007) used the Genetic Programming to create an approximate model of rotating beams. (Gunda et al., 2007) introduced a low degree of freedom model for dynamic analysis of rotating tapered beams based on a numerically efficient superelement, developed using a combination of polynomials and Fourier series as shape functions. (Kumar & Ganguli, 2009) looked for rotating beams whose eigenpair, frequency and mode-shape, is the same as that of uniform non rotating beams for a particular mode. An interesting paper (Ganesh & Ganguli, 2011) presented physics based basis function for vibration analysis of high speed rotating beams using the finite element method. The basis function gave rise to shape functions which depend on position of the element in the beam, material, geometric properties and rotational speed of the beam.

The present study tries to provide not only solutions for practical engineering situations but they also may be useful as benchmark for comparing other numerical models. The proposed differential quadrature method, offers a useful and accurate procedure for the solution of linear and non linear partial differential equations. It was used by Bellman in the 1970's. He used this method to calculate the natural frequencies of transverse vibration of a rotating cantilever beam. (Bellman & Casti, 1971). Other authors have used the differential quadrature method and recognized it as an effective technique for solving this kind of problems, (Bert & Malik, 1996; Shu & Chen, 1999; Choi et al., 2000; Liu & Wu, 2001; Shu, 2000).

Numerical results are obtained for the natural frequencies of transverse vibration and the mode shapes of rotating beams considering the elastic restraints, with non uniform variation of the cross-sectional area. Some of those cases have also been solved using the finite element method, and the sets of results are in excellent agreement.

## 2. Theory

Figure 1 shows the rotating tapered beam considered in the present paper. The beam could have step jumps in cross section and rotates at speed $\overline{\eta}$. The $\overline{X}$-axis coincides with the centroidal axis of the beam, the $\overline{Y}$-axis is parallel with the axis of rotation and the $\overline{Z}$-axis lies in the plane of rotation. $L$ is the length of the beam, $L_k$ is the length of the segment $k$ and $L_d$ is the length of the last segment of the beam. The displacement in the $\overline{Y}$ direction is denoted as $\overline{w}$ and the section rotation is denoted as $\overline{\psi}$. Only displacements in the $\overline{X}-\overline{Y}$ plane are taken into account and the Coriolis effects are not considered.

The centrifugal force of a beam element at a distance $\overline{R}_k+\overline{x}_k$ from the axis of rotation can be expressed as

$$d\overline{F}_k=\overline{\eta}^2\,(\overline{R}_k+\overline{x}_k)\,dm \tag{1}$$

where $dm=\rho\, A_k(\overline{x}_k)\,d\overline{x}_k$ is its mass, with $\rho$ the mass density of material, and $A_k(\overline{x}_k)$, is the cross-sectional area at $\overline{x}_k$. Figure 2. The centrifugal force $\overline{N}_k(\overline{x}_k)$ generated by $\overline{\eta}$ is

$$d\overline{N}_k(\overline{x}_k)=\overline{\eta}^2\rho\,(\overline{R}_k+\overline{x}_k)\,A_k(\overline{x}_k)\,d\overline{x}_k \tag{2}$$

The total axial force at the cross section located at $\bar{R}_k + \bar{x}_k$ is

$$\bar{N}_k(\bar{x}_k) = \bar{\eta}^2 \rho \int_{\bar{x}_k}^{L_k} (\bar{R}_k + \bar{x}_k)\, A_k(\bar{x}_k)\, d\bar{x}_k + \bar{F}_{k+1} = \bar{\eta}^2 \rho \left( \bar{R}_k \int_{\bar{x}_k}^{L_k} A_k(\bar{x}_k)\, d\bar{x}_k + \int_{\bar{x}_k}^{L_k} A_k(\bar{x}_k)\bar{x}_k\, d\bar{x}_k \right) + \bar{F}_{k+1} \quad (3)$$

$\bar{F}_{k+1}$ is the outboard force at the end of the segment $k$ , due to the adjacent segments $k$+1 to $d$.

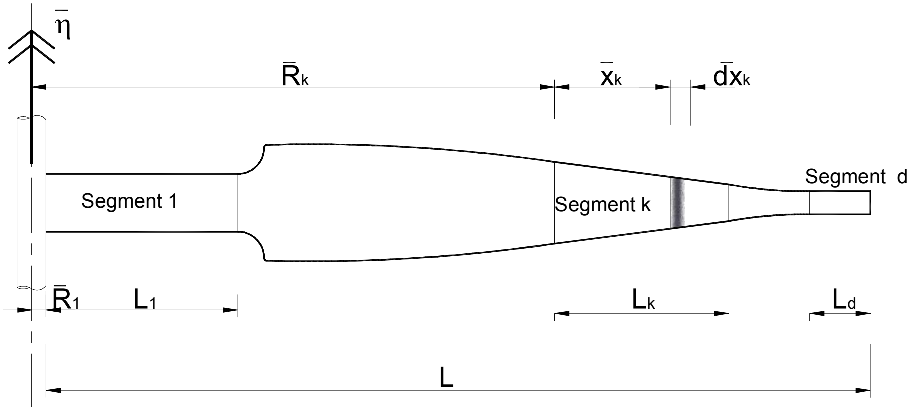



Fig. 1. Rotating beam model

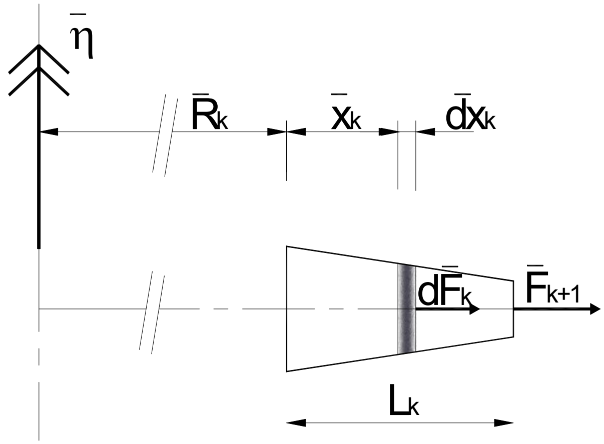



Fig. 2. Rotating beam segment $k$ of length $L_k$

Finally, the tensile force can be written as

$$\bar{N}_k(\bar{x}_k) = \bar{\eta}^2 \rho \left( \bar{R}_k V_k(L_k) + \Phi_k(L_k) - \bar{R}_k V_k(\bar{x}_k) - \Phi_k(\bar{x}_k) \right) + \bar{F}_{k+1} \quad (4)$$

with

$$V_k(\bar{x}_k) = \int_0^{\bar{x}_k} A_k(\bar{x}_k)\, d\bar{x}_k \ ; \ \Phi_k(\bar{x}_k) = \int_0^{\bar{x}_k} A_k(\bar{x}_k)\,\bar{x}_k\, d\bar{x}_k \tag{5a,b}$$

The expressions for shear force and bending moment at an instant $t$ in the rotating beam are

$$\bar{Q}_k^*(\bar{x}_k,t) = \bar{N}_k(\bar{x}_k)\frac{\partial \bar{w}_k(\bar{x}_k,t)}{\partial \bar{x}_k} + \kappa GA_k(\bar{x}_k)\left(\frac{\partial \bar{w}_k(\bar{x}_k,t)}{\partial \bar{x}_k} - \bar{\psi}_k(\bar{x}_k,t)\right) \tag{6}$$

$$\bar{M}_k^*(\bar{x}_k,t) = EI_k(\bar{x}_k)\frac{\partial \bar{\psi}_k(\bar{x}_k,t)}{\partial \bar{x}_k} \tag{7}$$

where $I_k(\bar{x}_k)$ is the second moment of area of the beam cross-section; $t$ the time; $\bar{w}_k(\bar{x},t)$ the transverse displacement; $\bar{\psi}_k(\bar{x},t)$ the section rotation; $E$ the Young's modulus; $\nu$ the Poisson's ratio; $G = E/2(1+\nu)$ the shear modulus and $\kappa$ is the shear factor.

The governing differential equations of motion of a rotating Timoshenko beams (Banerjee, 2001) are:

$$\frac{\partial \bar{Q}_k^*(\bar{x}_k,t)}{\partial \bar{x}_k} = \rho A_k(\bar{x}_k)\frac{\partial^2 \bar{w}_k(\bar{x}_k,t)}{\partial t^2}$$
$$\bar{Q}_k^*(\bar{x}_k,t) - N_k(\bar{x}_k)\frac{\partial \bar{w}_k(\bar{x}_k,t)}{\partial \bar{x}_k} + \frac{\partial \bar{M}_k^*(\bar{x}_k,t)}{\partial \bar{x}_k} + \rho I_k(\bar{x}_k)\bar{\eta}^2\bar{\psi}_k(\bar{x}_k,t) = \rho I_k(\bar{x}_k)\frac{\partial^2 \bar{\psi}_k(\bar{x}_k,t)}{\partial t^2} \tag{8a,b}$$

Assuming simple harmonic oscillation

$$\bar{w}_k(\bar{x}_k,t) = \bar{W}_k(\bar{x}_k)e^{i\omega t} \quad ; \quad \bar{\psi}_k(\bar{x}_k,t) = \bar{\Psi}_k(\bar{x}_k)e^{i\omega t} \tag{9a,b}$$

where $\omega$ is the circular frequency in radian per second.
The bending moment and the shear force are expressed as

$$\bar{Q}_k^*(\bar{x}_k,t) = \bar{Q}_k(\bar{x}_k)e^{i\omega t} ; \ \bar{M}_k^*(\bar{x}_k,t) = \bar{M}_k(\bar{x}_k)e^{i\omega t} \tag{10a,b}$$

where

$$\bar{Q}_k(\bar{x}_k) = \left(\bar{N}_k(\bar{x}_k) + \kappa GA_k(\bar{x}_k)\right)\frac{d\bar{W}_k(\bar{x}_k)}{d\bar{x}_k} - \kappa GA_k(\bar{x}_k)\bar{\Psi}_k(\bar{x}_k) \ ; \bar{M}_k(\bar{x}_k) = EI_k(\bar{x}_k)\frac{d\bar{\Psi}_k(\bar{x}_k)}{d\bar{x}_k} \tag{11a,b}$$

Substituting equations (9-10) into equations (8), the equations of motion for the free vibration of the segment $k$ of the rotating beam result in:

$$-\frac{d\bar{Q}_k(\bar{x}_k)}{d\bar{x}_k} = \rho A_k(\bar{x}_k)\,\omega^2\bar{W}_k(\bar{x}_k)$$
$$-\bar{Q}_k(\bar{x}_k) + \bar{N}_k(\bar{x}_k)\frac{d\bar{W}_k(\bar{x}_k)}{d\bar{x}_k} - \frac{d\bar{M}_k(\bar{x}_k)}{d\bar{x}_k} - \rho I_k(\bar{x}_k)\bar{\eta}^2\,\bar{\Psi}_k(\bar{x}_k) = \rho I_k(\bar{x}_k)\,\omega^2\,\bar{\Psi}_k(\bar{x}_k) \tag{12a,b}$$

Replacing equations (11) into equations (12), the differential equations of motion become:

$$-\frac{d\bar{N}_k(\bar{x}_k)}{d\bar{x}_k}\frac{d\bar{W}_k(\bar{x}_k)}{d\bar{x}_k}-\bar{N}_k(\bar{x}_k)\frac{d^2\bar{W}_k(\bar{x}_k)}{d\bar{x}_k^2}-\kappa G A_k(\bar{x}_k)\left(\frac{d^2\bar{W}_k(\bar{x}_k)}{d\bar{x}_k^2}-\frac{d\bar{\Psi}_k(\bar{x}_k)}{d\bar{x}}\right)-$$
$$\kappa G\frac{dA_k(\bar{x}_k)}{d\bar{x}_k}\left(\frac{d\bar{W}_k(\bar{x}_k)}{d\bar{x}_k}-\bar{\Psi}_k(\bar{x}_k)\right)=\rho A_k(\bar{x}_k)\,\omega^2\bar{W}_k(\bar{x}_k)$$

$$-\kappa G A_k(\bar{x}_k)\left(\frac{d\bar{W}_k(\bar{x}_k)}{d\bar{x}_k}-\bar{\Psi}_k(\bar{x}_k)\right)-EI_k(\bar{x}_k)\frac{d^2\bar{\Psi}_k(\bar{x}_k)}{d\bar{x}_k^2}-$$
$$E\frac{dI_k(\bar{x}_k)}{d\bar{x}_k}\frac{d\bar{\Psi}_k(\bar{x}_k)}{d\bar{x}_k}-\rho I_k(\bar{x}_k)\bar{\eta}^2\,\bar{\Psi}_k(\bar{x}_k)=\rho I_k(\bar{x}_k)\,\omega^2\,\bar{\Psi}_k(\bar{x}_k) \quad (13a,b)$$

The term $\rho I_k(\bar{x}_k)\bar{\eta}^2\bar{\Psi}_k(\bar{x}_k)$ included in equation (13.b) was introduced by Banerjee, 2001. This term generates more realistic results especially for high rotational speeds, $\bar{\eta}^2$ .

The conditions for displacements and forces between adjacent segments, *k* and *k*+1, are:

$$\bar{W}_k(L_k)-\bar{W}_{k+1}(0)=0\,;\;\; \bar{\Psi}_k(L_k)-\bar{\Psi}_{k+1}(0)=0 \quad (14a,b)$$

$$\bar{Q}_k(L_k)-\bar{Q}_{k+1}(0)=0\,;\;\; \bar{M}_k(L_k)-\bar{M}_{k+1}(0)=0 \quad (15a,b)$$

Figure 3 shows the beam elastically restrained at both ends.

The boundary conditions of the beam at its ends are, for the first segment *k*=1, at $\overline{x_1}=0$ :

$$\bar{Q}_1(0)-\bar{K}_{W1}\;\;\bar{W}_1(0)=0\,;\;\; \bar{M}_1(0)-\bar{K}_{\Psi 1}\;\;\bar{\Psi}_1(0)=0 \quad (16a,b)$$

and for the last segment *k*=*d*, at $\overline{x_d}=L_d$ :

$$\bar{Q}_d(L_d)-\bar{K}_{Wd}\;\;\bar{W}_d(0)=0\,;\;\; \bar{M}_d(L_d)-\bar{K}_{\Psi d}\;\bar{\Psi}_d(0)=0 \quad (17a,b)$$

The four spring constants are denoted as: $\bar{K}_{W1}, \bar{K}_{Wd}, \bar{K}_{\Psi 1}, \bar{K}_{\Psi d}$ .

The expressions and parameters in dimensionless form are defined as follows:

$$\Omega^2=\frac{\rho A_1(0)}{EI_1(0)}L^4\omega^2;\quad \eta^2=\frac{\rho A_1(0)}{EI_1(0)}L^4\bar{\eta}^2\,;$$

$$x=\frac{\bar{x}_k}{L_k};\, l_k=\frac{L_k}{L};\, r_k^2=\frac{I_k(0)}{A_k(0)};\, s_k=\frac{L}{r_k}\,;\;\, R_k=\frac{\bar{R}_k}{L};\, W_k(x)=\frac{\bar{W}_k(\bar{x}_k)}{L_k};\, \Psi_k(x)=\bar{\Psi}_k(\bar{x}_k)\,;$$

$$a_k(x)=\frac{A_k(\bar{x}_k)}{A_k(0)}\;;\; b_k(x)=\frac{I_k(\bar{x}_k)}{I_k(0)}\;;\; v_k(x)=\frac{V_k(\bar{x}_k)}{l_k\,A_k(0)}\;;\; \phi_k(x)=\frac{\Phi_k(\bar{x}_k)}{l_k^2\,A_k(0)}\;;$$

$$N_{k+1}=\frac{\bar{F}_{k+1}}{EA_k(0)};\quad N_k(x)=\frac{\bar{N}_k(\bar{x}_k)}{EA_k(0)};\quad Q_k(x)=\frac{\bar{Q}_k(\bar{x}_k)}{EA_k(0)};\quad M_k(x)=\frac{L_k}{EI_k(0)}\bar{M}_k(\bar{x}_k)\,;$$

$$\text{and } K_{Wj} = \overline{K}_{Wj} \frac{L}{EA_1(0)}; K_{\Psi j} = \overline{K}_{\Psi j} \frac{L}{EI_1(0)} \text{ ; with } j = 1 \text{ or } j = d.$$

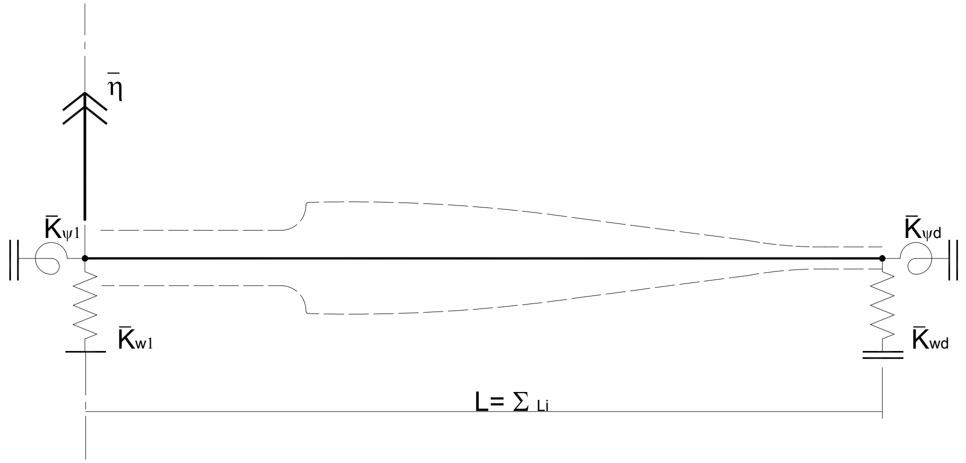



Fig. 3. Elastic restraints of the rotating beam

In each segment $k$ of the beam, $x$ varies between 0 and 1.

The axial force, the shear force and the bending moment in the adimensional form become:

$$N_k(x) = \eta^2 \frac{l_k^2}{s_1^2}\left(R_k\, v_k(1) + \phi_k(1) - R_k\, v_k(x) - \phi_k(x)\right) + N_{k+1};\ \text{with } s_1 = \frac{L}{r_1}\ ;\ r_1^2 = \frac{I_1(0)}{A_1(0)} \quad (18)$$

$$Q_k(x) = \left(N_k(x) + \frac{\kappa}{2(1+\nu)} a_k(x)\right)\frac{dW_k(x)}{dx} - \frac{\kappa}{2(1+\nu)} a_k(x)\Psi_k(x) \quad (19)$$

$$M_k(x) = b_k(x)\frac{d\Psi_k(x)}{dx} \quad (20)$$

And the equations of motion in dimensionless form are:

$$\eta^2 a_k(x)\left(R_k + x\right)\frac{dW_k(x)}{dx} - \frac{s_1^2}{l_k^2} N_k(x)\frac{d^2W_k(x)}{dx^2} - \frac{\kappa}{2(1+\nu)}\frac{s_1^2}{l_k^2} a_k(x)\left(\frac{d^2W_k(x)}{dx^2} - \frac{d\Psi_k(x)}{dx}\right) - \\ - \frac{\kappa}{2(1+\nu)}\frac{s_1^2}{l_k^2}\frac{da_k(x)}{dx}\left(\frac{dW_k(x)}{dx} - \Psi_k(x)\right) = \Omega^2\, a_k(x) W_k(x) \quad (21)$$

$$-s_1^2 \frac{\kappa}{2(1+\nu)} s_k^2\, a_k(x)\left(\frac{dW_k(x)}{dx} - \Psi_k(x)\right) - \frac{s_1^2}{l_k^2} b_k(x)\frac{d^2\Psi_k(x)}{dx^2} - \frac{s_1^2}{l_k^2}\frac{db_k(x)}{dx}\frac{d\Psi_k(x)}{dx} - \\ - \eta^2 b_k(x)\Psi_k(x) = \Omega^2 b_k(x)\Psi_k(x) \quad (22)$$

The equations (14), which satisfy continuity of displacement and rotation, can be expressed in dimensionless form as follows:

$$l_k W_k(1) - l_{k+1} W_{k+1}(0) = 0 \; ; \; \Psi_k(1) - \Psi_{k+1}(0) = 0 \quad (23a,b)$$

and the equations (15) of compatibility of the bending moment and the shear force, result in the following adimensional equations:

$$\alpha_k Q_k(1) - \alpha_{k+1} Q_{k+1}(0) = 0 \; ; \; \frac{\beta_k}{l_k} M_k(1) - \frac{\beta_{k+1}}{l_{k+1}} M_{k+1}(0) = 0$$

or

$$\alpha_k \left[ \left( N_k(x) + \frac{\kappa}{2(1+\nu)} a_k(x) \right) \frac{dW_k(x)}{dx} - \frac{\kappa}{2(1+\nu)} a_k(x) \Psi_k(x) \right] \Bigg|_{x=1} -$$

$$-\alpha_{k+1} \left[ \left( N_{k+1}(x) + \frac{\kappa}{2(1+\nu)} a_{k+1}(x) \right) \frac{dW_{k+1}(x)}{dx} - \frac{\kappa}{2(1+\nu)} a_{k+1}(x) \Psi_{k+1}(x) \right] \Bigg|_{x=0} = 0; \quad (24a,b)$$

$$\frac{\beta_k}{l_k} b_k(x) \frac{d\Psi_k(x)}{dx} \Bigg|_{x=1} - \frac{\beta_{k+1}}{l_{k+1}} b_{k+1}(x) \frac{d\Psi_{k+1}(x)}{dx} \Bigg|_{x=0} = 0$$

where $\alpha_k = \dfrac{A_k(0)}{A_1(0)}$; $\beta_k = \dfrac{I_k(0)}{I_1(0)}$.

The boundary conditions at the end closest to the axis of rotation, segment 1, $x$=0, are:

$$Q_1(0) - K_{W1} l_1 W_1(0) = 0 \; ; \left( N_1(0) + \frac{\kappa a_1(0)}{2(1+\nu)} \right) \frac{dW_1(x)}{dx} \Bigg|_{x=0} - \frac{\kappa a_1(0) \Psi_1(0)}{2(1+\nu)} - K_{W1} l_1 W_1(0) = 0$$

$$M_1(0) - K_{\Psi 1} l_1 \Psi_1(0) = 0 \; ; \; b_1(0) \frac{d\Psi_1(x)}{dx} \Bigg|_{x=0} - K_{\Psi 1} l_1 \Psi_1(0) = 0 \quad (25a,b)$$

and at the other end of the rotating beam, segment $d$, $x$=1, they are:

$$Q_d(1) - K_{Wd} \frac{l_d}{\alpha_d} W_d(1) = 0; \left( N_d(1) + \frac{\kappa \, a_d(1)}{2(1+\nu)} \right) \frac{dW_d(x)}{dx} \Bigg|_{x=1} - \frac{\kappa \, a_d(1)}{2(1+\nu)} \Psi_d(1) - \frac{K_{Wd} l_d}{\alpha_d} W_d(1) = 0$$

$$M_d(1) - \frac{K_{\Psi d} l_d}{\beta_d} \Psi_d(1) = 0 \; ; \; b_d(1) \frac{d\Psi_d(x)}{dx} \Bigg|_{x=1} - \frac{K_{\Psi d} \, l_d}{\beta_d} \Psi_d(1) = 0 \quad (26a,b)$$

where $N_d(1)$ is an outboard force at the end of the beam, farthest from the axis of rotation, that is equal to zero in the present study.

## 3. Differential Quadrature Method, DQM

In order to obtain the DQM analog equations from the governing equations of the rotating beam, the beam segment domain is discretized in a grid of $i$ points, using the Chebyshev – Gauss - Lobato expression, (Shu, 2000). (See Fig. A.1 in Appendix A)

Equations (18, 19, 20) assumed the form:

$$N_k(x_i) = \eta^2 \frac{l_k^2}{s_1^2}\left(R_k v_k(1) + \phi_k(1) - R_k v_k(x_i) - \phi_k(x_i)\right) + N_{k+1} \tag{27}$$

$$Q_k(x_i) = \left(N_k(x_i) + \frac{\kappa}{2(1+\nu)} a_k(x_i)\right)\sum_{j=1}^{n} A_{ij}^{(1)} W_{kj} - \frac{\kappa}{2(1+\nu)} a_k(x_i)\Psi_{ki} \tag{28}$$

$$M_k(x_i) = b_k(x_i)\sum_{j=1}^{n} A_{ij}^{(1)} \Psi_{kj} \tag{29}$$

The equations of motion (21) and (22) become:

$$\begin{aligned}&\left(\eta^2 a_k(x_i)(R_k + x_i) - \frac{\kappa}{2(1+\nu)}\frac{s_1^2}{l_k^2}\frac{da_k(x_i)}{dx}\right)\sum_{j=1}^{n}\left(A_{ij}^{(1)}\right) W_{kj} - \\ &-\left(\frac{s_1^2}{l_k^2} N_k(x_i) + \frac{\kappa}{2(1+\nu)}\frac{s_1^2}{l_k^2} a_k(x_i)\right)\sum_{j=1}^{n}\left(A_{ij}^{(2)}\right) W_{kj} + \frac{\kappa}{2(1+\nu)}\frac{s_1^2}{l_k^2} a_k(x_i)\sum_{j=1}^{n} A_{ij}^{(1)} \Psi_{kj} + \\ &+\frac{\kappa}{2(1+\nu)}\frac{s_1^2}{l_k^2}\frac{da_k(x_i)}{dx}\Psi_{ki} = \Omega^2 a_k(x_i) W_{ki}\end{aligned} \tag{30}$$

$$\begin{aligned}&-\frac{\kappa}{2(1+\upsilon)} s_1^2 s_k^2 a_k(x_i)\sum_{j=1}^{n} A_{ij}^{(1)} W_{kj} - \frac{s_1^2}{l_k^2} b_k(x_i)\sum_{j=1}^{n} A_{ij}^{(2)} \Psi_{kj} + \\ &+\left(\frac{\kappa}{2(1+\nu)} s_1^2 s_k^2 a_k(x_i) - \eta^2 b_k(x_i)\right)\Psi_{ki} - \frac{s_1^2}{l_k^2}\frac{db_k(x_i)}{dx}\sum_{j=1}^{n} A_{ij}^{(1)} \Psi_{kj} = \Omega^2 b_k(x_i)\Psi_{ki}\end{aligned} \tag{31}$$

where the $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ are the weighting coefficients of linear algebraic equations. (See Appendix A.1 for more details).

Finally, the conditions (23) and (24) are replaced by:

$$l_k W_{kn} - l_{k+1} W_{(k+1)1} = 0\,;\ \Psi_{kn} - \Psi_{(k+1)1} = 0\,; \tag{32a,b}$$

$$\begin{aligned}&\alpha_k\left(\left(N_k(1) + \frac{\kappa}{2(1+\nu)} a_k(1)\right)\sum_{j=1}^{n} A_{nj}^{(1)} W_{kj} - \frac{\kappa}{2(1+\nu)} a_k(1)\Psi_{kn}\right) \\ &-\alpha_{k+1}\left(\left(N_{k+1}(0) + \frac{\kappa}{2(1+\nu)} a_{k+1}(0)\right)\sum_{j=1}^{n} A_{1j}^{(1)} W_{(k+1)j} - \frac{\kappa}{2(1+\nu)} a_{k+1}(0)\Psi_{k1}\right) = 0;\end{aligned} \tag{33a,b}$$

$$\frac{\alpha_k}{l_k} b_k(1)\sum_{j=1}^{n} A_{nj}^{(1)} \Psi_{kj} - \frac{\alpha_{k+1}}{l_{k+1}} b_{k+1}(0)\sum_{j=1}^{n} A_{1j}^{(1)} \Psi_{(k+1)j} = 0$$

and the boundary conditions (25) and (26) replaced by:

$$\left(N_1(0)+\frac{\kappa}{2(1+\nu)}a_1(0)\right)\sum_{j=1}^{n}A_{1j}^{(1)}W_{1j}-\frac{\kappa}{2(1+\nu)}a_1(0)\Psi_{11}-l_1K_{W1}W_{11}=0\ ;$$
$$K_{\Psi 1}\Psi_{11}-\frac{b_1(0)}{l_1}\sum_{j=1}^{n}A_{1j}^{(1)}\Psi_{1j}=0 \qquad (34a,b)$$

$$\left(N_d(1)+\frac{\kappa}{2(1+\nu)}a_d(1)\right)\sum_{j=1}^{n}A_{nj}^{(1)}W_{dj}-\frac{\kappa}{2(1+\nu)}a_d(1)\Psi_{dn}-l_dK_{Wd}W_{dn}=0;$$
$$K_{\Psi n}\Psi_{dn}-\frac{b_d(1)}{l_d}\sum_{j=1}^{n}A_{nj}^{(1)}\Psi_{dj}=0 \qquad (35a,b)$$

The DQM linear equation system is used to determine the natural frequencies and mode shapes of the rotating beam.
The number of terms taken in the summations had been studied for many situations and the system has acceptable convergence by $n$= 21 terms. (See Table 1)

## 4. Finite element method, MEF

An independent set of results for the natural frequencies, was also obtained by a finite element code. (Bambill et al., 2010). The finite element model employed in the analysis has 3000 beam elements of two nodes in the longitudinal direction (Rossi, 2007). See Table 2. This number of elements was proved to be enough with a convergence analysis.

The beam model also takes into account the shear deformation (Timoshenko beam's theory) and the increase in bending stiffness induced by the centrifugal force.
The term $\rho I_k(\bar{x}_k)\bar{\eta}^2\bar{\Psi}_k(\bar{x}_k)$ of equation (13.b) was not included in the finite element formulation. Probably for this reason some small differences between both sets of numerical results (DQM and FEM) begin to appear when the rotational speed $\eta$ increases.

## 5. Numerical results

In the following examples some calculations were performed over elliptical cross sections. ($\kappa = 0.886364$). Without loss of generality, one may choose to keep constant width $e_k$=$e$ and vary the height $h_k(x)$ in each segment of the beam. The area and the second moment of area of the cross section of the beam will be $A_k(x)=\frac{\pi\, e h_k(x)}{4}$, $I_k(x)=\frac{\pi\, e h_k^3(x)}{64}$, and for this particular situation there are:

$$a_k(x)=\frac{h_k(x)}{h_k(0)}\ ;\ b_k(x)=\left(\frac{h_k(x)}{h_k(0)}\right)^3$$

The following formula is proposed to a quadratic variation of the height in each segment of beam:

$$h_k(x)=c_{0k}+c_{1k}\,x+c_{2k}\,x^2$$

And the slope is the derivative of this function

$$h_k'(x) = \frac{dh_k(x)}{dx} = c_{1k} + 2c_{2k}\,x$$

where $c_{0k}$, $c_{1k}$ and $c_{2k}$ are constants, which are defined by the heights and slopes at both ends of each segment *k*. The heights and slopes at each end are identified with the subscript *A* for *x*=0: $h_{Ak}$; $h'_{Ak}$ and with the subscript *B* for *x*=1: $h_{Bk}$; $h'_{Bk}$.

If the segment of the beam shows a linear variation of height , $c_{2k} = 0$ and

$$h_{Ak} = c_{0k}\,;\; h_{Bk} = c_{0k} + c_{1k}\,;\, h'_{Ak} = h'_{Bk} = c_{1k}$$

As it can be seen in Table 1, the frequency coefficients calculated by the Differential Quadrature Method, DQM, using a summation with $n \geq 19$ (*i*= 1, 2, 3, …, *n*) points, show none significant improvement.

| n | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|
| 5 | 15.6861 | 29.2939 | 49.1602 | 63.9792 | 112.610 |
| 7 | 15.1981 | 28.9907 | 46.9070 | 64.9219 | 88.8670 |
| 9 | 14.9057 | 29.5079 | 47.4960 | 64.7054 | 87.4079 |
| 11 | 14.8340 | 29.6332 | 47.6579 | 64.7247 | 87.6724 |
| 13 | 14.8281 | 29.6467 | 47.6811 | 64.7310 | 87.7047 |
| 15 | 14.8291 | 29.6464 | 47.6820 | 64.7319 | 87.7079 |
| 17 | 14.8295 | 29.6460 | 47.6816 | 64.7320 | 87.7080 |
| 19 | 14.8296 | 29.6459 | 47.6815 | 64.7320 | 87.7080 |
| 21 | 14.8296 | 29.6459 | 47.6815 | 64.7320 | 87.7080 |

Table 1. Convergence analysis of the DQM, for a two-span rotating Timoshenko beam elastically restrained al both ends, with a quadratic variation of height.

The frequency coefficients in Table 1, correspond to a beam of two segments, rotating at speed $\eta = 10$, whose characteristics are: elliptical cross section; $\nu = 0.3$; $\kappa = 0.886364$; $R_1$=0; $l_1 / L = l_2 / L = 1/2$; $s_1 = \sqrt{300}$; $h_{B1} / h_{A1} = 1/2$; $h'_{B1} = 0$; $h_{A2} / h_{B1} = 1/2$; $h_{B2} / h_{A2} = 1/2$; $h'_{A2} = 0$; $K_{W1} = 10$; $K_{\psi 1} = 5$; $K_{Wd} = 0.1$; $K_{\psi d} = 1$.

In Table 2 the values obtained for the natural frequency coefficients using the finite element method are presented for $\eta = \sqrt{\rho A_0 / EI_0}\, L^2 \,\overline{\eta} = 0$ and $\eta = 10$. The number of elements is increased from 10 to 3000.

The model of the rotating beam of Table 2 has the following characteristics: one segment; rectangular cross section; $\nu = 0.3$; $\kappa = 10(1+\nu)/(12+11\nu)$ =0.849673; $R_1$=0; $s_1 = \sqrt{300}$; $h_B / h_A = 1/4$; $h'_B$=0; $K_{W1} \to \infty$; $K_{\psi 1} \to \infty$; $K_{Wd} = 0$; $K_{\psi d} = 0$.

In the first examples it is assumed a perfect clamped condition at the axis of rotation, given by: $K_{W1} \to \infty$ and $K_{\psi 1} \to \infty$. (Tables 3, 4 and 5).

Table 3 presents the effect of the rotational speed parameter $\eta$ on the natural frequency coefficients of a rotating cantilever beam of one segment, ($K_{W1} \to \infty$; $K_{\psi 1} \to \infty$; $K_{Wd} = 0$; $K_{\psi d} = 0$). The results correspond to a linear variation of height and a comparison is made with

(Barnejee, 2006) when Banerjee´s parameter is n=1. As it can be observed the agreement is excellent.

| $\eta = 0$ | | | | | |
|---|---|---|---|---|---|
| Number of elements | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
| 10 | 3.38628165 | 11.7689336 | 26.5951854 | 46.6658427 | 71.0448001 |
| 100 | 3.37398143 | 11.7248502 | 26.4438604 | 46.1408176 | 69.5136708 |
| 1000 | 3.37385398 | 11.7243988 | 26.4423706 | 46.1357196 | 69.4986357 |
| 2000 | 3.37385302 | 11.7243954 | 26.4423593 | 46.1356810 | 69.4985219 |
| 3000 | 3.37385284 | 11.7243946 | 26.4423572 | 46.1356739 | 69.4985008 |
| $\eta = 10$ | | | | | |
| 10 | 11.6074237 | 25.8805102 | 44.0407905 | 66.3753084 | 92.6859627 |
| 100 | 11.6098042 | 25.7094320 | 43.5638284 | 65.4674874 | 90.8491237 |
| 1000 | 11.6098077 | 25.7074626 | 43.5585908 | 65.4579769 | 90.8301746 |
| 2000 | 11.6098078 | 25.7074476 | 43.5585511 | 65.4579049 | 90.8300310 |
| 3000 | 11.6098078 | 25.7074448 | 43.5585437 | 65.4578915 | 90.8300044 |

Table 2. Convergence analysis of the frequency coefficients $\Omega_i = \sqrt{\rho A_0 / EI_0}\, L^2\, \omega_i$ using MEF.

| $\eta$ | | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|---|
| 0 | DQM | 3.82377 | 18.3171 | 47.2638 | 90.4468 | 147.992 |
| | (Barnejee,2006) | 3.82379 | 18.3173 | 47.2648 | 90.4505 | 148.002 |
| 2 | DQM | 4.43680 | 18.9365 | 47.8706 | 91.0589 | 148.609 |
| | (Barnejee,2006) | 4.43680 | 18.9366 | 47.8717 | 91.0625 | 148.619 |
| 4 | DQM | 5.87874 | 20.6850 | 49.6446 | 92.8693 | 150.444 |
| | (Barnejee,2006) | 5.87877 | 20.6851 | 49.6456 | 92.8730 | 150.454 |
| 6 | DQM | 7.65512 | 23.3091 | 52.4622 | 95.8054 | 153.450 |
| | (Barnejee,2006) | 7.65514 | 23.3093 | 52.4632 | 95.8090 | 153.460 |
| 8 | DQM | 9.55392 | 26.5435 | 56.1584 | 99.7601 | 157.555 |
| | (Barnejee,2006) | 9.55396 | 26.5437 | 56.1595 | 99.7638 | 157.564 |
| 10 | DQM | 11.5015 | 30.1825 | 60.5628 | 104.608 | 162.668 |
| | (Barnejee,2006) | 11.5015 | 30.1827 | 60.5639 | 104.612 | 162.677 |

Table 3. Frequency coefficients $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2\, \omega_i$ for a one-span beam, $l_1 / L = 1$ ; $s_1 = \sqrt{1000}$ ; $h_B / h_A = 1/2$ ; $K_{W1} \to \infty$ ; $K_{\psi 1} \to \infty$ ; $K_{Wd} = 0$ ; $K_{\psi d} = 0$ .

All the calculations performed for the following Tables and Graphics used $R_1 = 0$ ; and $\nu = 0.30$ ; $\kappa = 0.886364$ (elliptical cross section).

The DQM results are determined using $n$ = 21 in each segment of the beam, and the MEF results were obtained with 3000 elements.

The beam considered in Table 4 has one segment and is elastically restrained at its outer end. The parameter of rotation speed $\eta$ is taken equal to 10. The Table presents the frequency coefficients for the first five mode shapes which correspond to different sets of elastically boundary conditions given by the spring constant parameters $K_{Wd}$ and $K_{\psi d}$. The other details of the beam are specified in the legend of the table.

The beam model considered in Table 5 has two segments of equal length and similar conditions and parameters as Table 4.

| $K_{\psi d}$ | $K_{Wd}$ | Method | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | DQM | 11.2148 | 27.6174 | 50.0089 | 77.5866 | 108.472 |
| | | FEM | 11.2375 | 27.6743 | 50.0711 | 77.6432 | 108.523 |
| | 0.1 | DQM | 15.4254 | 32.5178 | 52.8516 | 79.0733 | 109.357 |
| | | FEM | 15.4438 | 32.5548 | 52.9087 | 79.1298 | 109.408 |
| | 1 | DQM | 18.0157 | 40.3494 | 65.6538 | 92.2848 | 119.836 |
| | | FEM | 18.0465 | 40.3841 | 65.6882 | 92.3208 | 119.877 |
| | 10 | DQM | 18.3978 | 41.6361 | 69.0216 | 99.4111 | 131.859 |
| | | FEM | 18.4315 | 41.6757 | 69.0611 | 99.4484 | 131.893 |
| | $\to\infty$ | DQM | 18.4417 | 41.7750 | 69.3474 | 100.033 | 132.894 |
| | | FEM | 18.4757 | 41.8151 | 69.3878 | 100.071 | 132.929 |
| 1 | 0 | DQM | 11.3941 | 29.3678 | 53.0174 | 81.0192 | 112.024 |
| | | FEM | 11.4148 | 29.4104 | 53.0660 | 81.0662 | 112.068 |
| | 0.1 | DQM | 15.6233 | 32.8965 | 55.0307 | 82.2247 | 112.825 |
| | | FEM | 15.6400 | 32.9308 | 55.0763 | 82.2710 | 112.868 |
| | 1 | DQM | 19.2962 | 41.3980 | 65.9339 | 92.3365 | 120.822 |
| | | FEM | 19.3219 | 41.4295 | 65.9674 | 92.3723 | 120.859 |
| | 10 | DQM | 19.9179 | 43.3987 | 70.5662 | 100.622 | 132.723 |
| | | FEM | 19.9463 | 43.4345 | 70.6034 | 100.658 | 132.756 |
| | $\to\infty$ | DQM | 19.9899 | 43.6199 | 71.0558 | 101.509 | 134.136 |
| | | FEM | 20.0187 | 43.6562 | 71.0937 | 101.546 | 134.170 |
| 10 | 0 | DQM | 11.4913 | 30.3954 | 55.1815 | 84.0260 | 115.628 |
| | | FEM | 11.5115 | 30.4328 | 55.2229 | 84.0663 | 115.665 |
| | 0.1 | DQM | 15.7621 | 33.1503 | 56.5688 | 84.8630 | 116.210 |
| | | FEM | 15.7780 | 33.1835 | 56.6092 | 84.9031 | 116.247 |
| | 1 | DQM | 20.4765 | 42.5961 | 66.2635 | 92.3899 | 121.730 |
| | | FEM | 20.4994 | 42.6248 | 66.2963 | 92.4255 | 121.765 |
| | 10 | DQM | 21.3539 | 45.5548 | 72.8197 | 102.560 | 134.141 |
| | | FEM | 21.3795 | 45.5875 | 72.8543 | 102.594 | 134.174 |
| | $\to\infty$ | DQM | 21.4553 | 45.8807 | 73.5609 | 103.912 | 136.261 |
| | | FEM | 21.4813 | 45.9139 | 73.5962 | 103.947 | 136.294 |
| $\to\infty$ | 0 | DQM | 11.5091 | 30.5860 | 55.6105 | 84.6706 | 116.454 |
| | | FEM | 11.5291 | 30.6228 | 55.6510 | 84.7101 | 116.491 |
| | 0.1 | DQM | 15.7905 | 33.2010 | 56.8768 | 85.4233 | 116.975 |
| | | FEM | 15.8064 | 33.2340 | 56.9165 | 85.4625 | 117.012 |
| | 1 | DQM | 20.7557 | 42.9193 | 66.3549 | 92.4035 | 121.943 |
| | | FEM | 20.7782 | 42.9475 | 66.3875 | 92.4392 | 121.978 |
| | 10 | DQM | 21.6961 | 46.1510 | 73.5285 | 103.223 | 134.643 |
| | | FEM | 21.7214 | 46.1832 | 73.5626 | 103.257 | 134.675 |
| | $\to\infty$ | DQM | 21.8045 | 46.5039 | 74.3452 | 104.735 | 137.026 |
| | | FEM | 21.8302 | 46.5368 | 74.3801 | 104.769 | 137.059 |

Table 4. First natural frequencies $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2 \omega_i$ for a one-span rotating Timoshenko beam, with elliptical cross section and quadratic height variation along the axis. $\nu = 0.3$ ; $s_1 = \sqrt{300}$ ; $h_B / h_A = 1/2$ ; $h'_B = 0$ ; $K_{W1} \to \infty$ ; $K_{\psi 1} \to \infty$ ; $\eta = 10$ .

| $K_{\psi d}$ | $K_{Wd}$ | Method | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | DQM | 11.8651 | 24.5717 | 40.8347 | 59.8775 | 81.1573 |
| | | FEM | 11.8796 | 24.5914 | 40.8559 | 59.9110 | 81.1826 |
| | 0.1 | DQM | 15.2667 | 30.3903 | 49.8409 | 67.9228 | 90.5546 |
| | | FEM | 15.2858 | 30.4064 | 49.8638 | 67.9498 | 90.5743 |
| | 1 | DQM | 15.6938 | 31.4585 | 52.2093 | 72.0062 | 98.7038 |
| | | FEM | 15.7140 | 31.4754 | 52.2362 | 72.0330 | 98.7265 |
| | 10 | DQM | 15.7412 | 31.5756 | 52.4458 | 72.4214 | 99.4803 |
| | | FEM | 15.7616 | 31.5927 | 52.4732 | 72.4482 | 99.5038 |
| | $\to\infty$ | DQM | 15.7466 | 31.5887 | 52.4718 | 72.4669 | 99.5627 |
| | | FEM | 15.7669 | 31.6059 | 52.4993 | 72.4937 | 99.5862 |
| 1 | 0 | DQM | 11.9142 | 25.1342 | 42.8878 | 62.4877 | 85.6040 |
| | | FEM | 11.9288 | 25.1532 | 42.9079 | 62.5196 | 85.6258 |
| | 0.1 | DQM | 16.2121 | 31.6526 | 50.5459 | 67.9979 | 90.8436 |
| | | FEM | 16.2314 | 31.6672 | 50.5682 | 68.0245 | 90.8635 |
| | 1 | DQM | 16.9952 | 33.8476 | 55.0090 | 75.3283 | 102.166 |
| | | FEM | 17.0160 | 33.8634 | 55.0372 | 75.3520 | 102.190 |
| | 10 | DQM | 17.0842 | 34.0961 | 55.4704 | 76.2542 | 103.723 |
| | | FEM | 17.1052 | 34.1120 | 55.4993 | 76.2779 | 103.748 |
| | $\to\infty$ | DQM | 17.0942 | 34.1238 | 55.5205 | 76.3541 | 103.882 |
| | | FEM | 17.1152 | 34.1398 | 55.5496 | 76.3778 | 103.907 |
| 10 | 0 | DQM | 11.9157 | 25.1505 | 42.9498 | 62.5733 | 85.7690 |
| | | FEM | 11.9302 | 25.1695 | 42.9699 | 62.6051 | 85.7907 |
| | 0.1 | DQM | 16.2528 | 31.7152 | 50.5831 | 68.0018 | 90.8571 |
| | | FEM | 16.2721 | 31.7297 | 50.6053 | 68.0283 | 90.8770 |
| | 1 | DQM | 17.0528 | 33.9729 | 55.1728 | 75.5622 | 102.430 |
| | | FEM | 17.0737 | 33.9886 | 55.2011 | 75.5857 | 102.453 |
| | 10 | DQM | 17.1437 | 34.2281 | 55.6450 | 76.5200 | 104.034 |
| | | FEM | 17.1648 | 34.2440 | 55.6741 | 76.5435 | 104.059 |
| | $\to\infty$ | DQM | 17.1539 | 34.2566 | 55.6962 | 76.6231 | 104.197 |
| | | FEM | 17.1750 | 34.2725 | 55.7255 | 76.6466 | 104.222 |
| $\to\infty$ | 0 | DQM | 11.9158 | 25.1524 | 42.9569 | 62.5831 | 85.7880 |
| | | FEM | 11.9304 | 25.1713 | 42.9770 | 62.6149 | 85.8097 |
| | 0.1 | DQM | 16.2575 | 31.7225 | 50.5875 | 68.0023 | 90.8587 |
| | | FEM | 16.2768 | 31.7370 | 50.6097 | 68.0288 | 90.8786 |
| | 1 | DQM | 17.0595 | 33.9876 | 55.1921 | 75.5901 | 102.461 |
| | | FEM | 17.0804 | 34.0033 | 55.2205 | 75.6136 | 102.485 |
| | 10 | DQM | 17.1506 | 34.2436 | 55.6656 | 76.5517 | 104.071 |
| | | FEM | 17.1717 | 34.2595 | 55.6947 | 76.5752 | 104.096 |
| | $\to\infty$ | DQM | 17.1609 | 34.2722 | 55.7169 | 76.6551 | 104.234 |
| | | FEM | 17.1820 | 34.2881 | 55.7461 | 76.6786 | 104.259 |

Table 5. First natural frequencies $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2 \omega_i$ for a two-span elastically restrained rotating Timoshenko beam, with elliptical cross section and quadratic height variation along the axis. $l_1 / L = 1/2 \;\; l_2 / L = 1/2$ , $h_{B1} / h_{A1} = 1/2$ , $h'_{B1} = 0$ , $h_{A2} / h_{B1} = 1/2$ , $h_{B2} / h_{A2} = 1/2$ , $h'_{A2} = 0$ , $K_{W1} \to \infty \;\; K_{\psi 1} \to \infty$ , $\eta = 10$ .

Next Tables, 6 to 10, correspond to beams of two segments, elastically restrained at both ends and any particular details are expressed in each legend.

| $K_{\psi d}$ | $K_{Wd}$ | Method | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | DQM | 9.98841 | 21.2706 | 37.3110 | 54.9224 | 77.7336 |
| | | FEM | 10.0246 | 21.3074 | 37.3506 | 54.9699 | 77.7658 |
| | 0.1 | DQM | 12.4181 | 26.7466 | 45.4389 | 63.7717 | 87.4947 |
| | | FEM | 12.4641 | 26.7810 | 45.4827 | 63.8054 | 87.5227 |
| | 1 | DQM | 12.7051 | 27.6834 | 47.3154 | 67.8701 | 94.9448 |
| | | FEM | 12.7526 | 27.7193 | 47.3634 | 67.9035 | 94.9788 |
| | 10 | DQM | 12.7370 | 27.7869 | 47.5065 | 68.2901 | 95.6398 |
| | | FEM | 12.7847 | 27.8230 | 47.5548 | 68.3236 | 95.6747 |
| | $\to\infty$ | DQM | 12.7406 | 27.7985 | 47.5276 | 68.3362 | 95.7137 |
| | | FEM | 12.7883 | 27.8347 | 47.5760 | 68.3697 | 95.7487 |
| 1 | 0 | DQM | 10.0086 | 21.6505 | 39.0007 | 57.4853 | 82.1930 |
| | | FEM | 10.0451 | 21.6876 | 39.0412 | 57.5291 | 82.2227 |
| | 0.1 | DQM | 13.1047 | 28.0775 | 46.2626 | 63.9615 | 87.6768 |
| | | FEM | 13.1521 | 28.1101 | 46.3049 | 63.9941 | 87.7052 |
| | 1 | DQM | 13.6220 | 29.9765 | 49.8847 | 71.5084 | 98.2767 |
| | | FEM | 13.6718 | 30.0120 | 49.9325 | 71.5384 | 98.3117 |
| | 10 | DQM | 13.6812 | 30.1918 | 50.2664 | 72.4302 | 99.6646 |
| | | FEM | 13.7312 | 30.2278 | 50.3147 | 72.4604 | 99.7012 |
| | $\to\infty$ | DQM | 13.6879 | 30.2159 | 50.3082 | 72.5299 | 99.8074 |
| | | FEM | 13.7379 | 30.2519 | 50.3566 | 72.5601 | 99.8441 |
| 10 | 0 | DQM | 10.0092 | 21.6615 | 39.0505 | 57.5689 | 82.3537 |
| | | FEM | 10.0457 | 21.6987 | 39.0910 | 57.6127 | 82.3835 |
| | 0.1 | DQM | 13.1336 | 28.1416 | 46.3059 | 63.9714 | 87.6854 |
| | | FEM | 13.1811 | 28.1741 | 46.3481 | 64.0039 | 87.7138 |
| | 1 | DQM | 13.6618 | 30.0922 | 50.0321 | 71.7561 | 98.5256 |
| | | FEM | 13.7116 | 30.1278 | 50.0799 | 71.7860 | 98.5607 |
| | 10 | DQM | 13.7222 | 30.3131 | 50.4233 | 72.7081 | 99.9556 |
| | | FEM | 13.7723 | 30.3491 | 50.4716 | 72.7381 | 99.9923 |
| | $\to\infty$ | DQM | 13.7290 | 30.3379 | 50.4661 | 72.8107 | 100.102 |
| | | FEM | 13.7792 | 30.3739 | 50.5145 | 72.8408 | 100.139 |
| $\to\infty$ | 0 | DQM | 10.0093 | 21.6628 | 39.0562 | 57.5786 | 82.3722 |
| | | FEM | 10.0458 | 21.6999 | 39.0968 | 57.6224 | 82.4020 |
| | 0.1 | DQM | 13.1369 | 28.1491 | 46.3110 | 63.9726 | 87.6864 |
| | | FEM | 13.1844 | 28.1817 | 46.3532 | 64.0051 | 87.7149 |
| | 1 | DQM | 13.6664 | 30.1057 | 50.0495 | 71.7856 | 98.5555 |
| | | FEM | 13.7163 | 30.1414 | 50.0973 | 71.8155 | 98.5906 |
| | 10 | DQM | 13.7270 | 30.3273 | 50.4418 | 72.7411 | 99.9903 |
| | | FEM | 13.7771 | 30.3634 | 50.4901 | 72.7711 | 100.027 |
| | $\to\infty$ | DQM | 13.7338 | 30.3521 | 50.4847 | 72.8441 | 100.137 |
| | | FEM | 13.7840 | 30.3882 | 50.5331 | 72.8741 | 100.174 |

Table 6. First natural frequencies $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2 \omega_i$ for a two-span elastically restrained rotating Timoshenko beam, with elliptical cross section and quadratic height variation along the axis. $l_1 / L = 1/2$ $l_2 / L = 1/2$, $h_{B1} / h_{A1} = 1/2$, $h'_{B1} = 0$, $h_{A2} / h_{B1} = 1/2$, $h_{B2} / h_{A2} = 1/2$, $h'_{A2} = 0$, $K_{W1} \to \infty$, $K_{\psi 1} = 0.1$, $\eta = 10$.

| $K_{\psi d}$ | $K_{Wd}$ | Method | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | DQM | 11.3734 | 23.4059 | 39.2570 | 56.9363 | 78.4787 |
| | | FEM | 11.3904 | 23.4273 | 39.2820 | 56.9721 | 78.5041 |
| | 0.1 | DQM | 14.4551 | 28.9705 | 47.6260 | 65.1400 | 87.8678 |
| | | FEM | 14.4773 | 28.9890 | 47.6542 | 65.1664 | 87.8894 |
| | 1 | DQM | 14.8320 | 29.9641 | 49.6507 | 69.1005 | 95.0494 |
| | | FEM | 14.8553 | 29.9837 | 49.6827 | 69.1264 | 95.0757 |
| | 10 | DQM | 14.8738 | 30.0733 | 49.8536 | 69.5057 | 95.7100 |
| | | FEM | 14.8972 | 30.0931 | 49.8861 | 69.5316 | 95.7371 |
| | $\to\infty$ | DQM | 14.8785 | 30.0856 | 49.8761 | 69.5501 | 95.7801 |
| | | FEM | 14.9019 | 30.1054 | 49.9086 | 69.5761 | 95.8073 |
| 1 | 0 | DQM | 11.4126 | 23.8883 | 41.1048 | 59.4030 | 82.7965 |
| | | FEM | 11.4297 | 23.9092 | 41.1294 | 59.4361 | 82.8192 |
| | 0.1 | DQM | 15.3087 | 30.2367 | 48.3683 | 65.2804 | 88.0547 |
| | | FEM | 15.3312 | 30.2539 | 48.3957 | 65.3061 | 88.0766 |
| | 1 | DQM | 15.9925 | 32.2677 | 52.2109 | 72.5363 | 98.2033 |
| | | FEM | 16.0166 | 32.2865 | 52.2436 | 72.5592 | 98.2306 |
| | 10 | DQM | 16.0702 | 32.4974 | 52.6080 | 73.4315 | 99.5067 |
| | | FEM | 16.0945 | 32.5165 | 52.6413 | 73.4545 | 99.5353 |
| | $\to\infty$ | DQM | 16.0790 | 32.5231 | 52.6513 | 73.5282 | 99.6404 |
| | | FEM | 16.1032 | 32.5422 | 52.6847 | 73.5513 | 99.6691 |
| 10 | 0 | DQM | 11.4138 | 23.9023 | 41.1598 | 59.4837 | 82.9526 |
| | | FEM | 11.4309 | 23.9232 | 41.1844 | 59.5168 | 82.9753 |
| | 0.1 | DQM | 15.3450 | 30.2988 | 48.4074 | 65.2877 | 88.0636 |
| | | FEM | 15.3676 | 30.3159 | 48.4348 | 65.3134 | 88.0854 |
| | 1 | DQM | 16.0434 | 32.3866 | 52.3585 | 72.7730 | 98.4379 |
| | | FEM | 16.0676 | 32.4055 | 52.3914 | 72.7958 | 98.4653 |
| | 10 | DQM | 16.1228 | 32.6224 | 52.7651 | 73.6979 | 99.7796 |
| | | FEM | 16.1471 | 32.6415 | 52.7985 | 73.7208 | 99.8083 |
| | $\to\infty$ | DQM | 16.1317 | 32.6487 | 52.8094 | 73.7975 | 99.9164 |
| | | FEM | 16.1560 | 32.6679 | 52.8428 | 73.8204 | 99.9453 |
| $\to\infty$ | 0 | DQM | 11.4139 | 23.9039 | 41.1661 | 59.4929 | 82.9706 |
| | | FEM | 11.4310 | 23.9248 | 41.1908 | 59.5260 | 82.9933 |
| | 0.1 | DQM | 15.3493 | 30.3061 | 48.4120 | 65.2886 | 88.0646 |
| | | FEM | 15.3718 | 30.3232 | 48.4394 | 65.3142 | 88.0865 |
| | 1 | DQM | 16.0493 | 32.4005 | 52.3759 | 72.8012 | 98.4661 |
| | | FEM | 16.0735 | 32.4194 | 52.4088 | 72.8240 | 98.4934 |
| | 10 | DQM | 16.1289 | 32.6371 | 52.7836 | 73.7295 | 99.8122 |
| | | FEM | 16.1532 | 32.6562 | 52.8170 | 73.7524 | 99.8409 |
| | $\to\infty$ | DQM | 16.1378 | 32.6635 | 52.8280 | 73.8295 | 99.9493 |
| | | FEM | 16.1622 | 32.6827 | 52.8615 | 73.8524 | 99.9782 |

Table 7. First natural frequencies $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2 \omega_i$ for a two-span elastically restrained rotating Timoshenko beam, with elliptical cross section and quadratic height variation along the axis. $l_1 / L = 1/2$ $l_2 / L = 1/2$, $h_{B1} / h_{A1} = 1/2$, $h'_{B1} = 0$, $h_{A2} / h_{B1} = 1/2$, $h_{B2} / h_{A2} = 1/2$, $h'_{A2} = 0$, $K_{W1} = 10$, $K_{\psi 1} = 10$, $\eta = 10$.

| $K_{\psi d}$ | $K_{Wd}$ | Method | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | DQM | 11.0954 | 22.8658 | 38.6771 | 56.1987 | 78.0667 |
| | | FEM | 11.1149 | 22.8898 | 38.7052 | 56.2367 | 78.0931 |
| | 0.1 | DQM | 14.0189 | 28.3688 | 46.9190 | 64.5736 | 87.5306 |
| | | FEM | 14.0444 | 28.3902 | 46.9506 | 64.6011 | 87.5532 |
| | 1 | DQM | 14.3726 | 29.3408 | 48.8766 | 68.5581 | 94.6820 |
| | | FEM | 14.3992 | 29.3634 | 48.9120 | 68.5850 | 94.7094 |
| | 10 | DQM | 14.4118 | 29.4478 | 49.0737 | 68.9659 | 95.3401 |
| | | FEM | 14.4386 | 29.4706 | 49.1095 | 68.9930 | 95.3684 |
| | $\to\infty$ | DQM | 14.4162 | 29.4598 | 49.0954 | 69.0107 | 95.4100 |
| | | FEM | 14.4430 | 29.4826 | 49.1314 | 69.0377 | 95.4383 |
| 1 | 0 | DQM | 11.1299 | 23.3178 | 40.4680 | 58.6771 | 82.4016 |
| | | FEM | 11.1495 | 23.3414 | 40.4960 | 58.7122 | 82.4254 |
| | 0.1 | DQM | 14.8296 | 29.6459 | 47.6815 | 64.7320 | 87.7080 |
| | | FEM | 14.8556 | 29.6659 | 47.7122 | 64.7587 | 87.7309 |
| | 1 | DQM | 15.4691 | 31.6287 | 51.4150 | 72.0511 | 97.8483 |
| | | FEM | 15.4967 | 31.6508 | 51.4509 | 72.0751 | 97.8766 |
| | 10 | DQM | 15.5418 | 31.8531 | 51.8028 | 72.9497 | 99.1495 |
| | | FEM | 15.5696 | 31.8754 | 51.8392 | 72.9737 | 99.1791 |
| | $\to\infty$ | DQM | 15.5499 | 31.8782 | 51.8452 | 73.0468 | 99.2832 |
| | | FEM | 15.5778 | 31.9005 | 51.8817 | 73.0708 | 99.3129 |
| 10 | 0 | DQM | 11.1309 | 23.3309 | 40.5211 | 58.7582 | 82.5578 |
| | | FEM | 11.1505 | 23.3545 | 40.5491 | 58.7932 | 82.5816 |
| | 0.1 | DQM | 14.8640 | 29.7082 | 47.7217 | 64.7403 | 87.7163 |
| | | FEM | 14.8900 | 29.7282 | 47.7523 | 64.7669 | 87.7393 |
| | 1 | DQM | 15.5170 | 31.7461 | 51.5610 | 72.2905 | 98.0837 |
| | | FEM | 15.5447 | 31.7682 | 51.5970 | 72.3143 | 98.1121 |
| | 10 | DQM | 15.5912 | 31.9764 | 51.9582 | 73.2187 | 99.4233 |
| | | FEM | 15.6191 | 31.9987 | 51.9947 | 73.2426 | 99.4531 |
| | $\to\infty$ | DQM | 15.5996 | 32.0021 | 52.0016 | 73.3186 | 99.5601 |
| | | FEM | 15.6275 | 32.0245 | 52.0381 | 73.3426 | 99.5900 |
| $\to\infty$ | 0 | DQM | 11.1310 | 23.3324 | 40.5272 | 58.7675 | 82.5758 |
| | | FEM | 11.1506 | 23.3560 | 40.5552 | 58.8025 | 82.5996 |
| | 0.1 | DQM | 14.8680 | 29.7155 | 47.7264 | 64.7412 | 87.7173 |
| | | FEM | 14.8940 | 29.7355 | 47.7570 | 64.7679 | 87.7402 |
| | 1 | DQM | 15.5225 | 31.7598 | 51.5783 | 72.3191 | 98.1119 |
| | | FEM | 15.5503 | 31.7819 | 51.6143 | 72.3428 | 98.1403 |
| | 10 | DQM | 15.5970 | 31.9908 | 51.9765 | 73.2506 | 99.4560 |
| | | FEM | 15.6249 | 32.0132 | 52.0131 | 73.2745 | 99.4857 |
| | $\to\infty$ | DQM | 15.6053 | 32.0166 | 52.0200 | 73.3510 | 99.5931 |
| | | FEM | 15.6333 | 32.0391 | 52.0566 | 73.3749 | 99.6230 |

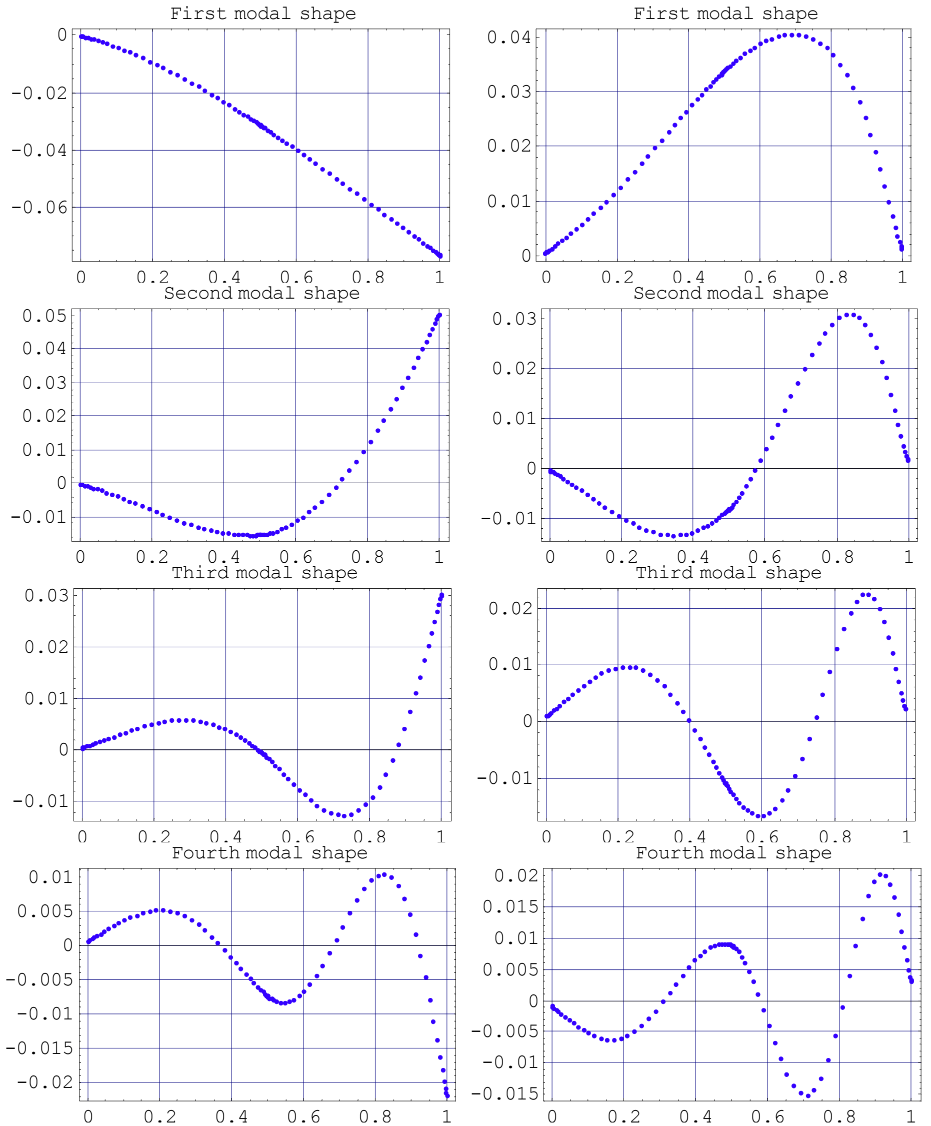
Table 8. First natural frequencies $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2 \omega_i$ for a two-span elastically restrained rotating Timoshenko beam, with elliptical cross section and quadratic height variation along the axis. $l_1 / L = 1/2$ $l_2 / L = 1/2$, $h_{B1} / h_{A1} = 1/2$, $h'_{B1} = 0$, $h_{A2} / h_{B1} = 1/2$, $h_{B2} / h_{A2} = 1/2$, $h'_{A2} = 0$, $K_{W1} = 10$, $K_{\psi 1} = 5$, $\eta = 10$.

| $K_{\psi d}$ | $K_{Wd}$ | Method | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | DQM | 10.3650 | 21.7083 | 37.5771 | 54.9536 | 77.3897 |
| | | FEM | 10.3937 | 21.7398 | 37.6121 | 54.9961 | 77.4183 |
| | 0.1 | DQM | 12.9366 | 27.1514 | 45.6367 | 63.6520 | 86.9652 |
| | | FEM | 12.9736 | 27.1805 | 45.6754 | 63.6816 | 86.9900 |
| | 1 | DQM | 13.2417 | 28.0899 | 47.4960 | 67.6756 | 94.0576 |
| | | FEM | 13.2800 | 28.1204 | 47.5385 | 67.7048 | 94.0875 |
| | 10 | DQM | 13.2755 | 28.1934 | 47.6848 | 68.0876 | 94.7110 |
| | | FEM | 13.3141 | 28.2242 | 47.7276 | 68.1169 | 94.7416 |
| | $\to\infty$ | DQM | 13.2794 | 28.2050 | 47.7057 | 68.1329 | 94.7804 |
| | | FEM | 13.3179 | 28.2358 | 47.7485 | 68.1621 | 94.8111 |
| 1 | 0 | DQM | 10.3892 | 22.1044 | 39.2705 | 57.4666 | 81.7463 |
| | | FEM | 10.4182 | 22.1360 | 39.3061 | 57.5054 | 81.7724 |
| | 0.1 | DQM | 13.6565 | 28.4599 | 46.4416 | 63.8427 | 87.1268 |
| | | FEM | 13.6946 | 28.4875 | 46.4789 | 63.8714 | 87.1519 |
| | 1 | DQM | 14.2060 | 30.3646 | 50.0205 | 71.2606 | 97.2421 |
| | | FEM | 14.2462 | 30.3947 | 50.0627 | 71.2866 | 97.2727 |
| | 10 | DQM | 14.2687 | 30.5803 | 50.3961 | 72.1634 | 98.5382 |
| | | FEM | 14.3091 | 30.6108 | 50.4388 | 72.1896 | 98.5700 |
| | $\to\infty$ | DQM | 14.2757 | 30.6044 | 50.4372 | 72.2610 | 98.6715 |
| | | FEM | 14.3161 | 30.6350 | 50.4800 | 72.2872 | 98.7035 |
| 10 | 0 | DQM | 10.3900 | 22.1160 | 39.3204 | 57.5486 | 81.9025 |
| | | FEM | 10.4190 | 22.1475 | 39.3560 | 57.5874 | 81.9286 |
| | 0.1 | DQM | 13.6869 | 28.5231 | 46.4839 | 63.8527 | 87.1344 |
| | | FEM | 13.7250 | 28.5506 | 46.5212 | 63.8813 | 87.1595 |
| | 1 | DQM | 14.2479 | 30.4798 | 50.1652 | 71.5043 | 97.4786 |
| | | FEM | 14.2881 | 30.5099 | 50.2075 | 71.5301 | 97.5092 |
| | 10 | DQM | 14.3119 | 30.7011 | 50.5501 | 72.4364 | 98.8132 |
| | | FEM | 14.3524 | 30.7317 | 50.5928 | 72.4624 | 98.8452 |
| | $\to\infty$ | DQM | 14.3191 | 30.7259 | 50.5922 | 72.5368 | 98.9498 |
| | | FEM | 14.3596 | 30.7565 | 50.6350 | 72.5629 | 98.9819 |
| $\to\infty$ | 0 | DQM | 10.3901 | 22.1173 | 39.3262 | 57.5580 | 81.9205 |
| | | FEM | 10.4190 | 22.1488 | 39.3618 | 57.5968 | 81.9466 |
| | 0.1 | DQM | 13.6904 | 28.5305 | 46.4889 | 63.8539 | 87.1353 |
| | | FEM | 13.7286 | 28.5580 | 46.5261 | 63.8825 | 87.1604 |
| | 1 | DQM | 14.2527 | 30.4933 | 50.1823 | 71.5333 | 97.5069 |
| | | FEM | 14.2930 | 30.5235 | 50.2245 | 71.5591 | 97.5375 |
| | 10 | DQM | 14.3169 | 30.7153 | 50.5682 | 72.4688 | 98.8460 |
| | | FEM | 14.3574 | 30.7458 | 50.6110 | 72.4948 | 98.8780 |
| | $\to\infty$ | DQM | 14.3241 | 30.7401 | 50.6105 | 72.5696 | 98.9829 |
| | | FEM | 14.3646 | 30.7707 | 50.6532 | 72.5957 | 99.0150 |

Table 9. First natural frequencies $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2 \omega_i$ for a two-span elastically restrained rotating Timoshenko beam, with elliptical cross section and quadratic height variation along the axis. $l_1 / L = 1/2$ $l_2 / L = 1/2$, $h_{B1} / h_{A1} = 1/2$, $h'_{B1} = 0$, $h_{A2} / h_{B1} = 1/2$, $h_{B2} / h_{A2} = 1/2$, $h'_{A2} = 0$, $K_{W1} = 10$, $K_{\psi 1} = 1$, $\eta = 10$.

| $K_{\psi d}$ | $K_{Wd}$ | Method | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | DQM | 9.94295 | 21.1789 | 37.1287 | 54.4936 | 77.1422 |
| | | FEM | 9.97877 | 21.2145 | 37.1668 | 54.5377 | 77.1717 |
| | 0.1 | DQM | 12.3497 | 26.6244 | 45.1340 | 63.3202 | 86.7544 |
| | | FEM | 12.3950 | 26.6574 | 45.1757 | 63.3506 | 86.7801 |
| | 1 | DQM | 12.6335 | 27.5522 | 46.9619 | 67.3571 | 93.8220 |
| | | FEM | 12.6802 | 27.5867 | 47.0072 | 67.3872 | 93.8529 |
| | 10 | DQM | 12.6651 | 27.6547 | 47.1481 | 67.7705 | 94.4732 |
| | | FEM | 12.7119 | 27.6894 | 47.1937 | 67.8007 | 94.5049 |
| | $\to\infty$ | DQM | 12.6686 | 27.6662 | 47.1687 | 67.8159 | 94.5425 |
| | | FEM | 12.7155 | 27.7009 | 47.2144 | 67.8461 | 94.5743 |
| 1 | 0 | DQM | 9.96265 | 21.5532 | 38.7860 | 57.0231 | 81.5044 |
| | | FEM | 9.99874 | 21.5890 | 38.8248 | 57.0634 | 81.5315 |
| | 0.1 | DQM | 13.0297 | 27.9504 | 45.9575 | 63.5235 | 86.9102 |
| | | FEM | 13.0763 | 27.9817 | 45.9976 | 63.5530 | 86.9362 |
| | 1 | DQM | 13.5409 | 29.8282 | 49.4887 | 70.9743 | 97.0122 |
| | | FEM | 13.5897 | 29.8623 | 49.5335 | 71.0012 | 97.0437 |
| | 10 | DQM | 13.5994 | 30.0410 | 49.8608 | 71.8782 | 98.3059 |
| | | FEM | 13.6484 | 30.0754 | 49.9060 | 71.9052 | 98.3387 |
| | $\to\infty$ | DQM | 13.6060 | 30.0648 | 49.9016 | 71.9759 | 98.4391 |
| | | FEM | 13.6550 | 30.0993 | 49.9469 | 72.0030 | 98.4720 |
| 10 | 0 | DQM | 9.96324 | 21.5641 | 38.8347 | 57.1056 | 81.6606 |
| | | FEM | 9.99933 | 21.5999 | 38.8736 | 57.1459 | 81.6876 |
| | 0.1 | DQM | 13.0583 | 28.0142 | 46.0008 | 63.5342 | 86.9175 |
| | | FEM | 13.1049 | 28.0454 | 46.0408 | 63.5635 | 86.9436 |
| | 1 | DQM | 13.5802 | 29.9428 | 49.6333 | 71.2194 | 97.2490 |
| | | FEM | 13.6291 | 29.9769 | 49.6781 | 71.2462 | 97.2806 |
| | 10 | DQM | 13.6399 | 30.1611 | 50.0148 | 72.1525 | 98.5813 |
| | | FEM | 13.6890 | 30.1956 | 50.0600 | 72.1794 | 98.6142 |
| | $\to\infty$ | DQM | 13.6467 | 30.1856 | 50.0566 | 72.2531 | 98.7177 |
| | | FEM | 13.6958 | 30.2201 | 50.1018 | 72.2800 | 98.7508 |
| $\to\infty$ | 0 | DQM | 9.96331 | 21.5653 | 38.8403 | 57.1151 | 81.6785 |
| | | FEM | 9.99940 | 21.6011 | 38.8792 | 57.1553 | 81.7056 |
| | 0.1 | DQM | 13.0616 | 28.0216 | 46.0058 | 63.5354 | 86.9184 |
| | | FEM | 13.1082 | 28.0529 | 46.0459 | 63.5648 | 86.9444 |
| | 1 | DQM | 13.5848 | 29.9563 | 49.6504 | 71.2486 | 97.2773 |
| | | FEM | 13.6337 | 29.9904 | 49.6952 | 71.2753 | 97.3090 |
| | 10 | DQM | 13.6447 | 30.1752 | 50.0329 | 72.1851 | 98.6141 |
| | | FEM | 13.6938 | 30.2097 | 50.0781 | 72.2120 | 98.6471 |
| | $\to\infty$ | DQM | 13.6514 | 30.1997 | 50.0748 | 72.2860 | 98.7509 |
| | | FEM | 13.7006 | 30.2342 | 50.1201 | 72.3129 | 98.7840 |

Table 10. First natural frequencies $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2 \omega_i$ for a two-span elastically restrained rotating Timoshenko beam, with elliptical cross section and quadratic height variation along the axis. $l_1 / L = 1/2$ $l_2 / L = 1/2$, $h_{B1} / h_{A1} = 1/2$, $h'_{B1} = 0$, $h_{A2} / h_{B1} = 1/2$, $h_{B2} / h_{A2} = 1/2$, $h'_{A2} = 0$, $K_{W1} = 10$, $K_{\psi 1} = 0.1$, $\eta = 10$.




a) $K_{W1}=10$, $K_{\psi 1}=5$, $K_{Wd}=0$, $K_{\psi d}=0$ b) $K_{W1}=10$, $K_{\psi 1}=5$, $K_{Wd}=0.1$, $K_{\psi d}=1$

Fig. 4. Natural frequencies mode shapes for a two-span elastically restrained rotating Timoshenko beams, with elliptical cross section and quadratic height variation along the axis. $l_1/L=l_2/L=1/2$ ; $h_{B1}/h_{A1}=1/2$; $h'_{B1}=0$; $h_{A2}/h_{B1}=1/2$; $h_{B2}/h_{A2}=1/2$; $h'_{A2}=0$; $\eta=10$

Figure 4 shows the first four natural frequency mode shapes for beams, with two different kinds of boundary conditions: a) corresponds to $K_{W1}=10$, $K_{\psi 1}=5$, $K_{Wd}=0$, $K_{\psi d}=0$, while b) corresponds to $K_{W1}=10$, $K_{\psi 1}=5$, $K_{Wd}=0.1$, $K_{\psi d}=1$.

The next Figures, 5 and 6, present the variation of the fundament frequency parameter $\Omega_1$ with the variation of the non-dimensional rotational speed $\eta$ and the spring constant $K\psi_1$ .

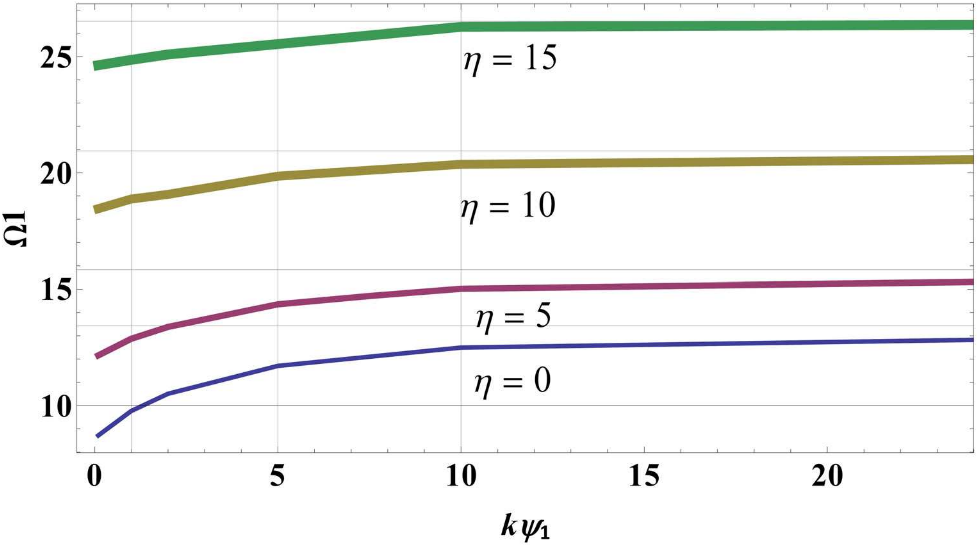



Fig. 5. The fundamental frequency coefficient $\Omega_1$ of a one-span elastically restrained rotating Timoshenko beam versus the spring constant parameter of the rotational spring $K_{\psi 1}$, for different rotational speed parameters $\eta$. $K_{w1}$= 10; $K_{wd}$=1; $K_{\psi d}$=10

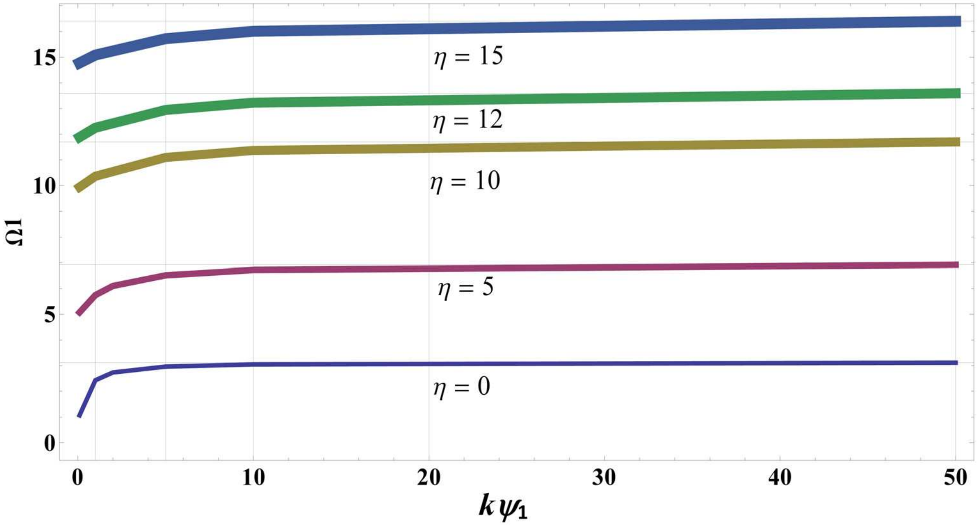



Fig. 6. The fundamental frequency coefficient $\Omega_1$ of a two-span elastically restrained rotating Timoshenko beam versus the spring constant parameter of the rotational spring $K_{\psi 1}$, for different rotational speed parameters $\eta$. $K_{w1}$= 10; $K_{wd}$=0; $K_{\psi d}$=0

## 6. Conclusion

The differential quadrature method proves to be very efficient to obtain frequencies and mode shapes of natural vibration, for the rotating Timoshenko beam model.

The versatility of the proposed beam model (variable cross section, step change in cross section, elastic restraints at both ends) allows to solve a large number of individual cases.

Something interesting to point out is that because the method directly solves two ordinary differential equations, additional restrictions are not generated. This does not happen in other methodologies, such as the dynamic stiffness method (Banerjee, 2000, 2001).

As a matter of fact, the differential quadrature method has the same advantage as the finite element method and it needs less computer memory requirements than the FEM.

In particular the present results show that the frequency coefficients vary more significantly when the translational spring stiffness changes at the end of the beam farthest from the axis of rotation $K_{\psi d}$.

## 7. Appendix A

As Shu presents in his book (Shu, 2000), the differential quadrature method, DQM, is a numerical technique for solving differential equations.

In order to obtain the DQM analog equations to the governing equations of the rotating beam and its boundary conditions, the beam domain is discretized in a grid of points using the Chebyshev – Gauss - Lobato expression, (Shu & Chen, 1999):

$$x_i = \frac{1-\cos\left[(i-1)\pi/(n-1)\right]}{2} \quad ; \; i = 1,2,\ldots,n$$

where $n$ is the number of discrete points or nodes and $x_i$ is the coordinate of node $i$.

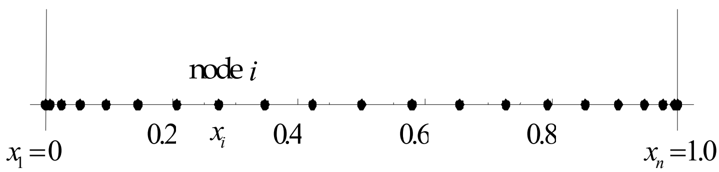



Fig. A1. Grid of $n$ points

The weighting coefficients $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$, which appeared in the linear algebraic equations of quadrature (28-35), were determined using the explicit expressions cited by (Bert & Malik, 1996).

The coefficients $A_{ij}^{(1)}$ correspond to first order derivatives and can be arranged in a square matrix of order $n$.

The matrix elements $A_{ij}^{(1)}$ with $i \neq j$, are determined by:

$$A_{ij}^{(1)} = \frac{\Pi(x_i)}{(x_i - x_j)\,\Pi(x_j)}$$

where

$$\Pi(x_i) = \prod_{\nu=1\ \nu\neq i}^{n} (x_i - x_\nu)\ ;\ \Pi(x_j) = \prod_{\nu=1\ \nu\neq i}^{n} (x_j - x_\nu)\ ;$$

The coefficients $A_{ij}^{(1)}$ with $i = j$ , will tend to infinity and need to be calculated in another way.

The coefficients $A_{ij}^{(2)}$ correspond to second-order derivatives and are obtained from

$$A_{ij}^{(2)} = 2\left[A_{ii}^{(1)} * A_{ij}^{(1)} - \frac{A_{ij}^{(1)}}{x_i - x_j}\right]$$

with $i \neq j$ and $i,\ j = 1, 2, 3, \ldots, n$.
Because the sum of the weighting coefficients of a row of the matrix is zero, it is easy to calculate the diagonal terms of derivatives of any order $q$, using the following expression:

$$A_{ii}^{(q)} = -\sum_{j=1\ j\neq i}^{n} A_{ij}^{(q)}$$

And the equations for $q$ equal to 1 and 2, corresponding to first and second order derivatives, are:

$$A_{ii}^{(1)} = -\sum_{j=1\ j\neq i}^{n} A_{ij}^{(1)}\ ;\ \ A_{ii}^{(2)} = -\sum_{j=1\ j\neq i}^{n} A_{ij}^{(2)}$$

## 8. Acknowledgment

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## 9. References

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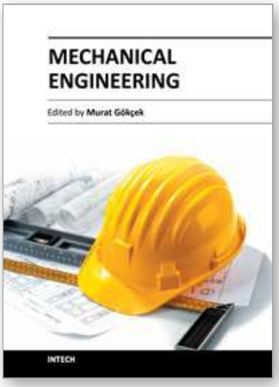


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