

A Reformulative Retouch on the Fourier Transform – “Unprincipled” Uncertainty Principle

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1. Introduction

According to a widespread notion, a monochromatic harmonic temporal wave's frequency ω becomes inherently “uncertain” to a degree $\Delta\omega$ if the wave lasts for only a limited time Δt owing to the “Uncertainty Principle” which states that $\Delta t \cdot \Delta\omega \geq \text{constant}$ (i.e. $\Delta\omega$ cannot be zero if Δt is finite). This frequency uncertainty is commonly thought of as also being reflected in the fact that the (“sinc”-shaped) frequency spectrum obtained by Fourier-transforming a finite-duration monochromatic wave spreads over a nonzero frequency range. Since, as often argued, the Fourier spectrum represents the “frequency components” of a temporal function (or signal), the nonzero width of the spectrum tells us that the “nominally” monochromatic ω frequency of a time-limited sinusoid is “effectively” polychromatic, with frequencies continuously distributed over a $\Delta\omega$ range in accord with the Uncertainty Principle.

A typical area where this argument shows up is Pulsed Fourier-Transform (PFT) nuclear magnetic resonance (NMR) spectroscopy. In NMR a molecular ensemble is placed in a strong homogeneous static magnetic field, whereupon the ensemble's NMR-active nuclei (those that act like spinning atomic magnets, say protons) will precess about the direction of the magnetic field. Those “spins” in the ensemble which precess (Larmor precession) with identical ω_L frequencies form a net macroscopic magnetization which is normally aligned along the external field, but which will also (Larmor-)precess with a frequency ω_L following a transient perturbation of the magnetization from its equilibrium state. Typically there are several such bundles of homogeneous spins in the analyzed sample, covering a range of Larmor frequencies $\Delta\omega_L$. PFT-NMR works on the principle that the sample's bulk magnetizations of interest can be excited into precession-mode (made to “resonate”) simultaneously in a wide $\Delta\omega_L$ (“off-resonance”) frequency range when perturbed by a “driving” electromagnetic wave of a single well-defined monochromatic radio-frequency (RF) ω_D if the wave is applied as a “pulse” in a sufficiently short time Δt . Because this phenomenon may appear counter-intuitive, the fundamental NMR literature is abundant in attempts to offer a preliminary explanation for why pulsed excitation works, typically invoking the “Uncertainty Principle” and the “sinc”-shaped FT-spectrum of the pulse according to the notions outlined above. Some representative examples from authoritative

sources are as follows: "...although the applied excitation may be precisely centred at a frequency ω_D , [...], our act of turning the excitation power on at time zero and off at time Δt effectively broadens the spectral range of the excitation (to a bandwidth of $\sim 1/\Delta t$)." (Marshall and Verdun, 1990); "A pulse of monochromatic RF with a rectangular envelope can be described in the frequency domain as a band of frequencies centred at the RF frequency. The Heisenberg principle states that there is a minimum uncertainty in the simultaneous specification of [...] the frequency of a system and the duration of the measurement. [...] this means that the [...] irradiation is spread over a wide frequency band. [...] the 'sinc' Fourier spectrum of a rectangular RF pulse shows that a shorter pulse gives a wider 'sinc' band and a longer pulse gives a narrower 'sinc' band." (King and Williams, 1989); "...if the pulse is made shorter, we will no longer have a truly monochromatic Fourier spectrum even though the source is still monochromatic. This is because many different frequencies have to be combined in order to form the rising and falling edges of the rectangular pulse." (Fukushima and Roeder, 1981); "... as the Uncertainty Principle indicates, a pulse [of carrier frequency ω_D] will contain, in effect, a range of frequencies centred on ω_D . [...] the distribution of RF magnetic field amplitudes takes the 'sinc' form [...] which is the frequency-domain equivalent of a short pulse in the time domain. The two domains are connected by the Fourier transform." (Harris, 1983); "... the RF source [...] is monochromatic, so we have to work out a way of using a single frequency to excite multiple frequencies. To see how this can be done we take our cue from the Uncertainty Principle. If the irradiation is applied for a time Δt , then [...] the nominally monochromatic irradiation is uncertain in frequency by about $1/\Delta t$." (Derome, 1987)

One should quickly note that at a macroscopic level the basic NMR phenomenon is a purely classical effect (Hoult, 1989 and Hanson, 2008) involving interactions between the oscillating magnetic component of the RF pulse and the sample's bulk magnetizations. Indeed, the fact that a monochromatic pulse can excite spins in a wide frequency range can be well explained classically using the laws of driven mechanical harmonic oscillators and magnetism (Bloch, 1946) without resorting to the vague idea that the ω_D RF pulse frequency should be "uncertain" or "effectively" polychromatic according to its Fourier spectrum. Furthermore, except for one case the above arguments are unclear on whether by "Uncertainty Principle" they mean the Heisenberg Uncertainty Principle (HUP) which is a probabilistic, quantum-mechanical statement, or the Fourier Uncertainty Principle (FUP) which is a deterministic theorem in the discipline of time-frequency analysis.

In a series of recent papers (Szántay, 2007, 2008a, 2008b, 2008c) the author of the present work explored the above problem in detail, showing that a) the idea that the frequency of a time-limited sinusoid would be uncertain in the Heisenberg sense is flawed; b) it is the FUP (and not the HUP) that is of relevance to the problem; c) the notion that a monochromatic pulse "effectively" comprises of a range of physically existing frequency components is erroneous. During that discourse the need had come up to contemplate the essence of the Fourier Transform (FT) itself, which may appear somewhat odd considering the fact that the FT is ubiquitously applied in a broad range of sciences and technology, and works explaining its mathematics and its manifold physical applications are available in almost countless abundance. However, it turns out that the FT is so commonplace, it has become so much consolidated into many fields of science and engineering, its basic principles and formulas are so commonly taken for granted, that, paradoxically, this can easily stop people

from thinking more deeply about how and why the FT works as a mathematical tool and what exactly it means when evoked to interpret physical phenomena. One of the central features of the present author’s approach in discussing the FT was the use of a “touched up” mathematical formalism to provoke a new way of looking at some key ideas of the FT and to avoid some hidden ambiguities in meaning that are otherwise inherent to the conventional formalism and can give rise to the misconceptions mentioned above. Besides coming out with a disambiguative but symbolically elaborate formalism for the FT involving the depiction of the domain intervals on which functions “exist” and introducing the concept of thinking about the FT in either “trigonometric-frequency-space” or “phasor-frequency-space”, those articles also expounded, from a new perspective, on several apparently simple but in practice elusive related concepts such as “periodicity”, “harmonic oscillation”, “phasor”, “negative frequency”, “uncertainty”, etc.; much emphasis was placed also on a comparison of Fourier analysis and the Fourier transform, on the pros and cons of metaphoric thinking and the dangers of unrecognized linguistic ambiguities (e.g., words and phrases like “*uncertainty*”, “*nominally* monochromatic” and “*effectively* polychromatic” deceptively appear to be meaningful, but on deeper examination their meaning is ambiguous and notoriously misunderstood).

Owing to the volume and thematic complexity of those papers the need has repeatedly arisen to present its main thrusts in a more compact and symbolically simplified form. The present work therefore attempts to condense the original discussion into a shorter communication by focusing only on two themes, namely the reformulation of the primary FT equations (other closely related topics such as Fourier analysis will be left out) and the main conclusions on the problem of the “Uncertainty Principle”. These thoughts are hoped to provide some fresh insights and a deeper understanding of the essence of the FT and the Uncertainty Principles. In that regard it should be emphasized that the applied mathematical formalism is not a *l’art pour l’art* design: it serves to facilitate a more conscientious way of thinking about the pertinent topics. For the full technical and philosophical details behind the ideas presented here the reader is encouraged to consult the original articles.

2. Nomenclature

2.1 General notation

- The term *Fourier analysis* will be used to refer to the Fourier series expansion of periodic functions and the term *Fourier transform* (FT) will refer to the generalization of Fourier analysis including nonperiodic functions. The FT will be discussed only in the context of the Fourier-pair variables time (t) and frequency (ω). The word “spectrum” will in general mean the ω -dimension (ω -D) representation of a time-dimension (t -D) function, more specifically as it has been obtained by the FT.
- Sets will be represented by bold, italicized letters while vectors will be denoted by bold, non-italicized letters. If in a set \mathbf{X} its members x can assume values on a continuous scale, I will use the symbol “@” as a primary generic index to tag an arbitrary value as “ $x_{@}$ ” so as to emphasize that we are focusing on that particular value in the set (“@” can be interpreted as: “at a given value”. Members x of the sets \mathbf{X} , \mathbf{X}^{+} and \mathbf{X}^{-} can in general take on negative as well as positive values, only nonnegative values, and only nonpositive values, respectively. If a set \mathbf{X} represents an interval for the variable

$x \in \mathbf{X}$, we will consider five main interval types labeled as follows: $\overleftrightarrow{\mathbf{X}} = (x = -\infty, x = \infty)$ is open and unbounded on both sides; $\overleftrightarrow{\mathbf{X}}^{\bullet\bullet} = [x_{@}, x_{@}']$ is closed and bounded (compact) at arbitrary $x_{@}$ and $x_{@}'$ values on both sides; $\overrightarrow{\mathbf{X}} = [x = 0, x = \infty)$ is closed and bounded at zero on the left, open and unbounded on the right and $\overleftarrow{\mathbf{X}} = (x = -\infty, x = 0]$ is closed and bounded at zero on the right and open and unbounded on the left; $\overset{\uparrow}{\mathbf{X}}_{@} = \{x | x_{@} \leq x \leq x_{@}', x_{@} \leftarrow x_{@}'\}$ will be associated with a function which has a finite function value at $x = \overset{\bullet}{x}_{@}$ but is zero if $x \neq \overset{\bullet}{x}_{@}$; $\overset{\uparrow}{\mathbf{X}}_{@}^{\bullet} = \{x | x_{@} \leq x \leq x_{@}', x_{@} \leftarrow x_{@}'\}$ will be used in connection with a Dirac delta whose function value is infinite at $x = \overset{\uparrow}{x}_{@}$ but zero if $x \neq \overset{\uparrow}{x}_{@}$.

- Underlined symbols are used to denote complex quantities having a real (\Re) and an imaginary (\Im) component, so a complex constant $\underline{A} = (\overset{\Re}{A}, \overset{\Im}{A})$ is defined and symbolized as $\underline{A} = \overset{\Re}{A} + i \overset{\Im}{A} = \overset{\Re}{A} + \overset{\Im}{A} i$ where $i \overset{\Im}{A} = \overset{\Im}{A} i$. Hence \underline{A} has the absolute value $|\underline{A}| = A = \sqrt{(\overset{\Re}{A})^2 + (\overset{\Im}{A})^2} = \sqrt{(\overset{\Re}{A})^2 - (\overset{\Im}{A})^2}$. A complex number \underline{A} expressed in the form of a complex exponential $\underline{A} = A \cdot \exp(i \cdot \phi)$ will be referred to as a *phasor* where the phase ϕ is measured from the positive real axis ($+\Re$) with a counterclockwise angle being positive and a clockwise angle being negative. A non-underlined character, if lacking the tags \Re or \Im , represents a real-valued quantity by default. Likewise, a function in general is represented as $f(x)$ if it is real-valued and as $\underline{f}(x)$ if it is complex-valued.
- The action of a mathematical formula (rule, operator, etc...) upon a set will be denoted as $\text{Formula} \overline{\text{set}}$. Thus, a rule acting on a variable x to give a function $\underline{f}(x)$ is written as $\underline{f}(x) = \text{Rule} \overline{x}$, and an operator that converts a function $\underline{f}(x)$ into $\underline{F}(y)$ is depicted as $\underline{F}(y) = \text{Operator} \overline{\underline{f}(x)}$.
- The natural (largest possible) domain of a function $\underline{f}(x) = \text{Rule} \overline{x}$ is the set of all allowable inputs that the function's argument x may assume. However, $\underline{f}(x)$ can of course be restricted to or can simply be of interest to us over a subset \mathbf{X} of the natural domain, and herein \mathbf{X} will be called the *operative interval* of $\underline{f}(x)$. A function's operative interval will be denoted as $\underline{f}(x) \overline{\mathbf{X}} = \text{Rule} \overline{x} \overline{\mathbf{X}}$ and should be interpreted either as having no values or zero function values for any $x \notin \mathbf{X}$. For example, for a single rectangular "bump" along x we have $\mathbf{X} = \overleftrightarrow{\mathbf{X}}^{\bullet\bullet}$ corresponding to the rectangle's width, while for a function that starts with a finite value at $x = 0$ but decays exponentially to zero at $x = \infty$ we have $\mathbf{X} = \overrightarrow{\mathbf{X}}$ ranging from zero to positive infinity. Care must be taken not to confuse the operative interval (which is always indicated as a subscript following a bar

placed after the function’s name or rule of association) with the function’s natural domain. For example, for a function $f(x) \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}}$ the natural domain may be $\overset{\cdot}{\underset{\cdot}{\cdot}}$ but the operative interval is $\overset{\cdot}{\underset{\cdot}{\cdot}} \in \overset{\cdot}{\underset{\cdot}{\cdot}}$; for $f(x) \overset{\cdot}{\underset{\cdot}{\cdot}}$ the operative interval is $\overset{\cdot}{\underset{\cdot}{\cdot}}$, implying that the natural domain must also be $\overset{\cdot}{\underset{\cdot}{\cdot}}$. A proper understanding of the FT requires that one should always be conscious of what domains and intervals are relevant to the time scales and frequency ranges in the pertinent mathematical expressions. These aspects are often far from being self-evident from the formulas themselves in their conventional notation, and are usually present as hidden assumptions or unstated premises. To overcome this problem herein the pertinent interval types will be represented as $f(x) \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}} = \text{Rule} \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}}$; $f(x) \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}} = \text{Rule} \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}}$; $f(x) \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}} = \text{Rule} \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}}$; $f(x) \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}} = \text{Rule} \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}}$, $f(x) \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}} = \text{Rule} \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}}$ if $x = x_{@}$, otherwise zero, and $f(x) \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}} = \text{Rule} \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}}$ if $x = x_{@}$, otherwise zero. If the properties of $f(x)$ are considered on different intervals simultaneously, the pertinent interval symbols will be separated by semicolons, such as $f(x) \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}} = \text{Rule} \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}} \overset{\cdot}{\underset{\cdot}{\cdot}}$.

2.2 Trigonometric and phasor frequencies

The symbol ω will be used as a general representation of (angular) frequency such that the oscillation frequencies associated with a temporal function have constant values ($\omega(t) = \text{constant}$) over the operative interval \mathbf{t} of the function. In the customary notation a sinusoid (trigonometric) harmonic wave with a complex amplitude (i.e., a 1D oscillation in the complex plane that gives a 2D wave as a function of time) has the general form $\underline{A}(\omega) \cdot \cos[(\omega t) + \phi^\circ(\omega)]$ and a *harmonic phasor* (i.e., a vector \mathbf{A} rotating uniformly either clockwise or anti-clockwise in the complex plane, and describing a 3D helix as a function of time) has the general form $A(\omega) \cdot \exp[i \cdot \phi^\circ(\omega)] \cdot \exp(i \cdot \omega t) = \underline{A}(\omega) \cdot \exp(i \cdot \omega t)$, where ϕ° is the wave’s initial phase at $t = 0$. For the harmonic phasor the phase angle $\varphi(t) = \omega t + \phi^\circ$ is measured from the positive real axis (+ \Re) with a counterclockwise angle being positive and a clockwise angle being negative. For simplicity both a sinusoid wave and a harmonic phasor will be referred to as a “harmonic wave”.

The sinusoid and phasor forms of a harmonic wave are connected by Euler’s famous identity, written in its conventional form as $\exp(i \cdot \omega t) = \cos(\omega t) + i \cdot \sin(\omega t)$, from which the trigonometric terms can be expressed by rearrangement as $\cos(\omega t) = 0.5 \cdot \exp(i \cdot \omega t) + 0.5 \cdot \exp(-i \cdot \omega t)$, and $\sin(\omega t) = -i \cdot 0.5 \cdot \exp(i \cdot \omega t) + i \cdot 0.5 \cdot \exp(-i \cdot \omega t)$. Using this conventional notation the concept of the sign of ω , in particular the idea of a negative frequency, can be a confusing issue. Thus, based on making a distinction between “trigonometric frequency” and “phasor frequency” the following notation is introduced: a *trigonometric frequency* (the frequency of a sinusoid oscillation) will be labeled as $\omega \in \boldsymbol{\omega}$; a *phasor frequency* (the frequency of harmonic phasor) will be labeled as $\omega \in \boldsymbol{\omega}$. Accordingly, a sinusoid harmonic wave $\underline{h}_{\omega}(t)$ and a harmonic phasor $\underline{h}_{\omega}(t)$ take the following general forms:

$$\underline{h}_{\omega}(t) = \underline{A}(\omega) \cdot \cos[(\omega t) + \varphi^{\circ}(\omega)], \quad (1)$$

$$\underline{h}_{\omega}(t) = A(\omega) \cdot e^{i \cdot \varphi^{\circ}(\omega)} \cdot e^{i \cdot \omega t} = \underline{A}(\omega) \cdot e^{i \cdot \omega t} = \Re A(\omega) \cdot e^{i \cdot \omega t} + i \cdot \Im A(\omega) \cdot e^{i \cdot \omega t}. \quad (2)$$

Note from (2) that the phasor's complex amplitude \underline{A} always corresponds to $t = 0$ since \underline{A} is defined by its magnitude (absolute value) and the phasor's initial phase, which are both time-independent quantities. From (2) we see that the phasor $\underline{A}(\omega) \cdot \exp(i \cdot \omega t)$ can be equivalently viewed also as a sum of two phasors of the same frequency, one with a real amplitude $\Re A$, and the other with a pure imaginary amplitude $i \Im A$.

For the trigonometric wave $h_{\omega}(t)$ the meaning of ω seems straightforward: it expresses the number of full periods (multiplied by 2π) completed by the wave in unit time, whereby ω is defined as having positive numerical values only since it would not make sense to speak of a negative number of oscillations per unit time. The algebraically positive nature of ω merits emphasis because in the context of the FT trigonometric frequencies are sometimes (mistakenly) given negative values in the literature [e.g. the fact that neither the cosine transform, nor the sine transform alone can distinguish between the two possible rotational directions of a harmonic phasor is often explained by the formal truths that $\cos[(-\omega)t] = \cos(\omega t)$ and $\sin[(-\omega)t] = -\sin(\omega t)$]. However, one should appreciate that the concept of $\cos[(-\omega)t]$ (or $\cos[(-\omega)t]$) or $\sin[(-\omega)t]$ (or $\sin[(-\omega)t]$) does not make any physical sense. This issue is partly rooted in the fact that the literature makes no symbolic distinction between “ ω ” and “ φ ”, hence the intended meaning (phasor vs. trigonometric) of “frequency”, labeled as “ ω ”, is typically an implicit consequence of the context in which it appears but is not implied or conveyed by the symbol “ ω ” itself. Thus the formulas $\cos[(-\omega)t]$ or $\sin[(-\omega)t]$ may easily escape mental objection when speaking about positive and negative phasor frequencies when the latter are also labeled as $+\omega$ and $-\omega$. There is in fact no need to allow sinusoid frequencies to be negative (see below) and we can adhere to the physically appropriate definition that trigonometric frequencies are always positive.

A harmonic phasor $h_{\omega}(t)$ has either a positive or a negative frequency ω^{+} and ω^{-} depending on the direction of “skew” (or “helicity”) of the 3D helix described by the phasor as a function of time. I employ the convention according to which for positive ω the phasor $\underline{h}_{\omega^{+}}(t) = A \cdot \exp(i \cdot \omega^{+} t)$ corresponds to the vector \mathbf{A} rotating in the complex plane in the $+\Re \rightarrow +\Im \rightarrow -\Re \rightarrow -\Im$ direction as time progresses, while for negative ω the phasor $\underline{h}_{\omega^{-}}(t) = A \cdot \exp(i \cdot \omega^{-} t) = A \cdot \exp(-i \cdot \omega^{+} t)$ represents rotation in the opposite sense.

Note that according to the above considerations there is an intrinsic difference between the concepts of the “sign” of the frequency for a harmonic sinusoid wave and a harmonic phasor. If, in a (\Re, \Im, t) space, we associate a positive skew with ω^{+} and a negative skew with ω^{-} (and this is what we mean by the positive and negative sign of ω), then a trigonometric frequency ω , which is “skew-less”, should also be “sign-less”. Thus our definition of ω being always positive carries the implicit understanding that a numerically positive frequency value requires a different physical interpretation for ω and ω^{+} .

All subsequent equations will be formulated in line with the above definitions on the concept of frequency. Forcing oneself to accordingly rethink the common text-book equations related to the FT is an instructive endeavour yielding added levels of insight.

Furthermore, the concept of a “frequency sweep” is introduced as follows: the symbols $\overleftrightarrow{\omega}$, $\overrightarrow{\omega}$ and $\overleftarrow{\omega}$ emphasize the fact that we take into account all values for the variable ω in the continuous sets $\overleftrightarrow{\omega}$, $\overrightarrow{\omega}$ and $\overleftarrow{\omega}$, respectively. Thus we may think of $\overleftrightarrow{\omega}$, $\overrightarrow{\omega}$ and $\overleftarrow{\omega}$ as representing a “sweep” of the frequency value from $-\infty$ to $+\infty$, from 0 to $+\infty$, and from 0 to $-\infty$, respectively, in phasor-frequency space. Likewise, the symbol $\overleftrightarrow{\omega}$ designates a trigonometric-frequency “sweep” of ω from 0 to $+\infty$ in the set $\overrightarrow{\omega}$. Frequency-sweep notation will be used, when needed for emphasis, in expressions describing temporal harmonic waves as well as in the function names of FT-spectra. For example, when stressing that we are considering the set of all sinusoid harmonic waves $\underline{h}_{\overleftrightarrow{\omega}}(t)$ covering the entire trigonometric frequency range $\overleftrightarrow{\omega}$ where each wave lasts from $-\infty$ to $+\infty$, we write $\underline{h}_{\overleftrightarrow{\omega}}(t)|_{\mathfrak{F}} = \underline{A}(\overleftrightarrow{\omega}) \cdot \cos[\overleftrightarrow{\omega}t + \varphi^\circ(\overleftrightarrow{\omega})]|_{\mathfrak{F}}$. Similarly, the set of all perpetual harmonic phasors in the infinite phasor-frequency range $\overleftrightarrow{\omega}$ can be written as $\underline{h}_{\overleftrightarrow{\omega}}(t)|_{\mathfrak{F}} = \underline{A}(\overleftrightarrow{\omega}) \cdot \exp(i \cdot \overleftrightarrow{\omega}t)|_{\mathfrak{F}}$. If however we want to focus on a harmonic wave at a given frequency value we write $\underline{h}_{\overleftrightarrow{\omega}_@}(t) = \underline{A}_@ \cdot \cos(\overleftrightarrow{\omega}_@t + \varphi^\circ_@)$ or $\underline{h}_{\overleftrightarrow{\omega}_@}(t) = \underline{A}_@ \cdot \exp(i \cdot \overleftrightarrow{\omega}_@t)$.

2.3 FT-contextualized function notation in the time and frequency dimensions

The ways in which the forward and inverse FT operators $\overrightarrow{\underline{F}\overline{\underline{T}}}$ and $\overleftarrow{\underline{F}\overline{\underline{T}}}$ connect a t -D function $\underline{f}(t)$ and its ω -D counterpart $\underline{F}(\omega)$ are denoted as $\underline{F}(\omega) = \overrightarrow{\underline{F}\overline{\underline{T}}}\underline{f}(t)$ and $\underline{f}(t) = \overleftarrow{\underline{F}\overline{\underline{T}}}\underline{F}(\omega)$, respectively. The so-called Fourier Inversion Theorem (FIT) (see below) states that the FT is an invertible transform: there is a one-to-one correspondence between all temporal and spectral function pairs that we consider here; to reflect this concept the forward and backward FT operator symbols are combined into the symbol $\overleftrightarrow{\underline{F}\overline{\underline{T}}}$, so the link between the two dimensions can be depicted as: $\underline{f}(t) \xleftrightarrow{\overleftrightarrow{\underline{F}\overline{\underline{T}}}} \underline{F}(\omega)$.

In the time dimension we generally consider complex-valued temporal functions $\underline{f}(t)|_{\mathfrak{F}}$ on a real timeline, with the operative duration \mathfrak{F} corresponding to the time frame over which the (forward) FT is performed. This means that $\underline{f}(t)|_{\mathfrak{F}} = \text{“something”}$ for $t \in \mathfrak{F}$ (that “something” may contain values of $\underline{f}(t) = \text{zero}$ besides nonzero values) but $\underline{f}(t)|_{\mathfrak{F}} = \text{“nothing”}$ (i.e., $\underline{f}(t)$ is either zero or nonexistent) for all $t \notin \mathfrak{F}$. All $\underline{f}(t)|_{\mathfrak{F}}$ functions that are of practical relevance to us can be Fourier-transformed, i.e., their spectrum exists and $\underline{f}(t)|_{\mathfrak{F}}$ can be recovered exactly

(for all practical purposes) from its spectrum, and all nonperiodic temporal functions labeled here as $\underline{f(t)}|_{\mathbf{t}}$ have the extra feature that their function values are zero in the infinite past and in the infinite future - a property needed so that they can be absolute integrable (the integral of $|\underline{f(t)}|_{\mathbf{t}}$ over all t exists and is finite) which in turn is required for the spectrum of $\underline{f(t)}|_{\mathbf{t}}$ to exist in a form that contains only finite function values.

In the frequency dimension, when thinking of the spectrum $\underline{F(\omega)}|_{\omega}$ as a mathematical object that has been obtained by the forward FT of $\underline{f(t)}|_{\mathbf{t}}$, it will prove useful to extend the name $\underline{F(\omega)}|_{\omega}$ such that it can remind us of what specific operative temporal interval $\underline{f(t)}|_{\mathbf{t}}$ existed on. To that end the temporal function's operative interval \mathbf{t} will be indicated in the spectrum's name as $\underline{F_{\mathbf{t}}(\omega)}|_{\omega}$. Although $\underline{f(t)}|_{\mathbf{t}}$ and $\underline{F_{\mathbf{t}}(\omega)}|_{\omega}$ are equivalent representations of the same mathematical object, $\underline{f(t)}|_{\mathbf{t}}$, we may regard $\underline{f(t)}|_{\mathbf{t}}$ as the primary mathematical entity since the physical events that we are interested in and which are modelled by $\underline{f(t)}|_{\mathbf{t}}$ actually "happen" in time. A function $\underline{f(t)}|_{\mathbf{t}}$ describing an "event" of duration \mathbf{t} is defined as zero outside of \mathbf{t} , and in that sense we can ignore all $t \notin \mathbf{t}$ without actually having to "sweep through" the natural domain from $t = -\infty$ to $t = +\infty$. However, most $\underline{F_{\mathbf{t}}(\omega)}|_{\omega}$ spectra that we will encounter have an infinite operative interval $\overset{\curvearrowright}{\omega}$ or $\overset{\curvearrowleft}{\omega}$ such that the spectrum is continuous and has nonzero function values (allowing for zero values at points where $\underline{F_{\mathbf{t}}(\omega)}|_{\omega; \overset{\curvearrowright}{\omega}}$ may intercept or touch the ω axis) over its entire domain. Thus for the mathematical equivalence between $\underline{f(t)}|_{\mathbf{t}}$ and $\underline{F_{\mathbf{t}}(\omega)}|_{\omega; \overset{\curvearrowright}{\omega}}$ to hold, the forward FT must be performed by accounting for all possible frequencies in the infinite natural domains $\overset{\curvearrowright}{\omega}$ or $\overset{\curvearrowleft}{\omega}$, depending on whether we want to express the spectrum in ϕ -space or ϖ -space. To reflect this concept the notation for the pertinent frequency sweep (as described in the previous section) will be indicated in the spectrum's name. Thus, spectra with an infinite support are written as $\underline{F_{\mathbf{t}}(\bar{\omega})}|_{\omega; \overset{\curvearrowright}{\omega}}$ or $\underline{F_{\mathbf{t}}(\bar{\omega})}|_{\omega; \overset{\curvearrowleft}{\omega}}$.

An ϕ -space spectrum can always be equated with a suitable, complex combination of two ϖ -space spectra, one obtained through the "cosine transform" and one through the "sine transform" of $\underline{f(t)}|_{\mathbf{t}}$. Phasor representation usually being more convenient to work with, herein preference will generally be given to ϕ -space formalism (however, the trigonometric form will also be discussed to some extent since this gives some interesting added insight into the nature of the FT). In all, the FT relationship will most generally be thought of as:

$$f(t) \Big|_{\mathfrak{f}} \xleftrightarrow{\text{FT}} F_{\mathfrak{f}}(\bar{\omega}) \Big|_{\omega} \tag{3}$$

Finally, we will find that some spectra of the type $F_{\mathfrak{f}}(\bar{\omega}) \Big|_{\omega}$ are centred about a specific frequency $\omega_{@}$. Although in such cases the value of $\omega_{@}$ is contained in the relevant rule, Rule $\overline{\omega - \omega_{@}} \Big|_{\omega}$, it will often prove informative to indicate $\omega_{@}$ in the function’s name as well, which will be done by writing $F_{\mathfrak{f}}(\bar{\omega}_{>\omega_{@}}) \Big|_{\omega} = \text{Rule} \overline{\omega - \omega_{@}} \Big|_{\omega}$.

3. “Enhanced” Fourier-Transform equations

3.1 Euler’s identities

With the above considerations in mind we can reformulate Euler’s identities as follows. For a given frequency we have

$$\underline{A}_{@} \cdot e^{i \cdot \omega_{@} t} \Big|_{\mathfrak{f}} = \Re \underline{A}_{@} \cdot e^{i \cdot \omega_{@} t} \Big|_{\mathfrak{f}} + i \Im \underline{A}_{@} \cdot e^{i \cdot \omega_{@} t} \Big|_{\mathfrak{f}} = \underline{A}_{@} \cdot \cos(\varpi_{@} t) \Big|_{\mathfrak{f}} \pm i \cdot \underline{A}_{@} \cdot \sin(\varpi_{@} t) \Big|_{\mathfrak{f}} \tag{4}$$

$$\underline{A}_{@} \cdot \cos(\varpi_{@} t) \Big|_{\mathfrak{f}} = 0.5 \underline{A}_{@} \cdot e^{i \cdot \omega_{@}^+ t} \Big|_{\mathfrak{f}} + 0.5 \underline{A}_{@} \cdot e^{-i \cdot \omega_{@}^+ t} \Big|_{\mathfrak{f}} \tag{5}$$

$$\underline{A}_{@} \cdot \sin(\varpi_{@} t) \Big|_{\mathfrak{f}} = -i \cdot 0.5 \underline{A}_{@} \cdot e^{i \cdot \omega_{@}^+ t} \Big|_{\mathfrak{f}} + i \cdot 0.5 \underline{A}_{@} \cdot e^{-i \cdot \omega_{@}^+ t} \Big|_{\mathfrak{f}} \tag{6}$$

On the other hand the entire positive, entire negative, or complete positive and negative phasor frequency range can be embraced via positive trigonometric frequency notation which may be represented by writing Equations (4) – (6) as

$$\underline{A}(\bar{\omega}) \cdot e^{i \cdot \bar{\omega} t} \Big|_{\mathfrak{f}} = \Re \underline{A}(\bar{\omega}) \cdot e^{i \cdot \bar{\omega} t} \Big|_{\mathfrak{f}} + i \Im \underline{A}(\bar{\omega}) \cdot e^{i \cdot \bar{\omega} t} \Big|_{\mathfrak{f}} = \underline{A}(\bar{\omega}) \cdot \cos(\bar{\omega} t) \Big|_{\mathfrak{f}} \pm i \cdot \underline{A}(\bar{\omega}) \cdot \sin(\bar{\omega} t) \Big|_{\mathfrak{f}} \tag{7}$$

$$\underline{A}(\bar{\omega}) \cdot \cos(\bar{\omega} t) \Big|_{\mathfrak{f}} = 0.5 \underline{A}(\bar{\omega}) \cdot e^{i \cdot \bar{\omega} t} \Big|_{\mathfrak{f}} + 0.5 \underline{A}(\bar{\omega}) \cdot e^{-i \cdot \bar{\omega} t} \Big|_{\mathfrak{f}} \tag{8}$$

$$\underline{A}(\bar{\omega}) \cdot \sin(\bar{\omega} t) \Big|_{\mathfrak{f}} = -i \cdot 0.5 \underline{A}(\bar{\omega}) \cdot e^{i \cdot \bar{\omega} t} \Big|_{\mathfrak{f}} + i \cdot 0.5 \underline{A}(\bar{\omega}) \cdot e^{-i \cdot \bar{\omega} t} \Big|_{\mathfrak{f}} \tag{9}$$

These equations reflect the concept that we can describe a harmonic phasor’s frequency and sign of rotation in both phasor notation and complex trigonometric form, so long as in the latter case we use both the (real) cosine and (imaginary) sine waves - both of which have algebraically positive ϖ frequencies. There is no contradiction in the fact that we can obtain a negative phasor frequency ω^- by combining a real and an imaginary sinusoid, both of

which have positive ω trigonometric frequencies. Equations (5,6) and (8,9) show that while a given sinusoid has no frequency sign ambiguity (it is always positive) in ω -space, it exhibits a sign duality when expressed in ϕ -space, i.e., a sinusoid is always equated with the sum of two phasors, one with a positive and one with a negative frequency.

3.2 Interrelationships in temporal and spectral representations

In our formalism a key stepping stone to the FT involves the consideration of Euler’s identities from both a t -D and ω -D perspective. Let us take a harmonic phasor $\underline{A}_@ \cdot \exp(i \cdot \omega_@^\pm t) \Big|_{\mathbf{t}}$, called herein the *principle phasor*, which can be viewed as a vector rotating in time as shown in temporal representation for a positive phasor $\underline{A}_@ \cdot \exp(i \cdot \omega_@^\pm t) \Big|_{\mathbf{t}}$ in Figure 1(a). We can also find it informative to represent our phasor as an ω -D entity $\underline{A}_@ \Big|_{\dot{\omega}_@^\pm}$ as shown for $\underline{A}_@ \cdot \exp(i \cdot \omega_@^\pm t) \Big|_{\mathbf{t}}$ in Figure 2(A). The temporal and spectral representations emphasize different aspects of the phasor: the temporal form shows its progression in time, from which the numerical value of $\omega_@^\pm$ is not directly accessible, while the spectral representation displays the $\omega_@^\pm$ value in ϕ -space, but offers no direct sense of the phasor’s temporal behavior or of its operative interval \mathbf{t} . Considering the principle phasor $\underline{A}_@ \cdot \exp(i \cdot \omega_@^\pm t) \Big|_{\mathbf{t}}$ in the light of (4)–(6), we can formulate the following scheme:

$$\text{Temporal:} \quad \underline{A}_@ \cdot e^{i \cdot \omega_@^\pm t} \Big|_{\mathbf{t}} = \Re \underline{A}_@ \cdot e^{i \cdot \omega_@^\pm t} \Big|_{\mathbf{t}} + i \Im \underline{A}_@ \cdot e^{i \cdot \omega_@^\pm t} \Big|_{\mathbf{t}} \quad (10t)$$

$$\underline{A}_@ \cdot e^{i \cdot \omega_@^\pm t} \Big|_{\mathbf{t}} = \underbrace{\underline{A}_@ \cdot \cos(\omega_@ t) \Big|_{\mathbf{t}}}_{\parallel} + \underbrace{\pm i \cdot \underline{A}_@ \cdot \sin(\omega_@ t) \Big|_{\mathbf{t}}}_{\parallel} \quad (11t)$$

$$\underline{A}_@ \cdot e^{i \cdot \omega_@^\pm t} \Big|_{\mathbf{t}} = \left\{ \begin{array}{c} \overbrace{0.5 \underline{A}_@ \cdot e^{i \cdot \omega_@^\pm t} \Big|_{\mathbf{t}}} \\ + \\ \overbrace{0.5 \underline{A}_@ \cdot e^{i \cdot \omega_@^\mp t} \Big|_{\mathbf{t}}} \end{array} \right\} + \left\{ \begin{array}{c} \overbrace{\pm 0.5 \underline{A}_@ \cdot e^{i \cdot \omega_@^\pm t} \Big|_{\mathbf{t}}} \\ + \\ \overbrace{\mp 0.5 \underline{A}_@ \cdot e^{i \cdot \omega_@^\mp t} \Big|_{\mathbf{t}}} \end{array} \right\} \quad (12t)$$

$$\text{Spectral:} \quad \underline{A}_@ \Big|_{\dot{\omega}_@^\pm} = \Re \underline{A}_@ \Big|_{\dot{\omega}_@^\pm} + i \Im \underline{A}_@ \Big|_{\dot{\omega}_@^\pm} \quad (10s)$$

$$\underline{A}_@ \Big|_{\dot{\omega}_@^\pm} \leftrightarrow \underbrace{\underline{A}_@ \Big|_{\omega_@}}_{\downarrow} \quad \& \quad \underbrace{\pm i \cdot \underline{A}_@ \Big|_{\omega_@}}_{\uparrow} \quad (11s)$$

$$\underline{A}_@ \Big|_{\dot{\omega}_@^\pm} = \left\{ \begin{array}{c} 0.5 \underline{A}_@ \Big|_{\dot{\omega}_@^+} \\ + \\ 0.5 \underline{A}_@ \Big|_{\dot{\omega}_@^-} \end{array} \right\} + \left\{ \begin{array}{c} \pm 0.5 \underline{A}_@ \Big|_{\dot{\omega}_@^\pm} \\ + \\ \mp 0.5 \underline{A}_@ \Big|_{\dot{\omega}_@^\pm} \end{array} \right\} \tag{12s}$$

Equation (10t) tells us that we can decompose the principle phasor into a sum of two component phasors, ${}^{\Re}A_@ \cdot \exp(i \cdot \omega_@^\pm t)$ and $i \cdot {}^{\Im}A_@ \cdot \exp(i \cdot \omega_@^\pm t)$, having at $t = 0$ a real and a pure imaginary amplitude, respectively, as shown for a positive phasor in Figure 1(b,c). In spectral form Eq. (10t) becomes (10s), and ${}^{\Re}A_@ \Big|_{\dot{\omega}_@^\pm}$ and $i \cdot {}^{\Im}A_@ \Big|_{\dot{\omega}_@^\pm}$ [Figure 2(B,C)]. This tells us that if we take the real and imaginary parts of the principle phasor’s complex spectrum $\underline{A}_@ \Big|_{\dot{\omega}_@^\pm}$ [Figure 2(A)], then the real spectrum ${}^{\Re}A_@ \Big|_{\dot{\omega}_@^\pm}$ [Figure 2(B)] corresponds to the real-amplitude component ${}^{\Re}A_@ \cdot \exp(i \cdot \omega_@^\pm t)$ [Figure 1(b)], and the imaginary part $i \cdot {}^{\Im}A_@ \Big|_{\dot{\omega}_@^\pm}$ [Figures 2(C)] corresponds to the pure imaginary-amplitude component $i \cdot {}^{\Im}A_@ \cdot \exp(i \cdot \omega_@^\pm t)$ [Figure 1(c)] of the principle phasor.

Neither the real, nor the imaginary spectrum are normally represented in a 3D complex amplitude vs. frequency graph as in Figure 2(B,C), but as 2D real-amplitude vs. frequency, or imaginary-amplitude vs. frequency graphs [the analogous graphical schemes for a negative principle phasor can be seen in the original work (Szantay, 2007)]. Although in this representation the concepts of “real” and “imaginary” spectrum may seem almost self-evident, one should note that the ordinate value at a given frequency of the real and imaginary spectrum is often erroneously understood as meaning the amplitude of a cosine and sine trigonometric harmonic wave, respectively.

Consider the decomposition of $\underline{A}_@ \cdot \exp(i \cdot \omega_@^\pm t)$ into a cosine and a sine wave according to Eq. (11t). Because their amplitudes are related as $\underline{A}_@$ and $i \cdot \underline{A}_@$, the planes of oscillation of the cosine and sine wave are perpendicular to each other in the complex plane as shown in Figure 1(d,e). In spectral form [Expression (11s)] the respective complex amplitudes are depicted in ω -space according to Figure 2(D,E). Note that when we translate (11t) into spectral form, (11s) ceases to be an equality. This is because: a) the spectral amplitudes $\underline{A}_@ \Big|_{\dot{\omega}_@}$ for the cosine and sine components [Figure 2(D,E)] belong to two different mathematical “species”, therefore they cannot simply be added; b) the spectrum of the principle phasor [Figure 2(A)] is a function in ω -space, while the spectra of its component sinusoids [Figures 2(D,E)] are functions in ω -space. (Again, this is a fundamental difference which can easily escape attention in the absence of a symbolism that differentiates between phasor and trigonometric frequencies.) Note also that we have no analogous problem with the relevant temporal forms of Figures 1(a,d,e), since they have a common argument: time. Thus the amplitudes depicted in Figure 2(D,E) do not add up to those shown in Figure 2(A).

Temporal representation

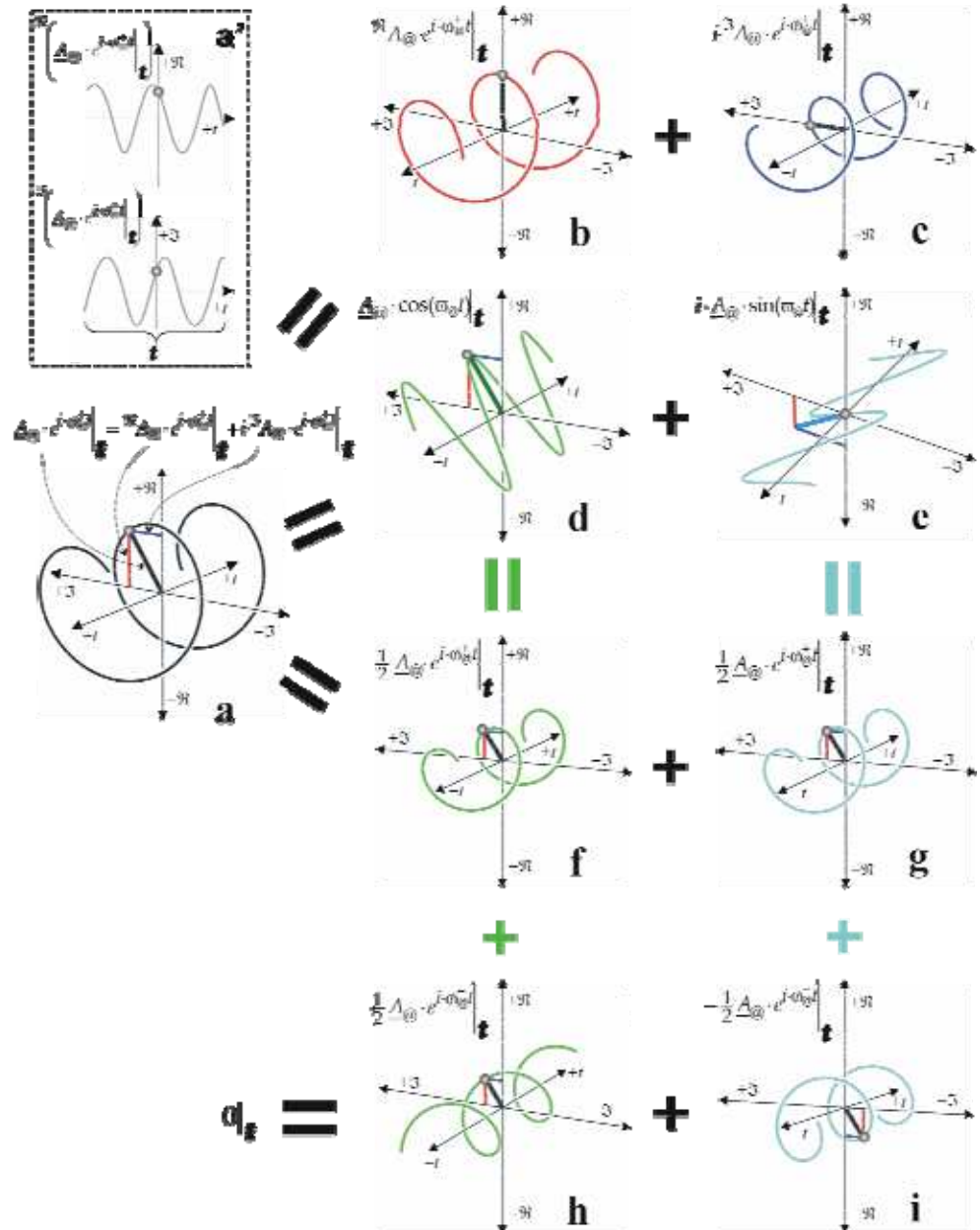


Fig. 1. Graphical representation of Equations (10t)-(12t) for a positive principle phasor. For added clarity the side-view projections of the principle phasor onto the real and imaginary axis are shown in (a')

Spectral representation

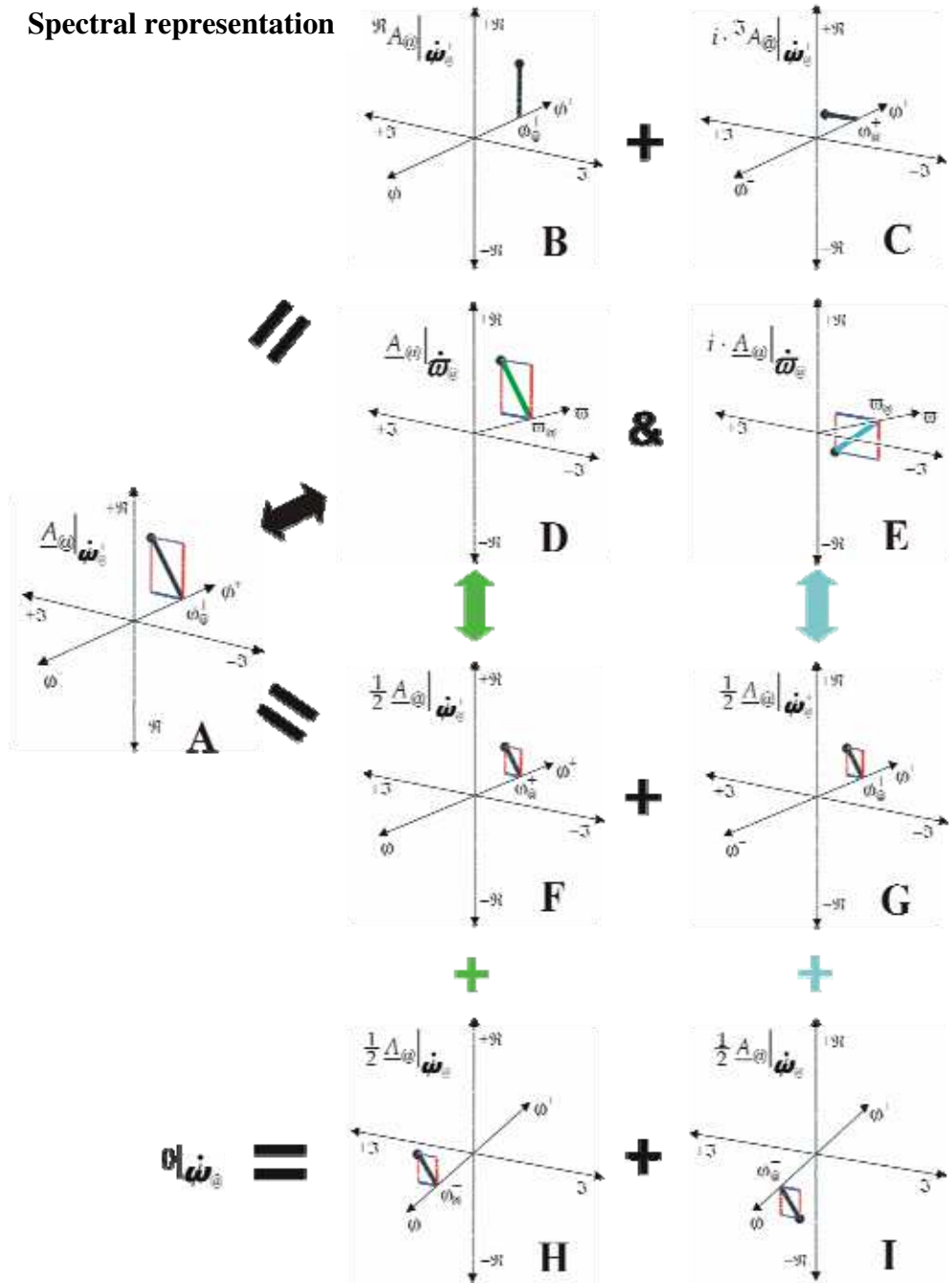


Fig. 2. The spectral representational analogue of Figure 1 according to Equations (10S)-(12S)

In order to bring $\underline{A}_{@} \Big|_{\dot{\omega}_{@}^{\pm}}$, $\underline{A}_{@} \Big|_{\dot{\omega}_{@}}$, and $i \cdot \underline{A}_{@} \Big|_{\dot{\omega}_{@}}$ onto a “common ground” we need to translate the ϖ -space spectra into ϕ -space. In the temporal domain this means that the principle phasor’s cosine component [Figure 1(d)] and sine component [Figure 1(e)] are decomposed into a sum of oppositely rotating phasors [Figure 1(f,h)] according to (12t). The decomposition of the cosine term reflects Equations (5) while that of the sine term corresponds to (6) multiplied by “ i ”. If we now add up the four phasors in (12t) we re-gain our principle phasor $\underline{A}_{@} \cdot \exp(i \cdot \phi_{@}^{\pm} t) \Big|_{\dot{\omega}_{@}}$ [Figures 1(a)]: the two negative phasor-frequency components have identical amplitudes but are in opposite phase [Figures 1(h,i)] and hence cancel each other at every point in time, while the two positive phasor-frequency components, both having an amplitude $0.5 \underline{A}_{@}$, are in-phase [Figure 1(f,g)] and thus reinforce each other to give the original principle phasor.

In spectral notation the relationships of (12t) are expressed according to (12s) and the pertinent ϕ -space spectra are shown in Figure 2(F,G,H,I). In analogy to the correlation between the left and right sides of (11s), there is a correlation, but not an equation, between the pertinent terms on the right side of (11s) and (12s).

By adding up the four amplitudes in [12s] we obtain the principle phasor’s spectrum $\underline{A}_{@} \Big|_{\dot{\omega}_{@}^{\pm}}$ [Figures 2(A)]. We may think of these “secondary-phasor” spectra [Figures 2(F,G,H,I)] as “interfaces” that allow us to connect the principle phasor’s ϕ -space spectrum [Figures 2(A)] with its ϖ -space spectra [Figures 2(D,E)].

From a broader viewpoint, the whole basis of the FT rests on the idea that a temporal function can be constructed from suitable harmonic “principle phasors”. Any given point on the complex FT-spectrum will represent a constituent harmonic principle phasor. It is according to the above scheme that we can conceptually envisage the way the spectrum of a temporal function (or signal) is obtained.

3.3 The Fourier Transform

The FT is based on the theorem that $\underline{f}(t) \Big|_{\dot{\omega}_{@}}$ can be uniquely expressed in terms of its projections onto an infinite frequency set of basis phasors $\underline{h}_{\dot{\omega}_{@}}^{\leftrightarrow}(t) \Big|_{\dot{\omega}_{@}} = \underline{A}(\dot{\omega}) \exp(i \cdot \dot{\omega} t) \Big|_{\dot{\omega}_{@}}$. In general, the infinite set of unit phasors $\exp(i \cdot \dot{\omega} t) \Big|_{\dot{\omega}_{@}}$ constitutes a unitary orthogonal basis set for all Fourier-transformable $\underline{f}(t) \Big|_{\dot{\omega}_{@}}$ functions, which can therefore be represented as a weighted sum of the unit harmonic waves. The spectrum $\underline{F}_{\dot{\omega}_{@}}(\dot{\omega}) \Big|_{\dot{\omega}_{@}}$ specifies the weighting coefficient for the unit basis phasors at all $\dot{\omega} \in \dot{\omega}_{@}$ frequencies. Using both trigonometric and phasor notation [cf. Equations (7)-(9)], the essence of the FT can conceptually be written as:

$$f(t) \Big|_{\mathfrak{t}} = \int_0^{\infty} \underline{h_{\overline{\omega}}^{\rightarrow}(t)} \Big|_{\mathfrak{t}} \cdot d\overline{\omega} = \int_{-\infty}^{\infty} \underline{h_{\overline{\omega}}^{\leftrightarrow}(t)} \Big|_{\mathfrak{t}} \cdot d\overline{\omega} \tag{13}$$

Employing the nomenclature and concepts outlined above, the Fourier integrals can be expressed as follows:

$$\underline{f(t)} \Big|_{\mathfrak{t}} = \int_0^{\infty} \underline{h_{\overline{\omega}}^{\rightarrow}(t)} \Big|_{\mathfrak{t}} \cdot d\overline{\omega} = \frac{1}{\pi} \cdot \underbrace{\int_0^{\infty} \underline{F_{\mathfrak{t}}^{\text{COS}}(\overline{\omega})} \Big|_{\overline{\omega}} \cdot \cos(\overline{\omega}t) \Big|_{\mathfrak{t}} \cdot d\overline{\omega}}_{\overleftarrow{\text{FT}}(\text{cos}) \Big|_{\mathfrak{t}} \Big|_{\overline{\omega}} \Big|_{\mathfrak{t}} = \underline{f(t)}^{\text{cos}} \Big|_{\mathfrak{t}}} + \frac{1}{\pi} \cdot \underbrace{\int_0^{\infty} \underline{F_{\mathfrak{t}}^{\text{SIN}}(\overline{\omega})} \Big|_{\overline{\omega}} \cdot \sin(\overline{\omega}t) \Big|_{\mathfrak{t}} \cdot d\overline{\omega}}_{\overleftarrow{\text{FT}}(\text{sin}) \Big|_{\mathfrak{t}} \Big|_{\overline{\omega}} \Big|_{\mathfrak{t}} = \underline{f(t)}^{\text{sin}} \Big|_{\mathfrak{t}}} = \tag{14a}$$

$$= \int_{-\infty}^{\infty} \underline{h_{\overline{\omega}}^{\leftrightarrow}(t)} \Big|_{\mathfrak{t}} \cdot d\overline{\omega} = \frac{1}{2\pi} \cdot \underbrace{\int_{-\infty}^{\infty} \underline{F_{\mathfrak{t}}(\overline{\omega})} \Big|_{\overline{\omega}} \cdot e^{i \cdot \overline{\omega}t} \Big|_{\mathfrak{t}} \cdot d\overline{\omega}}_{\overleftarrow{\text{FT}} \Big|_{\mathfrak{t}} \Big|_{\overline{\omega}} \Big|_{\mathfrak{t}} = \underline{f(t)} \Big|_{\mathfrak{t}}} \tag{14b}$$

$$\underline{F_{\mathfrak{t}}^{\text{COS}}(\overline{\omega})} \Big|_{\overline{\omega}} = \int_{-\infty}^{\infty} \underline{f(t)} \Big|_{\mathfrak{t}} \cdot \cos(\overline{\omega}t) \Big|_{\mathfrak{t}} \cdot dt = \int_{\mathfrak{t}} \underline{f(t)} \Big|_{\mathfrak{t}} \cdot \cos(\overline{\omega}t) \Big|_{\mathfrak{t}} \cdot dt = \overrightarrow{\text{FT}}(\text{cos}) \Big|_{\mathfrak{t}} \Big|_{\overline{\omega}} \underline{f(t)} \Big|_{\mathfrak{t}} \tag{15a^{\text{cos}}}$$

$$\underline{F_{\mathfrak{t}}^{\text{SIN}}(\overline{\omega})} \Big|_{\overline{\omega}} = \int_{-\infty}^{\infty} \underline{f(t)} \Big|_{\mathfrak{t}} \cdot \sin(\overline{\omega}t) \Big|_{\mathfrak{t}} \cdot dt = \int_{\mathfrak{t}} \underline{f(t)} \Big|_{\mathfrak{t}} \cdot \sin(\overline{\omega}t) \Big|_{\mathfrak{t}} \cdot dt = \overrightarrow{\text{FT}}(\text{sin}) \Big|_{\mathfrak{t}} \Big|_{\overline{\omega}} \underline{f(t)} \Big|_{\mathfrak{t}} \tag{15a^{\text{sin}}}$$

$$\underline{F_{\mathfrak{t}}(\overline{\omega})} \Big|_{\overline{\omega}} = \int_{-\infty}^{\infty} \underline{f(t)} \Big|_{\mathfrak{t}} \cdot e^{-i \cdot \overline{\omega}t} \Big|_{\mathfrak{t}} \cdot dt = \int_{\mathfrak{t}} \underline{f(t)} \Big|_{\mathfrak{t}} \cdot e^{-i \cdot \overline{\omega}t} \Big|_{\mathfrak{t}} \cdot dt = \overrightarrow{\text{FT}} \Big|_{\mathfrak{t}} \Big|_{\overline{\omega}} \underline{f(t)} \Big|_{\mathfrak{t}} \tag{15b}$$

Equations (15) and (14) express the forward and inverse FT, respectively. In the spectra $\underline{F_{\mathfrak{t}}^{\text{COS}}(\overline{\omega})} \Big|_{\overline{\omega}}$, $\underline{F_{\mathfrak{t}}^{\text{SIN}}(\overline{\omega})} \Big|_{\overline{\omega}}$ and $\underline{F_{\mathfrak{t}}(\overline{\omega})} \Big|_{\overline{\omega}}$ each $\overline{\omega}$ or $\overline{\omega}$ value corresponds to a pure harmonic wave and the unit basis functions $\exp(i \cdot \overline{\omega}t) \Big|_{\mathfrak{t}}$ are declared on an infinite time scale $\overleftrightarrow{\mathfrak{t}}$, which means that irrespective of what operative temporal interval $\underline{f(t)} \Big|_{\mathfrak{t}}$ “exists” upon, it can be decomposed into an infinite set of eternal harmonic waves. The eternity of the unitary orthogonal basis set $\exp(i \cdot \overline{\omega}t) \Big|_{\mathfrak{t}}$ is a generally not fully appreciated, but fundamentally important aspect of the FT, easily escaping attention in the standard mathematical symbolism. For example, in the conventional notation Equation (14b) is simply written as $f(t) = 1/2\pi \cdot \int_{-\infty}^{\infty} F(\omega) \cdot \exp(i \cdot \omega t) \cdot d\omega$, and this expression offers little in the way

of showing us that even if $f(t)$ is finite in time, the basis phasors $e^{i \cdot \omega t}$ are perpetual. Further insight may be gained into the FT by noting that the right-hand side of Equation (14b) represents an infinite collection of eternal “principle phasors” on the phasor frequency scale. Thus, in analogy to Equations (10)-(12) we can formulate the following scheme:

Temporal:
$$\frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \underline{F}_{\mathbf{f}}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \cdot e^{i \cdot \vec{\omega} t} \Big|_{\vec{\mathbf{f}}} \cdot d\omega =$$

$$= \underbrace{\left(\frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \underline{F}_{\mathbf{f}}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \cdot e^{i \cdot \vec{\omega} t} \Big|_{\vec{\mathbf{f}}} \cdot d\omega \right)}_{\Re f(t) \Big|_{\vec{\mathbf{f}}}} + i \cdot \underbrace{\left(\frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \underline{F}_{\mathbf{f}}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \cdot e^{i \cdot \vec{\omega} t} \Big|_{\vec{\mathbf{f}}} \cdot d\omega \right)}_{i \cdot \Im f(t) \Big|_{\vec{\mathbf{f}}}} \quad (16t)$$

$$= \underbrace{\frac{1}{\pi} \cdot \int_0^{\infty} \underline{F}_{\mathbf{f}}^{\cos}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \cdot \cos(\vec{\omega} t) \Big|_{\vec{\mathbf{f}}} \cdot d\omega}_{f(t)^{\cos} \Big|_{\vec{\mathbf{f}}}} + \underbrace{\frac{1}{\pi} \cdot \int_0^{\infty} \underline{F}_{\mathbf{f}}^{\sin}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \cdot \sin(\vec{\omega} t) \Big|_{\vec{\mathbf{f}}} \cdot d\omega}_{f(t)^{\sin} \Big|_{\vec{\mathbf{f}}}} \quad (17t)$$

$$= \left\{ \begin{array}{l} \frac{1}{2\pi} \cdot \int_0^{\infty} \underline{F}_{\mathbf{f}}^{\cos}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \cdot e^{i \cdot \vec{\omega} t} \Big|_{\vec{\mathbf{f}}} \cdot d\omega \\ + \\ \frac{1}{2\pi} \cdot \int_{-\infty}^0 \underline{F}_{\mathbf{f}}^{\cos}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \cdot e^{i \cdot \vec{\omega} t} \Big|_{\vec{\mathbf{f}}} \cdot d\omega \end{array} \right\} + \left\{ \begin{array}{l} \frac{1}{2\pi} \cdot \int_0^{\infty} \underline{F}_{\mathbf{f}}^{\sin}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \cdot e^{i \cdot \vec{\omega} t} \Big|_{\vec{\mathbf{f}}} \cdot d\omega \\ + \\ \frac{1}{2\pi} \cdot \int_{-\infty}^0 \underline{F}_{\mathbf{f}}^{\sin}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \cdot e^{i \cdot \vec{\omega} t} \Big|_{\vec{\mathbf{f}}} \cdot d\omega \end{array} \right\} \quad (18t)$$

Spectral:
$$\underline{F}_{\mathbf{f}}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} = \Re \underline{F}_{\mathbf{f}}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} + i \cdot \Im \underline{F}_{\mathbf{f}}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \quad (16s)$$

$$\underline{F}_{\mathbf{f}}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \leftrightarrow \underline{F}_{\mathbf{f}}^{\cos}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \quad \& \quad \underline{F}_{\mathbf{f}}^{\sin}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \quad (17s)$$

$$\underline{F}_{\mathbf{f}}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} = \left\{ \begin{array}{l} \underline{F}_{\mathbf{f}}^{\cos}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \\ + \\ \underline{F}_{\mathbf{f}}^{\cos}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \end{array} \right\} + \left\{ \begin{array}{l} \underline{F}_{\mathbf{f}}^{\sin}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \\ + \\ \underline{F}_{\mathbf{f}}^{\sin}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}} \end{array} \right\} \quad (18s)$$

From Equations (4) and (5) we can express the forward FT for the phasor-space spectral form also as:

$$\overbrace{\int_{-\infty}^{\infty} \underline{f}(t) \Big|_{\vec{\mathbf{f}}} \cdot e^{-i \cdot \vec{\omega} t} \Big|_{\vec{\mathbf{f}}} \cdot dt}_{\underline{F}_{\mathbf{f}}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}}} = \underbrace{\left(\int_{-\infty}^{\infty} \underline{f}(t) \Big|_{\vec{\mathbf{f}}} \cdot e^{-i \cdot \vec{\omega} t} \Big|_{\vec{\mathbf{f}}} \cdot dt \right)}_{\Re \underline{F}_{\mathbf{f}}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}}} + i \cdot \underbrace{\left(\int_{-\infty}^{\infty} \underline{f}(t) \Big|_{\vec{\mathbf{f}}} \cdot e^{-i \cdot \vec{\omega} t} \Big|_{\vec{\mathbf{f}}} \cdot dt \right)}_{i \cdot \Im \underline{F}_{\mathbf{f}}(\vec{\omega}) \Big|_{\vec{\mathbf{f}}}} \quad (19s)$$

$$\begin{aligned}
 &= \underbrace{\int_{-\infty}^{\infty} \underline{f}(t) \Big|_{\underline{t}} \cdot \cos(\overline{\omega}t) \Big|_{\overline{\omega}} \cdot dt}_{\underline{F}_{\underline{t}}^{\text{COS}}(\overline{\omega}) \Big|_{\overline{\omega}}} \quad (\overline{\omega}^{\pm}) \quad \mp \quad \underbrace{i \cdot \int_{-\infty}^{\infty} \underline{f}(t) \Big|_{\underline{t}} \cdot \sin(\overline{\omega}t) \Big|_{\overline{\omega}} \cdot dt}_{\underline{F}_{\underline{t}}^{\text{SIN}}(\overline{\omega}) \Big|_{\overline{\omega}}} \quad (20s) \\
 &\quad \parallel \qquad \qquad \qquad \parallel
 \end{aligned}$$

$$\begin{aligned}
 &= \left. \begin{aligned}
 &\underbrace{0.5 \cdot \int_{-\infty}^{\infty} \underline{f}(t) \Big|_{\underline{t}} \cdot e^{i \cdot \overline{\omega}t} \Big|_{\overline{\omega}} \cdot dt}_{0.5 \cdot \underline{F}_{\underline{t}}(\overline{\omega}) \Big|_{\overline{\omega}}} \\
 &+ \\
 &\underbrace{0.5 \cdot \int_{-\infty}^{\infty} \underline{f}(t) \Big|_{\underline{t}} \cdot e^{i \cdot \overline{\omega}t} \Big|_{\overline{\omega}} \cdot dt}_{0.5 \cdot \underline{F}_{\underline{t}}(\overline{\omega}) \Big|_{\overline{\omega}}}
 \end{aligned} \right\} (\overline{\omega}^{\pm}) \quad \mp \quad i \cdot \left. \begin{aligned}
 &\underbrace{-i \cdot 0.5 \cdot \int_{-\infty}^{\infty} \underline{f}(t) \Big|_{\underline{t}} \cdot e^{i \cdot \overline{\omega}t} \Big|_{\overline{\omega}} \cdot dt}_{-i \cdot 0.5 \cdot \underline{F}_{\underline{t}}(\overline{\omega}) \Big|_{\overline{\omega}}} \\
 &+ \\
 &\underbrace{i \cdot 0.5 \cdot \int_{-\infty}^{\infty} \underline{f}(t) \Big|_{\underline{t}} \cdot e^{i \cdot \overline{\omega}t} \Big|_{\overline{\omega}} \cdot dt}_{i \cdot 0.5 \cdot \underline{F}_{\underline{t}}(\overline{\omega}) \Big|_{\overline{\omega}}}
 \end{aligned} \right\} \quad (21s)
 \end{aligned}$$

In Equations (16) - (21) the relationships between the various $\overline{\omega}$ -space and $\overline{\omega}$ -space spectral forms are as follows:

$$\left. \begin{aligned}
 &\underline{F}_{\underline{t}}^{\text{COS}}(\overline{\omega}) \Big|_{\overline{\omega}} = \underline{F}_{\underline{t}}^{\text{COS}}(\overline{\omega}) \Big|_{\overline{\omega}} = \underline{F}_{\underline{t}}^{\text{COS}}(\overline{\omega}) \Big|_{\overline{\omega}}; \underline{F}_{\underline{t}}^{\text{SIN}}(\overline{\omega}) \Big|_{\overline{\omega}} = -i \cdot \underline{F}_{\underline{t}}^{\text{SIN}}(\overline{\omega}) \Big|_{\overline{\omega}}; \underline{F}_{\underline{t}}^{\text{SIN}}(\overline{\omega}) \Big|_{\overline{\omega}} = i \cdot \underline{F}_{\underline{t}}^{\text{SIN}}(\overline{\omega}) \Big|_{\overline{\omega}} \\
 &\underline{F}_{\underline{t}}(\overline{\omega}) \Big|_{\overline{\omega}} = \underline{F}_{\underline{t}}^{\text{COS}}(\overline{\omega}) \Big|_{\overline{\omega}} + \underline{F}_{\underline{t}}^{\text{SIN}}(\overline{\omega}) \Big|_{\overline{\omega}} = \underline{F}_{\underline{t}}^{\text{COS}}(\overline{\omega}) \Big|_{\overline{\omega}} - i \cdot \underline{F}_{\underline{t}}^{\text{SIN}}(\overline{\omega}) \Big|_{\overline{\omega}} \\
 &\underline{F}_{\underline{t}}(\overline{\omega}) \Big|_{\overline{\omega}} = \underline{F}_{\underline{t}}^{\text{COS}}(\overline{\omega}) \Big|_{\overline{\omega}} + \underline{F}_{\underline{t}}^{\text{SIN}}(\overline{\omega}) \Big|_{\overline{\omega}} = \underline{F}_{\underline{t}}^{\text{COS}}(\overline{\omega}) \Big|_{\overline{\omega}} + i \cdot \underline{F}_{\underline{t}}^{\text{SIN}}(\overline{\omega}) \Big|_{\overline{\omega}} \\
 &\underline{F}_{\underline{t}}^{\text{COS}}(\overline{\omega}) \Big|_{\overline{\omega}} = 0.5 \cdot \underline{F}_{\underline{t}}(\overline{\omega}) \Big|_{\overline{\omega}} + 0.5 \cdot \underline{F}_{\underline{t}}(\overline{\omega}) \Big|_{\overline{\omega}} = 0.5 \cdot \underline{F}_{\underline{t}}^{\text{COS}}(\overline{\omega}) \Big|_{\overline{\omega}} + 0.5 \cdot \underline{F}_{\underline{t}}^{\text{COS}}(\overline{\omega}) \Big|_{\overline{\omega}} \\
 &\underline{F}_{\underline{t}}^{\text{SIN}}(\overline{\omega}) \Big|_{\overline{\omega}} = i \cdot 0.5 \cdot \underline{F}_{\underline{t}}(\overline{\omega}) \Big|_{\overline{\omega}} - i \cdot 0.5 \cdot \underline{F}_{\underline{t}}(\overline{\omega}) \Big|_{\overline{\omega}} = i \cdot 0.5 \cdot \underline{F}_{\underline{t}}^{\text{SIN}}(\overline{\omega}) \Big|_{\overline{\omega}} - i \cdot 0.5 \cdot \underline{F}_{\underline{t}}^{\text{SIN}}(\overline{\omega}) \Big|_{\overline{\omega}}
 \end{aligned} \right\} \quad (22)$$

From this formalism one can draw a number of subtle insights into the FT as was detailed in the original papers which also included a discussion on a geometrical interpretation of the FT and on its capability to unravel the oscillating frequency components that are otherwise obscured in a complicated time signal (Szántay, 2007, 2008a, 2008b, 2008c). These aspects will not be elaborated here. Suffice to note only that if a signal, represented by the function $\underline{f}(t) \Big|_{\underline{t}}$, comprises of harmonic phasor components (such is the case with the NMR “free induction decay” except that it contains exponentially damped harmonic phasors), then the signal must be detected from two orthogonal directions simultaneously to be able to tell the sense of rotation of the pertinent phasors. Letting these directions correspond to the $+\Re$ and $+\Im$ axes we obtain the functions $\Re f(t) \Big|_{\underline{t}}$ and $i \cdot \Im f(t) \Big|_{\underline{t}}$ [cf. Equation (20t)] as the “side projections” of $\underline{f}(t) \Big|_{\underline{t}}$ which are analogous to the principle phasor’s real and imaginary

“side-view” components as shown in Figure 1 (a’). Accordingly, in general $\Re f(t)|_{\mathbf{t}}$ and $i \cdot \Im f(t)|_{\mathbf{t}}$ both comprise of a mixture of cosine and sine waves, thus when we speak of the “complex forward” FT of $\underline{f(t)}|_{\mathbf{t}}$, then this is actually accomplished through the FT of $\Re f(t)|_{\mathbf{t}}$ combined with the FT of $i \cdot \Im f(t)|_{\mathbf{t}}$. The relationship between the right-hand sides of (20s) and (21s), or (17s) and (18s), reflects the way in which the ω -space outputs of the cosine and sine transforms $\vec{F}_{\mathbf{t}}(\cos)$ and $\vec{F}_{\mathbf{t}}(\sin)$ assume a sign duality in ω -space. The addition of the “secondary” ω -space spectra $\underline{F}_{\mathbf{t}}^{\cos}(\bar{\omega})|_{\omega}$, $\underline{F}_{\mathbf{t}}^{\cos}(\bar{\omega})|_{\omega}$, $\underline{F}_{\mathbf{t}}^{\sin}(\bar{\omega})|_{\omega}$ and $\underline{F}_{\mathbf{t}}^{\sin}(\bar{\omega})|_{\omega}$ gives the final FT-spectrum $\underline{F}_{\mathbf{t}}(\bar{\omega})|_{\omega}$, i.e., only the $\vec{F}_{\mathbf{t}}(\cos)$ and $\vec{F}_{\mathbf{t}}(\sin)$ transforms together can properly identify the sign of the phasor frequency components within $\underline{f(t)}|_{\mathbf{t}}$.

A few basic FT relationships, represented graphically as well as in terms of function rules written in the style discussed above are shown in Figure 3 (all of these formulas are of course available in conventional notation from standard textbooks).

4. Uncertainty principle(s)

4.1 The Heisenberg and Fourier Uncertainty Principles

With reference to the issue of whether or not the frequency of a time-limited sinusoid wave becomes inherently “fuzzy” due to the Uncertainty Principle (as was discussed in the Introduction), let us now take a closer look at the meaning of the term “Uncertainty Principle”. One mental trap here is that for most people the phrase “Uncertainty Principle” invokes a single famous theorem, while there are actually several uncertainty principles that have come to constitute their independent branches of mathematics, physics and signal analysis theories. In the context of our present discussion we can speak of two main types of uncertainty principle: Heisenberg’s Uncertainty Principle (HUP) which is a quantum-mechanical concept grounded in the physical world, and the Fourier Uncertainty Principle (FUP) which is a classical concept and stems from a mathematical necessity inherent to the way the FT works.

Within our framework of discussion the HUP can be written for the noncommuting variables time and frequency through the expression:

$$\Delta t \cdot \Delta_{sd}\omega \geq \text{constant} \tag{23}$$

where the symbol Δ_{sd} denotes the standard deviation of frequency.

In the classical realm of time-frequency signal analysis there exists another fundamental inequality commonly referred to as the Fourier Uncertainty Principle (FUP). In qualitative terms the FUP states that the “broader” $\underline{\hat{f}(t)}|_{\mathbf{t}}$ is in time, the “narrower” $\underline{F_{\mathbf{t}}(\bar{\omega})}|_{\omega}$ is in frequency and vice versa. Since there is no universally applicable mathematical or semantic

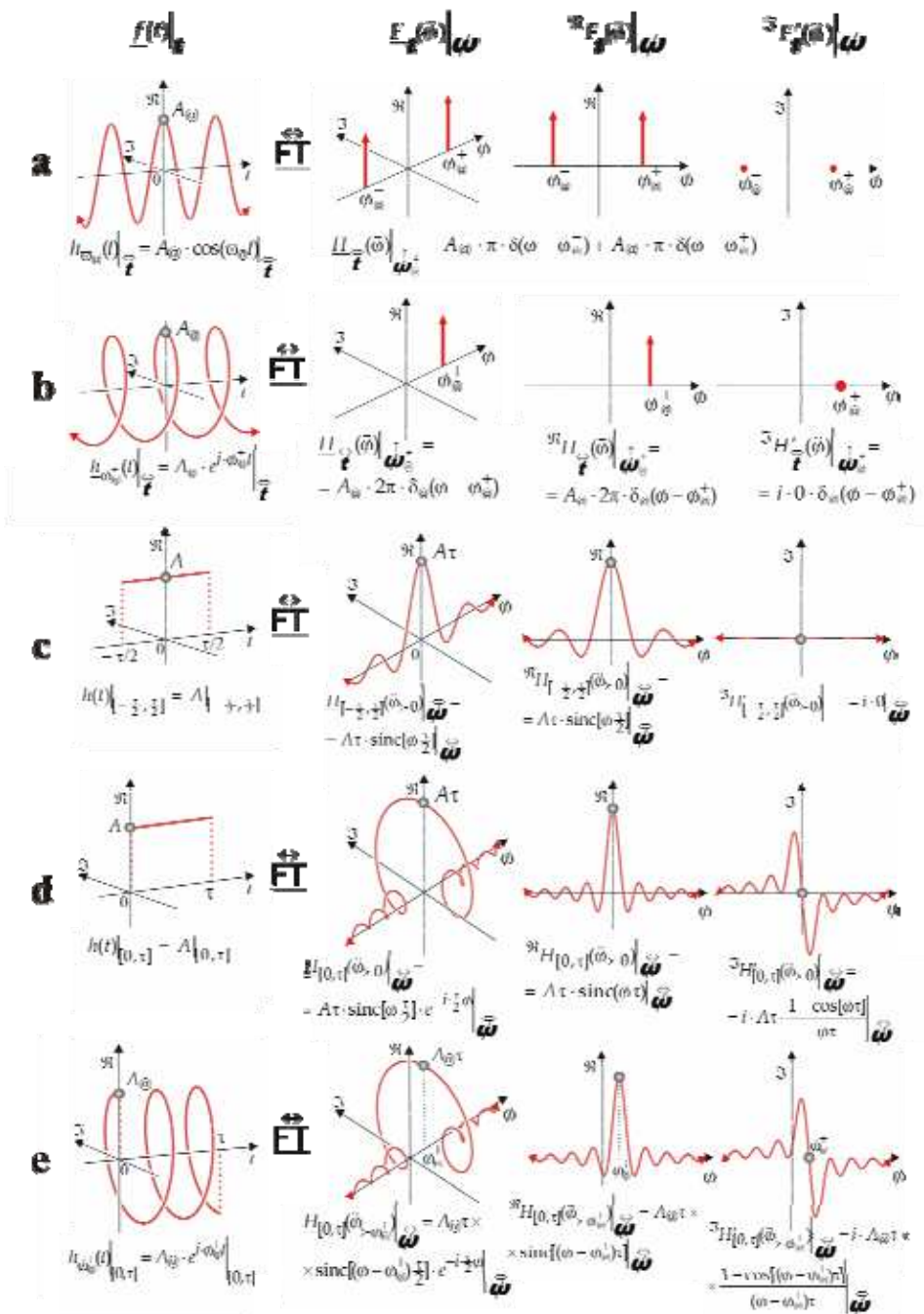


Fig. 3. Some special FT relationships

definition that would adequately quantify or delineate the concept of a function's "breadth" (Cohen, 1995 and Claerbout, 2004), we adopt from Claerbout the generic symbol " \wedge ", herein called "*characteristic span*", connoting the "compressedness" or "elongatedness" of a function along its domain, the idea being that " \wedge " symbolizes any mathematical definition, including Δ_{sd} , that may be suitably applied to the specific function at hand. Using the symbol " \neg " to denote "of", so " $\wedge x \neg f(x)$ " means the characteristic span of $f(x)$, the FUP can thus be expressed as

$$\wedge t \neg \underline{\hat{f}}(t) \Big|_{\mathcal{F}} \cdot \wedge \omega \neg \underline{F}_{\mathcal{F}}(\tilde{\omega}) \Big|_{\mathcal{W}} = \text{TBP} \geq \Xi \quad (24a)$$

$$\wedge t \cdot \wedge \omega = \text{TBP} \geq \Xi \quad (24b)$$

where TBP is the so-called "time-bandwidth product", Ξ is the minimum value of the TBP, and (24b) is just a short-hand for (24a). The actual values of TBP and Ξ are specific to the exact way in which $\wedge t \neg \underline{\hat{f}}(t) \Big|_{\mathcal{F}}$ and $\wedge \omega \neg \underline{F}_{\mathcal{F}}(\tilde{\omega}) \Big|_{\mathcal{W}}$ have been defined for the particular $\underline{\hat{f}}(t) \Big|_{\mathcal{F}}$ and $\underline{F}_{\mathcal{F}}(\tilde{\omega}) \Big|_{\mathcal{W}}$ functions at hand. The FUP is an equality in the sense that for any given absolute integrable function $\underline{\hat{f}}(t) \Big|_{\mathcal{F}}$ the TBP is constant, therefore increasing $\wedge t \neg \underline{\hat{f}}(t) \Big|_{\mathcal{F}}$ entails a concurrent decrease in $\wedge \omega \neg \underline{F}_{\mathcal{F}}(\tilde{\omega}) \Big|_{\mathcal{W}}$ and vice versa; the FUP however is an inequality in the sense that the TBP must be greater than the value of Ξ . The FUP asserts nothing more than: a) a t -D function can only be "squeezed" at the expense of expanding its ω -D counterpart and vice versa; b) $\underline{\hat{f}}(t) \Big|_{\mathcal{F}}$ and $\underline{F}_{\mathcal{F}}(\tilde{\omega}) \Big|_{\mathcal{W}}$ cannot both be made arbitrarily "narrow".

The FUP and HUP share a common mathematical form and terminology which is why, as it was pointed out by Cohen (Cohen, 1995), they are easily mixed up, especially since both are often written simply in the form " $\Delta t \cdot \Delta \omega \geq \text{constant}$ ". However, the physical interpretations of (23) and (24) are different: in the HUP the word "uncertainty" reflects probabilistic uncertainties in the measurement of quantum-mechanical physical observables, while the FUP is a deterministic theorem resulting from the innate nature of the FT, as was illustrated geometrically earlier (Szántay, 2008c).

The notion that by time-limiting a monochromatic sine wave we impose some "uncertainty" on its frequency stems either from confusing the HUP and the FUP, or from misunderstanding the FUP itself. With regard to the deterministic sine wave (such as the NMR pulse), it is obviously the FUP that is of relevance and not the HUP. As for the FUP, one reason it is often misinterpreted is because "uncertainty" is a dubious word with several possible interpretations in various situations, as was discussed in detail before (Szántay, 2008c). The concept of the characteristic span $\wedge x \neg \underline{\hat{f}}(x) \Big|_{\mathcal{X}}$ simply reflects "broadness" which is not a probabilistic quantity per se; just because a function has a large characteristic span, it is not "uncertain". In that sense, referring to (24) as an "Uncertainty Principle" is a misnomer, as was also emphasized by Cohen (Cohen, 1995).

4.2 The law of “Conjugate Physical Equivalence”

The Fourier Inversion Theorem (FIT) symbolized by (3) tells us that $\underline{f(t)}|_{\mathcal{F}}$ and $\underline{F_{\mathcal{F}}(\bar{\omega})}|_{\mathcal{W}}$, as being functions of the conjugate variables time and frequency, represent the same mathematical object $\underline{f(t)}|_{\mathcal{F}}$ expressed in two different “languages”, temporal and spectral.

The mathematical truth of the FIT implies that if $\underline{f(t)}|_{\mathcal{F}}$ is a model of some physical process, then whatever physical meaning we assign to $\underline{f(t)}|_{\mathcal{F}}$, we must assign the same physical meaning and information content to $\underline{F_{\mathcal{F}}(\bar{\omega})}|_{\mathcal{W}}$. For the sake of emphasis I will herein refer to this concept as the “Law of Conjugate Physical Equivalence” (LCPE).

Although, when phrased in this way the LCPE may seem rather self-evident, it is actually all too often overlooked when it comes to the physical interpretation of the mathematics of the FIT. The reason for this stems from the innate “linguistic” difference between $\underline{f(t)}|_{\mathcal{F}}$ and $\underline{F_{\mathcal{F}}(\bar{\omega})}|_{\mathcal{W}}$. The temporal form is intuitively more familiar since it reflects, and is more congruent with, our natural perception of physical events in the way they actually “happen” in time. The spectral “language” however is an artificial construct which is non-intuitive in the sense that when $\underline{f(t)}|_{\mathcal{F}}$ is transposed according to (13) onto frequency space, we lose direct information on its temporal progression of; furthermore, the spectral form is highly abstract in that its “alphabet” consists of an infinite number of eternally sounding “vowels” in the form of the scaled perpetual basis harmonic waves $\exp(i \cdot \omega t)|_{\mathcal{F}}$. It is the entirety of these spectral “letters” that describes $\underline{f(t)}|_{\mathcal{F}}$ in the frequency dimension (in other words the concerted sound, i.e. the superposition of the vowels returns $\underline{f(t)}|_{\mathcal{F}}$ in its exact shape and at its precise time location; however, the vowels cancel each other exactly to produce utter “silence” for all $t \notin \mathcal{F}$). This kind of abstraction being inherent to $\underline{F_{\mathcal{F}}(\bar{\omega})}|_{\mathcal{W}}$ is something that is often not fully appreciated, and the fact that the FT involves transposition between the physical variables time and frequency invites a strong reflexive (and often unjustified) association of its mathematics with physical properties. In particular, since the “letters” of the spectrum are formed of harmonic waves, one may be inclined to think (in violation of the LCPE) that each of the (abstract) letters $\exp(i \cdot \omega t)|_{\mathcal{F}}$ represents a physically existing oscillation perceived as being somehow inherent in the temporal function but “revealed” by the FT. However, the spectral letters happen to be harmonic waves simply as a consequence of the mathematics of the FT. We can envisage the basis set $\exp(i \cdot \bar{\omega} t)|_{\mathcal{F}}$ on which the forward FT transposes $\underline{f(t)}|_{\mathcal{F}}$ as an abstract “ocean” of harmonic waves stretching infinitely in both the frequency and time dimensions. The forward FT projects the properties of $\underline{f(t)}|_{\mathcal{F}}$

onto this “ocean” by adjusting the amplitudes and phases of the waves according to the “coding instructions” of the FT, and the backward FT “decodes” this information to reconstruct $\underline{f}(t) \Big|_{\mathcal{F}}$. However, $\exp(i \cdot \bar{\omega}t) \Big|_{\mathcal{F}}$ is just (a rather convenient and useful) mathematical vehicle whose waves act as carriers of the FT codes but do not *ab ovo* represent physical oscillations (in principle it would be possible to “translate” $\underline{f}(t) \Big|_{\mathcal{F}}$ into an infinite number of other mathematical “languages” whose letters would be based on functional forms other than $\exp(i \cdot \bar{\omega}t) \Big|_{\mathcal{F}}$, in which case the letters would less readily suggest some association with a physical meaning).

To better understand this problem it can be useful to think of $\underline{f}(t) \Big|_{\mathcal{F}}$ simply as a curve detached initially from any physical meaning which is “morphed” by the FT into the “oceanic profile” of $\underline{F}_{\mathcal{F}}(\bar{\omega}) \Big|_{\omega}$. There are certain features of the temporal curve that we, for convenience, can call by specific names (such as ‘oscillation frequency’, ‘phase’, ‘amplitude’, in the case of, say, a sinusoid). If we render specific physical properties through these features to our temporal signal, then we must assign the same physical properties to the spectral representation according to the LCPE since the FT does not “know” anything about the physics represented by those curves (thus it cannot “create” or “reveal” e.g. any frequency “uncertainty” in the frequency domain that was not there in the time domain). Whatever meaning we therefore attribute to $\underline{F}_{\mathcal{F}}(\bar{\omega}) \Big|_{\omega}$, that meaning emerges through a process of its “a posteriori” physical interpretation, and this must be in keeping with our physical interpretation of the relevant temporal “curve”.

Consider e.g. the simple monochromatic real-valued eternal cosine wave $h_{\omega_{@}}(t) \Big|_{\mathcal{F}}$ and its spectrum $\underline{H}_{\mathcal{F}}(\bar{\omega}) \Big|_{\omega_{@}}$ [Figure 3(d)]. In $h_{\omega_{@}}(t) \Big|_{\mathcal{F}}$ the wave’s frequency $\omega_{@}$ is known exactly by definition and $\underline{H}_{\mathcal{F}}(\bar{\omega}) \Big|_{\omega_{@}}$ is localized to $\bar{\omega}_{@}^{\pm}$, representing the limit where $\wedge t - h_{\omega_{@}}(t) \Big|_{\mathcal{F}} = \infty$ and so $\wedge \bar{\omega} - \underline{H}_{\mathcal{F}}(\bar{\omega}) \Big|_{\omega_{@}}^{\uparrow} = 0$ in (24). Take, next, the time-windowed harmonic waves $h_{\omega_{@}}(t) \Big|_{\mathcal{F}}^{\bullet}$ and their spectra $\underline{H}_{\mathcal{F}}^{\bullet}(\bar{\omega}_{>\omega_{@}}) \Big|_{\omega}$ (Figure 4).

Again, in $h_{\omega_{@}}(t) \Big|_{\mathcal{F}}^{\bullet}$ the frequency $\omega_{@}$ has an exactly known value. Since now $\wedge t - h_{\omega_{@}}(t) \Big|_{\mathcal{F}}^{\bullet}$ is finite, the FUP requires that $\wedge \bar{\omega} - \underline{H}_{\mathcal{F}}^{\bullet}(\bar{\omega}_{>\omega_{@}}) \Big|_{\omega}$ must also broaden to some finite value. However, just because we have time-windowed $h_{\omega_{@}}(t) \Big|_{\mathcal{F}}$ its frequency $\omega_{@}$ has not become “uncertain”. The fact that $\wedge \bar{\omega} - \underline{H}_{\mathcal{F}}^{\bullet}(\bar{\omega}_{>\omega_{@}}) \Big|_{\omega} > 0$ is not a reflection of any “uncertainty” in $\omega_{@}$ but a mathematical necessity ensuing from the fact that all information that specifies the temporal function or signal must be uniquely coded into the spectrum. One way of thinking about this is that while for $h_{\omega_{@}}(t) \Big|_{\mathcal{F}}$ there was no need for the spectrum to code for the

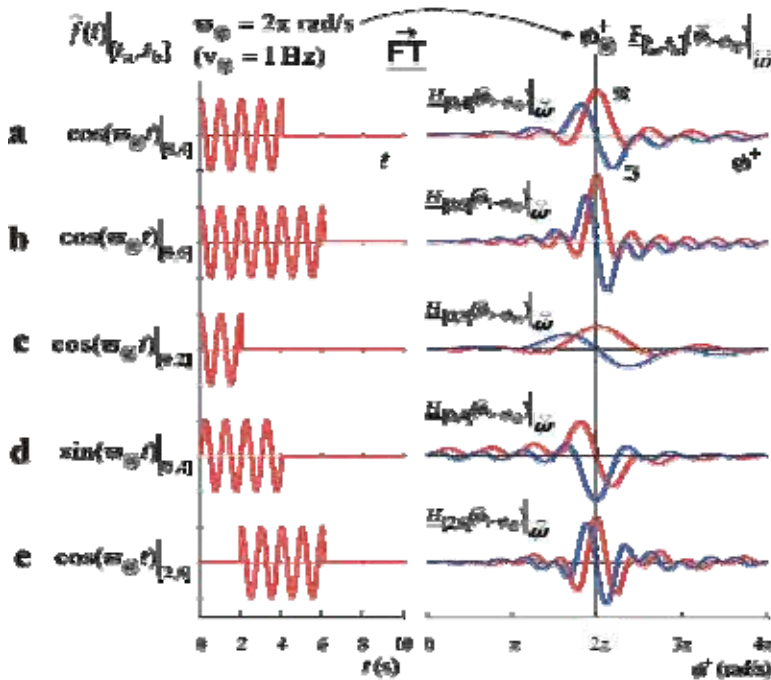


Fig. 4. Simple time-windowed monochromatic temporal waves and their ω -space spectra (for simplicity, only the positive phasor-frequency side of the spectra is shown). Each wave’s frequency is the same value, $\omega_{@}$, but their duration, time location, and initial phase are different. The value of $\omega_{@}$ “codes into” the frequency location of the center of the sinc-shaped spectra, therefore this location is the same in each spectrum. Differences in the spectral profiles reflect differences in the waves’ other features, because those are coded exactly by the FT into the (complex) FT-spectrum. Spectral broadening is due to that fact that the FT-spectrum carries all that coding information, and not to any “uncertainty” in $\omega_{@}$

duration and time location of the wave (which is why the spectrum is a Dirac-delta $\underline{H}_{\mathbf{f}}^{\omega}(\bar{\omega}) \uparrow_{\omega_{@}}$), for $h_{\omega_{@}}(t) \cdot \mathbf{f}$ a Dirac-delta would not provide enough degrees of freedom in its spectral features to be able to code for the wave’s duration and time location. It is, then, this encoding of the “window” into the FT-spectrum that manifests itself in a spectral profile that covers the infinite frequency set $\hat{\omega}$, and not some kind of “uncertainty” in $\omega_{@}$. Thinking in terms of “curves”, we can render to the curve of $h_{\omega_{@}}(t) \cdot \mathbf{f}$ [Figure 4(a)] some distinct physical “aspects”, such as a “frequency aspect” defined through the term $\cos(\omega_{@}t)$, and a “window aspect” defined through the location and duration of the operative interval \mathbf{f} . All of these features are coded exactly into the shape of the spectrum $\underline{H}_{\mathbf{f}}^{\omega}(\bar{\omega} > \omega_{@}) \uparrow_{\omega_{@}}$ [Figure 4(a)]. The “frequency aspect” of $h_{\omega_{@}}(t) \cdot \mathbf{f}$ translates into the (exact) central

frequency value $\omega_{@}^{\pm}$ of $\underline{H}_{\mathbf{f}}^{\star}(\tilde{\omega}_{>\omega_{@}})|_{\omega}$, while the “window aspect” translates into the broadened complex profile of the spectrum. One must therefore understand that the scaled basis elements $\exp(i \cdot \omega t)|_{\mathbf{f}}$ in $\underline{H}_{\mathbf{f}}^{\star}(\tilde{\omega}_{>\omega_{@}})|_{\omega}$ do not necessarily represent physical vibrations that are actually or “effectively” present in $h_{\omega_{@}}(t)|_{\mathbf{f}}$. Rather, each eternal basis harmonic wave’s (complex) amplitude carries a single piece of the entire set of codes that is needed to reconstruct the original temporal function $h_{\omega_{@}}(t)|_{\mathbf{f}}$. The physical interpretation that we should, then, correctly assign to $\underline{H}_{\mathbf{f}}^{\star}(\tilde{\omega}_{>\omega_{@}})|_{\omega}$ is simply that the frequency of the central basis element of the spectrum corresponds to the monochromatic frequency of $h_{\omega_{@}}(t)|_{\mathbf{f}}$, and the rest of the basis elements code for all of its other features such as its “window aspect”.

The popular notion that the forward FT disseminates a temporal function (signal) into its “frequency components” is incorrect if by “frequency component” one understands a physical vibration contained in, and lasting for the duration of, the signal. Rather, the spectrum represents a transposition of all properties of the temporal signal into frequency space, and some of those properties may happen to be periodic events whose frequency will show up as peaks in the spectrum, but other features of the signal also become scrambled into the spectrum. Claiming therefore that any given point of the FT-spectrum represents a “frequency component” of the temporal signal is misleading and violates the LCPE. Indeed, Figures 4(b-e) illustrate this point by showing that if we keep the monochromatic “frequency aspect” of $h_{\omega_{@}}(t)|_{\mathbf{f}}$ constant but vary its other “aspects”, the spectral profiles will change so as to code for every alteration in the temporal curve, but the (unchanged) value of $\omega_{@}$ remains “coded into” the same frequency position for each spectrum.

4.3 Back to NMR

If one reconsiders the statements on PFT-NMR quoted in the Introduction, the flaws in those arguments should now be evident. An RF pulse can be treated as a deterministic, time-windowed harmonic phasor in that it acts as a classical magnetic field vector rotating with a specific monochromatic frequency. The assertion that “a nominally monochromatic short pulse effectively has a (Heisenberg) uncertainty-broadened frequency band” simply reflects a misuse of the HUP and/or a physical misinterpretation of the FUP. The FUP does not warrant the causal inference that “wide frequency band” should mean a wide range of physically existing driving RF oscillations, as is often deemed necessary for explaining the phenomenon of off-resonance excitation. Actually, according to the LCPE the sinc-shaped spectral profile of the pulse is synonymous in physical meaning with its temporal form, therefore the statement that a “short pulse has a wide frequency band” does not tell us any more “physics” about the pulse than saying that a “short pulse is a short pulse”. In fact there is nothing paradoxical about the fact that a monochromatic pulse can excite off-resonant spins: the phenomenon is solely due to the pulse’s large amplitude, akin to the way a classical mechanical resonator also gives nonzero response to a sufficiently large off-

resonance driving force. Nota bene: a) the common notion that “a shorter pulse excites a wider frequency band” is misleading in that it is valid if by ‘excites’ one specifically means ‘gives absorption-shaped measured signals’, but is not valid, and thus does not conflict with the above statements, if by ‘excites’ one just means ‘perturbs’ in general (Szántay, 2008c); b) the response of a linear system to an input signal can in fact be calculated from the spectrum of the input signal by way of the so-called Superposition Principle (the NMR system is nonlinear in general, and the linearity approximation is valid only for special limiting conditions), but that concept should not be confused with the ideas discussed above.

5. Conclusion

The Fourier Transform (FT) is often thought of as a mathematical device capable of “revealing” the “frequency components” of a temporal signal. Although this notion is correct with regard to any sinusoid oscillatory feature that may be present in the signal, it is deceptive in general, easily leading to the belief that the overall spectral profile represents all the physical “frequency components” that are present in the signal. This idea can give rise to false interpretations such as the concept that the frequency of a time-limited monochromatic sinusoid is “uncertain” or “effectively polychromatic”, in line with the “Uncertainty Principle”. In fact the Heisenberg Uncertainty Principle (HUP) has nothing to do with this problem; it is the Fourier Uncertainty Principle (FUP) that is of relevance, but the word “uncertainty” is also misleading in that regard: the FUP does not imply that time-windowing a deterministic wave with a monochromatic frequency makes its frequency “uncertain” or that the forward FT “reveals” such uncertainty just because the spectrum has a profile that covers a broad range of frequencies. Whatever physical attributes we associate with a temporal signal (such as that it has an exact monochromatic frequency) must be carried over exactly into the frequency dimension according to the “Law of Conjugate Physical Equivalence” (LCPE).

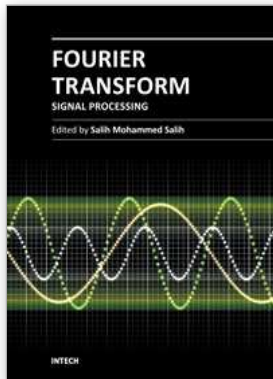
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The field of signal processing has seen explosive growth during the past decades; almost all textbooks on signal processing have a section devoted to the Fourier transform theory. For this reason, this book focuses on the Fourier transform applications in signal processing techniques. The book chapters are related to DFT, FFT, OFDM, estimation techniques and the image processing techniques. It is hoped that this book will provide the background, references and the incentive to encourage further research and results in this area as well as provide tools for practical applications. It provides an applications-oriented to signal processing written primarily for electrical engineers, communication engineers, signal processing engineers, mathematicians and graduate students will also find it useful as a reference for their research activities.

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