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Spacecraft Relative Orbital Motion

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1. Introduction

The relative orbital motion problem may now be considered classic, because of so many scientific papers written on this subject in the last few decades. This problem is also quite important, due to its numerous applications: spacecraft formation flying, rendezvous operations, distributed spacecraft missions.

The model of the relative motion consists in two spacecraft flying in Keplerian orbits due to the influence of the same gravitational attraction center (see Fig. 1). The main problem is to determine the position and velocity vectors of the Deputy satellite with respect to a reference frame originated in the Leader satellite center of mass. This non-inertial reference frame, traditionally named LVLH (Local-Vertical-Local-Horizontal) is chosen as follows: the C_x axis has the same orientation as the position vector of the Leader with respect to an inertial reference frame originated in the attraction center; the C_z axis has the same orientation as the Leader orbit angular momentum; the C_y axis completes a right-handed frame.

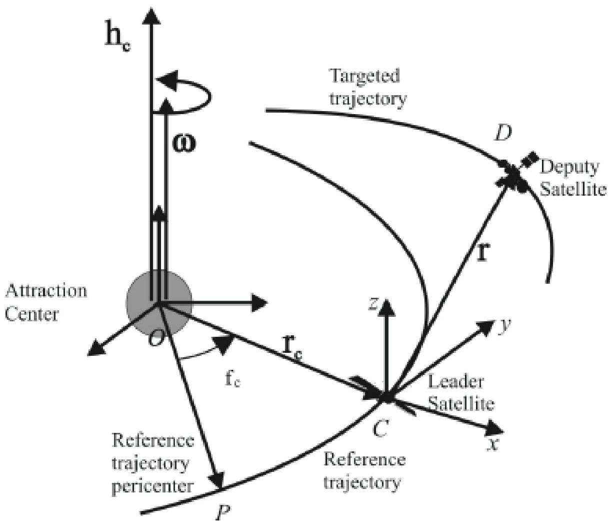

Fig. 1. The model of the relative orbital motion.

Consider $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ the angular velocity of the LVLH reference frame with respect to an inertial frame originated in the attraction center. By denoting $\mathbf{r}_c$ the Leader position vector with respect to an inertial frame originated in O (the attraction center), $f_c = f_c(t)$ the true anomaly, e_c the eccentricity and p_c the semilatus rectum of the Leader orbit, it follows that vector ω has the expression:

$$\omega = \dot{f}_c \frac{\mathbf{h}_c}{h_c} = \frac{1}{r_c^2}\mathbf{h}_c = \left[\frac{1 + e_c \cos f_c(t)}{p_c}\right]^2 \mathbf{h}_c, \tag{1}$$

where vector $\mathbf{r}_c$ is expressed with respect to the LVLH frame and has the form

$$\mathbf{r}_c = \frac{p_c}{1 + e_c \cos f_c(t)} \frac{\mathbf{r}_c^0}{r_c^0}, \tag{2}$$

and $\mathbf{h}_c$ is the angular momentum of the leader which will be named in the following satellite chief (or chief).

Vector $\mathbf{r}_c^0$ points to the initial position of the Leader spacecraft with respect to the inertial reference frame originated in the attraction center O. The initial value problem that models the motion of the Deputy satellite with respect to the LVLH reference frame is

$$\boxed{\begin{cases} \ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\omega} \times \mathbf{r} + \frac{\mu}{|\mathbf{r}_c + \mathbf{r}|^3}(\mathbf{r}_c + \mathbf{r}) - \frac{\mu}{r_c^3}\mathbf{r}_c = 0 \\ \mathbf{r}(t_0) = \Delta\mathbf{r},\ \dot{\mathbf{r}}(t_0) = \Delta\mathbf{v} \end{cases}} \tag{3}$$

where $\mu > 0$ is the gravitational parameter of the attraction center and $\Delta\mathbf{r}; \Delta\mathbf{v}$ represent the relative position and relative velocity vectors of the Deputy spacecraft with respect to LVLH at the initial moment of time $t_0 \geq 0$.

The analysis of relative motion began in the early 1960s with the paper of Clohessy and Wiltshire (Clohessy & Wiltshire (1960)), who obtained the equations that model the relative motion in the situation in which the chief spacecraft has a circular orbit and the attraction force is not affected by the Earth oblateness. They linearized the nonlinear initial value problem that models the relative motion by assuming that the relative distance between the two spacecraft remains small during the mission. The Clohessy - Wiltshire equations are still used today in rendezvous maneuvers, but they cannot offer a long-term accuracy because of the secular terms present in the expression of the relative position vector. Independently, Lawden (Lawden (1963)), Tschauner and Hempel (Tschauner & Hempel (1964)), and Tschauner (Tschauner (1966)) obtained the solution to the linearized equations of motion in the situation in which the chief orbit is elliptic, but their solutions still involved secular terms and also had singularities. The singularities in the Tschauner - Hempel equations were removed firstly by Carter (Carter (1990)) and also by Yamanaka and Andersen (Yamanaka & Andersen (2002)). Later on, the formation flying concept began to be considered, and the problem of deriving equations for the relative motion with a long-term accuracy degree raised, together with the need to obtain a more accurate solution to the relative orbital motion problem (Alfriend et al. (2009)). Gim and Alfriend (Gim & Alfriend (2003)) used the state transition matrix in the study of the relative motion.

The main goal was to express the linearized equations of motion with respect to the initial conditions, with applications in formation initialization and reconfiguration. Attempts to offer more accurate equations of motion starting from the nonlinear initial value problem that models the motion were made. Gurfil and Kasdin (Gurfil & N.J.Kasdin (2004)) derived closed-form expression of the relative position vector, but only when the reference trajectory is circular. Similar expressions for the law of relative motion starting from the nonlinear model are presented in (Alfriend et al. (2009); Balaji & Tatnall (2003); Ketema (2006); Lee et al. (2007)). The relative orbital motion problem was also studied from the point of view of the associated differential manifold. Gurfil and Kholshevnikov (Gurfil & Kholshevnikov (2006)) introduced a metric which helps to study the relative distance between Keplerian orbits. Gronchi (Gronchi (2006),Gronchi (2005)) also introduced a metric between two confocal Keplerian orbits and used this instrument in problems of asteroid and comet collisions.

In 2007, Condurache and Martinusi (Condurache & Martinusi (2007b;c)) offered the closed-form solution to the nonlinear unperturbed model of the relative orbital motion. The method led to closed-form vectorial coordinate-free expressions for the relative law of motion and relative velocity and it was based on an approach first introduced in 1995 (Condurache (1995)). It involves the Lie group of proper orthogonal tensor functions and its associated Lie algebra of skew-symmetric tensor functions. Then, the solution was generalized to the problem of the relative motion in a central force field (Condurache & Martinusi (2007e; 2008a;b)). An inedite solution to the Kepler problem by using the algebra of hypercomplex numbers was offered in (Condurache & Martinusi (2007d)). Based on this solution and by using the hypercomplex eccentric anomaly, a unified closed-form solution to the relative orbital motion was determined (Condurache & Martinusi (2010a)).

The present approach offers a tensor procedure to obtain exact expressions for the relative law of motion and the relative velocity between two Keplerian confocal orbits. The solution is obtained by pure analytical methods and it holds for any chief and deputy trajectories, without involving any secular terms or singularities. The relative orbital motion is reduced, by an adequate change of variables, into the classic Kepler problem. It is proved that the relative orbital motion problem is superintegrable. The tensor play only a catalyst role, the final solution being expressed in a vectorial form.

To obtain this solution, one has to know only the inertial motion of the chief spacecraft and the initial conditions (position and velocity) of the deputy satellite in the local-vertical-local-horizontal (LVLH) frame. Both the relative law of motion and the relative velocity of the deputy are obtained, by using the tensor instrument that is developed in the first part of the paper. Another contribution is the expression of the solution to the relative orbital motion by using universal functions, in a compact and unified form. Once the closed form solution is given a comprehensive analysis of the relative orbital motion of satellites is presented. Next the periodicity conditions in the relative orbital motion are revealed and in the end a tensor invariant in the relative motion is highlighted. The tensor invariant is a very useful propagator for the state of the deputy spacecraft in the LVLH frame.

2. Mathematical preliminaries

The key notions that are studied in this Section are proper orthogonal tensorial maps and a Sundman-like vectorial regularization, the latter introduced via a vectorial change of variable. The proper orthogonal tensorial maps are related with the skew-symmetric tensorial maps via the Darboux equation. The results presented in this section appeared for the first time in (Condurache (1995)). The section related to orthogonal tensorial maps after a powerful instrument in the study of the motion with respect to a non-inertial reference frames.

2.1 Proper orthogonal tensorial maps

We denote $SO_3^{\mathbb{R}}$ the set of maps defined on the set of real numbers $\mathbb{R}$ with values in the set of proper orthogonal tensors SO_3:

$$SO_3^{\mathbb{R}} = \left\{ \mathbf{R} : \mathbb{R} \to SO_3 | \mathbf{R}\mathbf{R}^T = \mathbf{I}_3, \det \mathbf{R} = 1 \right\} \quad (4)$$

We denote $s\mathbf{o}_3^R$ the set of maps defined on the set of real numbers $\mathbb{R}$ with values in the set of skew-symmetric tensors $s\mathbf{o}_3^R$:

$$s\mathbf{o}_3^R = \left\{ \widetilde{\boldsymbol{\omega}} : \mathbb{R} \to s\mathbf{o}_3 | \widetilde{\boldsymbol{\omega}}^T = -\widetilde{\boldsymbol{\omega}} \right\} \quad (5)$$

We denote $\mathbb{V}_3^{\mathbb{R}}$ to be the set of applications that can be on $\mathbb{R}$ with values in the free vectors set with dimension 3 ($\mathbb{V}_3$).

Theorem 1. *The initial value problem:*

$$\dot{\mathbf{Q}} + \widetilde{\boldsymbol{\omega}}\mathbf{Q} = \mathbf{0}_2, \mathbf{Q}(t_0) = \mathbf{I}_3 \quad (6)$$

has a unique solution $\mathbf{Q} \in SO_3^{\mathbb{R}}$ *for any continuous map* $\widetilde{\boldsymbol{\omega}} \in s\mathbf{o}_3^{\mathbb{R}}$.

Proof. Denote $\mathbf{Q}^T$ the transpose of tensor $\mathbf{Q}$. Computing:

$$\frac{d}{dt}(\mathbf{Q}\mathbf{Q}^T) = \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T = \mathbf{Q}\widetilde{\omega}\mathbf{Q}^T - \mathbf{Q}\widetilde{\omega}\mathbf{Q}^T = \mathbf{0}_3 \quad (7)$$

it follows that

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}\mathbf{Q}^T(t_0) = \mathbf{I}_3 \quad (8)$$

Since $\mathbf{Q} = \mathbf{Q}(t)$ is a continuous map, $t \geq t_0$, it follows that $\det(\mathbf{Q})$ is a continuous map too. From Eq. (8) it results $\det(\mathbf{Q}) \in [-1, 1]$. Since $\det(\mathbf{Q}(t_0)) = \det \mathbf{I}_3 = 1$, it follows that:

$$\begin{cases} \mathbf{Q}\mathbf{Q}^T = \mathbf{I}_3 \\ \det(\mathbf{Q}) = 1 \end{cases} \quad (9)$$

therefore $\mathbf{Q} \in SO_3^{\mathbb{R}}$ is a proper orthogonal tensor map.

Equation (6) represents the tensor form of the Darboux equation (Condurache & Martinusi (2010b); Darboux (1887)). Its solution will be denoted $\mathbf{R}_{-\boldsymbol{\omega}}$. It models the rotation with instantaneous angular velocity $-\boldsymbol{\omega}$ ($\boldsymbol{\omega}$ is the vectorial map associated to the skew-symmetric tensor $\widetilde{\boldsymbol{\omega}}$). The link between them is given by: $\widetilde{\boldsymbol{\omega}}\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}, \forall \mathbf{x} \in \mathbb{V}_3^{\mathbb{R}}$; where $\mathbb{V}_3$ is the three-dimensional linear space of free vectors and " $\times$ " denotes the cross product.

The inverse (in this case the transpose) of tensor $\mathbf{R}_{-\boldsymbol{\omega}}$ is denoted:

$$\mathbf{R}_{-\boldsymbol{\omega}}^T = \mathbf{F}_{\boldsymbol{\omega}} \quad (10)$$

Theorem 2. *The tensor map* $\mathbf{F}_{\boldsymbol{\omega}}$ *satisfies:*

1. $\mathbf{F}_{\boldsymbol{\omega}}$ *is invertible and* $\mathbf{F}_{\boldsymbol{\omega}}^{-1} = \mathbf{F}_{\boldsymbol{\omega}}^{T}$
2. $\mathbf{F}_{\boldsymbol{\omega}}\mathbf{u} \cdot \mathbf{F}_{\boldsymbol{\omega}}\mathbf{u} = \mathbf{u} \cdot \mathbf{v}, \forall \mathbf{u}, \mathbf{v} \in \mathbb{V}_3^{\mathbb{R}}$
3. $|\mathbf{F}_{\boldsymbol{\omega}}\mathbf{u}| = |\mathbf{u}|, \forall \mathbf{u} \in \mathbb{V}_3^{\mathbb{R}}$
4. $\mathbf{F}_{\boldsymbol{\omega}}(\mathbf{u} \times \mathbf{u}) = \mathbf{F}_{\boldsymbol{\omega}}\mathbf{u} \times \mathbf{F}_{\boldsymbol{\omega}}\mathbf{v}, \forall \mathbf{u}, \mathbf{v} \in \mathbb{V}_3^{\mathbb{R}}$
5. $\frac{d}{dt}\mathbf{F}_{\boldsymbol{\omega}}\mathbf{u} = \mathbf{F}_{\boldsymbol{\omega}}(\dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{u}), \forall \mathbf{u} \in \mathbb{V}_3^{\mathbb{R}}$, *differentiable*
6. $\frac{d^2}{dt^2}\mathbf{F}_{\boldsymbol{\omega}}\mathbf{u} = \mathbf{F}_{\boldsymbol{\omega}}(\ddot{\mathbf{u}} + 2\boldsymbol{\omega} \times \dot{\mathbf{u}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{u}) + \dot{\boldsymbol{\omega}} \times \mathbf{u}), \forall \mathbf{u} \in \mathbb{V}_3^{\mathbb{R}}$.

If vector $\boldsymbol{\omega}$ has fixed direction, given by the unit vector $\mathbf{u}; \boldsymbol{\omega} = \omega(t)\mathbf{u}$ with ω a continuous real valued map, the Darboux equation (6) has the explicit solution:

$$\mathbf{R}_{-\boldsymbol{\omega}} = \mathbf{I}_3 - (\sin\varphi)\widetilde{\mathbf{u}} + (1 - \cos\varphi)\widetilde{\mathbf{u}}^2 \tag{11}$$

where $\varphi(t) = \int_{t_0}^{t} \omega(s)ds$

Following from Eq (11), if vector ω is constant and nonzero, the solution to the Darboux equation (6) is written as:

$$\mathbf{R}_{-\boldsymbol{\omega}} = \mathbf{I}_3 - [\sin\omega(t - t_0)]\frac{\widetilde{\boldsymbol{\omega}}}{\omega} + [1 - \cos\omega(t - t_0)]\frac{\widetilde{\boldsymbol{\omega}}^2}{\omega}. \tag{12}$$

3. Closed-form solution to the relative orbital motion problem

3.1 Vectorial solutions

In this section we present the closed-form exact solution to Eq. (3). In the initial value problem (3), we make the change of variable:

$$\mathbf{r}_* = \mathbf{F}_{\boldsymbol{\omega}}(\mathbf{r} + \mathbf{r}_c) \tag{13}$$

where $\mathbf{r}_c$ is the solution of the problem:

$$\begin{cases} \ddot{\mathbf{r}}_c + 2\boldsymbol{\omega} \times \dot{\mathbf{r}}_c + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_c) + \dot{\omega} \times \mathbf{r}_c - \frac{\mu}{r_c^3}\mathbf{r}_c = 0 \\ \mathbf{r}_c(t_0) = \mathbf{r}_c^0,\ \dot{\mathbf{r}}_c(t_0) = \dot{\mathbf{r}}_c^0 \end{cases} \tag{14}$$

After some algebra, it follows that

$$\ddot{\mathbf{r}}_* = \mathbf{F}_{\boldsymbol{\omega}}\left\{(\ddot{\mathbf{r}} + \ddot{\mathbf{r}}_c) + 2\boldsymbol{\omega} \times (\dot{\mathbf{r}} + \dot{\mathbf{r}}_c) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r} + \mathbf{r}_c)) + \dot{\boldsymbol{\omega}} \times (\mathbf{r} + \mathbf{r}_c)\right\} \tag{15}$$

and furthermore

$$\ddot{\mathbf{r}}_* = \mathbf{F}_{\boldsymbol{\omega}}\left\{\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r}\right\} + \mathbf{F}_{\boldsymbol{\omega}}\left\{\ddot{\mathbf{r}}_c + 2\boldsymbol{\omega} \times \dot{\mathbf{r}}_c + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_c) + \dot{\boldsymbol{\omega}} \times \mathbf{r}_c\right\} \tag{16}$$

Using Eqs. (3) and (14) we obtain:

$$\ddot{\mathbf{r}}_* = \mathbf{F}_{\boldsymbol{\omega}}\left[\frac{\mu}{r_c^3}\mathbf{r}_c - \frac{\mu}{|\mathbf{r} + \mathbf{r}_c|^3}(\mathbf{r} + \mathbf{r}_c) - \frac{\mu}{r_c^3}\mathbf{r}_c\right] = -\frac{\mu}{|\mathbf{r} + \mathbf{r}_c|^3}\mathbf{F}_{\boldsymbol{\omega}}(\mathbf{r} + \mathbf{r}_c) \tag{17}$$

which leads to:

$$\ddot{\mathbf{r}}_* + \frac{\mu}{r_*^3}\mathbf{r}_* = \mathbf{0} \tag{18}$$

The initial conditions for equation (18) are deduced by taking into account that $\mathbf{F}_{\boldsymbol{\omega}}(t_0) = \mathbf{I}_3$ and Eq. (13):

$$\mathbf{r}_*(t_0) = \mathbf{r}_c^0 + \Delta\mathbf{r} \tag{19}$$

$$\dot{\mathbf{r}}_*(t_0) = \mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r} \tag{20}$$

where $\mathbf{r}_c^0 = \mathbf{r}_c(t_0)$, $\mathbf{v}_c^0 = \dot{\mathbf{r}}_c(t_0) + \boldsymbol{\omega}(t_0) \times \mathbf{r}_c^0$.

From (10) and (13) we deduce:

$$\mathbf{r} = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{r}_* - \mathbf{r}_c \tag{21}$$

The above considerations lead to the main result of this paper. This is stated thus: the solution to the relative orbital motion problem, described by the initial value problem (3) is:

$$\boxed{\mathbf{r} = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{r}_* - \frac{p_c}{1 + e_c \cos f_c(t)} \frac{\mathbf{r}_c^0}{r_c^0}} \tag{22}$$

where $\mathbf{R}_{-\boldsymbol{\omega}}$ is the solution of Eq. (6) and $\mathbf{r}_*$ is the solution to the initial value problem:

$$\ddot{\mathbf{r}}_* + \frac{\mu}{r_*^3}\mathbf{r}_* = \mathbf{0};\ \mathbf{r}_*(t_0) = \mathbf{r}_c^0 + \Delta\mathbf{r};\ \dot{\mathbf{r}}_*(t_0) = \mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r} \tag{23}$$

and the relative velocity may be computed as:

$$\boxed{\mathbf{v} = \mathbf{R}_{-\boldsymbol{\omega}}\dot{\mathbf{r}}_* - \widetilde{\boldsymbol{\omega}}\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{r}_* - \frac{e_c|\mathbf{h}_c| \sin f_c(t)}{p_c} \frac{\mathbf{r}_c^0}{r_c^0}} \tag{24}$$

This result shows a very interesting property of the relative orbital motion problem (3). We have proven that this problem is super-integrable, by reducing it to the classic Kepler problem (23). The solution of the relative orbital motion problem is expressed thus:

$$\mathbf{r} = \mathbf{r}(t, t_0, \Delta\mathbf{r}, \Delta\mathbf{v});\ \ \mathbf{v} = \mathbf{v}(t, t_0, \Delta\mathbf{r}, \Delta\mathbf{v}) \tag{25}$$

The Kepler problem (23) satisfies the prime integral of energy:

$$\frac{\dot{\mathbf{r}}_*^2}{2} - \frac{\mu}{r_*} = \zeta. \tag{26}$$

Taking into account (22) , (24) and (26) results that the problem which models the motion of the Deputy satellite with respect to the LVLH frame Eq. (3) has the following prime integral

$$\frac{\mathbf{v}^2}{2} - V(\mathbf{r}, \dot{\mathbf{r}}, t) = \zeta \tag{27}$$

where $V = V(\mathbf{r}, \dot{\mathbf{r}}, t)$ is the generalized potential defined by:

$$V(\mathbf{r}, \dot{\mathbf{r}}, t) = (\boldsymbol{\omega}, \mathbf{r}, \dot{\mathbf{r}}) + \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{r})^2 + \frac{\mu}{|\mathbf{r} + \mathbf{r}_c|} - \frac{\mu}{r_c^3}\mathbf{r} \cdot \mathbf{r}_c \tag{28}$$

and ζ

$$\zeta = \frac{1}{2}|\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}|^2 - \frac{\mu}{|\mathbf{r}_c^0 + \Delta\mathbf{r}|}. \tag{29}$$

The prime integral (27) generates in the phase space a differential manifold associated to the relative orbital motion. The solutions (22) and (24) are a parametrization of this manifold.

3.2 An unified solution for relative orbital motion

Here, we present another formulation of the solution to the relative orbital motion. Let $U_k, k = \{0,1,2,3\}, U_k = U_k(\chi,\alpha)$ be the universal functions defined in (Battin (1999)), pp. 175-179, with

$$\alpha = \frac{2}{|\mathbf{r}_c^0 + \Delta\mathbf{r}|} - \frac{|\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}|^2}{\mu} = -\mu\zeta \tag{30}$$

and χ a Sudman-like independent universal variable that satisfies

$$\frac{dt}{d\chi} = \frac{1}{\sqrt{\mu}} r_* \tag{31}$$

Then, the solution to the initial value problem (23) may be expressed as Eq. (25):

$$\begin{aligned}\mathbf{r}_* = &\left\{U_0 + \left[\frac{1}{|\mathbf{r}_c^0 + \Delta\mathbf{r}|} - \frac{|\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}|^2}{\mu}\right] U_2\right\}(\mathbf{r}_c^0 + \Delta\mathbf{r}) + \\ &+ \left[U_1 \frac{|\mathbf{r}_c^0 + \Delta\mathbf{r}|}{\sqrt{\mu}} + U_2 \frac{(\mathbf{r}_c^0 + \Delta\mathbf{r}) \cdot (\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})}{\mu}\right] \times (\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})\end{aligned} \tag{32}$$

and the magnitude of the solution is:

$$r_* = |\mathbf{r}_c^0 + \Delta\mathbf{r}| U_0 + \frac{(\mathbf{r}_c^0 + \Delta\mathbf{r}) \cdot (\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})}{\mu} U_1 + U_2 \tag{33}$$

The velocity of the motion governed by Eq.(23) is

$$\begin{aligned}\dot{\mathbf{r}}_* = &-\frac{\sqrt{\mu}}{r_*} U_1 \frac{\mathbf{r}_c^0 + \Delta\mathbf{r}}{|\mathbf{r}_c^0 + \Delta\mathbf{r}|} + \frac{\sqrt{\mu}}{r_*}\left[U_0 \frac{|\mathbf{r}_c^0 + \Delta\mathbf{r}|}{\sqrt{\mu}}\right. \\ &\left.+ U_1 \frac{(\mathbf{r}_c^0 + \Delta\mathbf{r}) \cdot (\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})}{\mu}\right] \times (\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})\end{aligned} \tag{34}$$

Then, using (22) and (24) together with (32) and (34), the solution to the initial value problem (3) may be written as:

$$\boxed{\begin{aligned}\mathbf{r} = \mathbf{R}_{-\omega} &\left\{\left\{U_0 + \left[\frac{1}{|\mathbf{r}_c^0 + \Delta\mathbf{r}|} - \frac{|\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}|^2}{\mu}\right] U_2\right\}(\mathbf{r}_c^0 + \Delta\mathbf{r}) + \right. \\ &+ \left[U_1 \frac{|\mathbf{r}_c^0 + \Delta\mathbf{r}|}{\sqrt{\mu}} + U_2 \frac{(\mathbf{r}_c^0 + \Delta\mathbf{r}) \cdot (\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})}{\mu}\right] \\ &\left.\times(\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})\right\} - \frac{p_c}{1 + e_c \cos f_c(t)} \frac{\mathbf{r}_0}{r_0}.\end{aligned}} \tag{35}$$

$$\boxed{\begin{aligned}\mathbf{v} = \mathbf{R}_{-\omega} &\left\{\frac{\sqrt{\mu}}{r_*} U_1 \frac{\mathbf{r}_c^0 + \Delta\mathbf{r}}{|\mathbf{r}_c^0 + \Delta\mathbf{r}|} + \frac{\sqrt{\mu}}{r_*}\left[U_0 \frac{|\mathbf{r}_c^0 + \Delta\mathbf{r}|}{\sqrt{\mu}}\right.\right. \\ &\left.\left.+ U_1 \frac{(\mathbf{r}_c^0 + \Delta\mathbf{r}) \cdot (\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})}{\mu}\right] \times (\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})\right\} \\ -\widetilde{\omega}\mathbf{R}_{-\omega} &\left\{\left\{U_0 + \left[\frac{1}{|\mathbf{r}_c^0 + \Delta\mathbf{r}|} - \frac{|\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}|^2}{\mu}\right] U_2\right\}(\mathbf{r}_c^0 + \Delta\mathbf{r}) + \right. \\ &+ \left[U_1 \frac{|\mathbf{r}_c^0 + \Delta\mathbf{r}|}{\sqrt{\mu}} + U_2 \frac{(\mathbf{r}_c^0 + \Delta\mathbf{r}) \cdot (\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})}{\mu}\right] \\ &\left.\times(\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})\right\} - \frac{e_c |\mathbf{h}_c| \sin f_c(t)}{p_c} \frac{\mathbf{r}_c^0}{r_c^0}.\end{aligned}} \tag{36}$$

where $\mathbf{R}_{-\boldsymbol{\omega}} = \mathbf{I}_3 - \sin f_c^0 \frac{\widetilde{\mathbf{h}}_c}{h_c} + (1 - \cos f_c^0) \frac{\widetilde{\mathbf{h}}_c^2}{h_c^2}$ and $f_c^0 = f_c(t) - f_c(t_0)$.

The universal functions U_k are linked by a Kepler-like equation (Battin (1999)):

$$\sqrt{\mu}(t - t_0) = U_1(\chi;\alpha)|\mathbf{r}_c^0 + \Delta\mathbf{r}| + U_2(\chi;\alpha) \frac{(\mathbf{r}_c^0 + \Delta\mathbf{r}) \cdot (\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})}{\sqrt{\mu}} + U_3(\chi;\alpha) \tag{37}$$

Equations (35) and (36) offer the closed-form compact solution to the relative orbital motion problem. They hold for all types of reference trajectories of the chief and deputy (elliptic, parabolic, hyperbolic).

4. Comprehensive analysis of the relative orbital motion of satellites

By using the results presented in the previous sections, we are about to offer the closed-form solution to the relative orbital motion in all possible particular cases. In this approach, the chief inertial trajectory is less important than the deputy inertial trajectory, and the study will focus on the nature of the latter. We must make here the remark that in fact the initial value problem (23) models the motion of the deputy spacecraft in the inertial frame. This equation is deduced by knowing only the chief motion and the initial conditions of the deputy in the LVLH frame. From this point, when referring to the deputy inertial motion, we refer in fact to the motion governed by the initial value problem (23).

It is possible to obtain a closed-form solution to the nonlinear model of the relative orbital motion (3) in the situation where the inertial deputy trajectory is an ellipse, a parabola, or a hyperbola. These situations are delimited by the sign of the generalized specific energy of the deputy spacecraft (Battin (1999); Condurache & Martinusi (2007a)). It was proven that in the conditions that are given above, the sign of the quantity

$$\zeta = \frac{1}{2}|\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}|^2 - \frac{\mu}{|\mathbf{r}_c^0 + \Delta\mathbf{r}|} \tag{38}$$

gives the type of the Keplerian inertial trajectory of the deputy spacecraft, i.e., if $\zeta < 0$ the inertial trajectory of the deputy is an ellipse, if $\zeta = 0$ it is a parabola, and if $\zeta > 0$ it is a hyperbola. An accurate observer would remark that the previous phrase is mathematically correct only if the angular momentum $\mathbf{h}$ of the deputy inertial orbit is nonzero, $\mathbf{h} \neq \mathbf{0}$,

$$\mathbf{h} = (\mathbf{r}_c^0 + \Delta\mathbf{r}) \times (\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}). \tag{39}$$

Only this situation will be taken into consideration in this approach.

In the following the elliptic inertial deputy trajectory ($\zeta < 0, \mathbf{h} \neq \mathbf{0}$) will be analyzed. The inertial trajectory of the deputy spacecraft is an ellipse (or a circle). The motion on this orbit is modeled by the position vector $\mathbf{r}_*$, which is the solution to the initial value problem (23). The expressions for the vectors $\mathbf{r}_*$ and $\dot{\mathbf{r}}_*$ are:

$$\mathbf{r}_* = \mathbf{a}[\cos E(t) - e] + \mathbf{b} \sin E(t) \tag{40}$$

$$\dot{\mathbf{r}}_* = \frac{n}{1 - e\cos E(t)} [-\mathbf{a} \sin E(t) + \mathbf{b} \cos E(t)] \tag{41}$$

where **e** represents the vector corresponding to the vectorial eccentricity of the Keplerian motion described by Eq. (23); its expression is

$$\mathbf{e} = \frac{1}{\mu}(\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}) \times \mathbf{h} - \frac{\mathbf{r}_c^0 + \Delta\mathbf{r}}{|\mathbf{r}_c^0 + \Delta\mathbf{r}|}. \tag{42}$$

If $\mathbf{e} = \mathbf{0}$, the inertial trajectory of the deputy spacecraft is circular; $\mathbf{h}$ is defined in Eq. (31); n is the mean motion of the motion described by Eq. (23); $\mathbf{a}$ and $\mathbf{b}$ represent the vectors that model the semimajor and semiminor axis of the deputy inertial trajectory respectively; their expressions are (Condurache & Martinusi (2007a)):

$$n = \frac{(2|\zeta|)^{\frac{3}{2}}}{\mu}; \mathbf{a} = \begin{cases} \frac{\mu}{2e|\zeta|}\mathbf{e}, & \mathbf{e} \neq \mathbf{0} \\ \mathbf{r}_c^0 + \Delta\mathbf{r}, & \mathbf{e} = \mathbf{0} \end{cases}; \mathbf{b} = \begin{cases} \frac{1}{e\sqrt{2|\zeta|}}(\mathbf{h} \times \mathbf{e}), & \mathbf{e} \neq \mathbf{0} \\ \frac{1}{n}(\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}), & \mathbf{e} = \mathbf{0} \end{cases}; \tag{43}$$

$E(t)$ represents the deputy spacecraft eccentric anomaly; it is the solution to the Kepler equation:

$$E(t) - e\sin E(t) = n(t - t_p), \quad t \in [t_0, +\infty] \tag{44}$$

where t_P denotes the time of periapsis passage of the deputy spacecraft and it is computed from (Condurache & Martinusi (2007a))

$$t_p = t_0 - \frac{1}{n}[E(t_0) - e\sin E(t_0)] \tag{45}$$

while

$$\cos E(t_0) = \frac{1}{e}\left(1 - n\frac{|\mathbf{r}_c^0 + \Delta\mathbf{r}|}{\sqrt{2|\zeta|}}\right) \tag{46}$$

$$\sin E(t_0) = n\frac{\Delta\mathbf{v} \cdot (\mathbf{r}_c^0 + \Delta\mathbf{r})}{2e|\zeta|}\left[1 - \frac{\boldsymbol{\omega}(t_0) \cdot \mathbf{h}}{\mu}|\mathbf{r}_c^0 + \Delta\mathbf{r}|\right]. \tag{47}$$

From (22) and (24) combined with (40) and (41) the relative law of motion and the relative velocity are modeled by:

$$\boxed{\begin{aligned} \mathbf{r} = [\cos E(t) - e]&\left\{\frac{\mathbf{h}_c \cdot \mathbf{a}}{|\mathbf{h}_c|^2}\mathbf{h}_c - \sin f_c^0 \frac{\widetilde{\mathbf{h}}_c \cdot \mathbf{a}}{|\mathbf{h}_c|} - \cos f_c^0 \frac{\widetilde{\mathbf{h}}_c^2 \cdot \mathbf{a}}{|\mathbf{h}_c|^2}\right\} \\ + \sin E(t)&\left\{\frac{\mathbf{h}_c \cdot \mathbf{b}}{|\mathbf{h}_c|^2}\mathbf{h}_c - \sin f_c^0 \frac{\widetilde{\mathbf{h}}_c \cdot \mathbf{b}}{|\mathbf{h}_c|} - \cos f_c^0 \frac{\widetilde{\mathbf{h}}_c^2 \cdot \mathbf{b}}{|\mathbf{h}_c|^2}\right\} - \frac{p_c}{1 + e_c\cos f_c(t)}\frac{\mathbf{r}_0}{r_0} \end{aligned}} \tag{48}$$

$$\boxed{\begin{aligned} \mathbf{v} = \frac{-n\sin E(t)}{1 - e\cos E(t)}&\left\{\frac{\mathbf{h}_c \cdot \mathbf{a}}{|\mathbf{h}_c|^2}\mathbf{h}_c - \sin f_c^0 \frac{\widetilde{\mathbf{h}}_c \cdot \mathbf{a}}{|\mathbf{h}_c|} - \cos f_c^0 \frac{\widetilde{\mathbf{h}}_c^2 \cdot \mathbf{a}}{|\mathbf{h}_c|^2}\right\} \\ + \frac{n\cos E(t)}{1 - e\cos E(t)}&\left\{\frac{\mathbf{h}_c \cdot \mathbf{b}}{|\mathbf{h}_c|^2}\mathbf{h}_c - \sin f_c^0 \frac{\widetilde{\mathbf{h}}_c \cdot \mathbf{b}}{|\mathbf{h}_c|} - \cos f_c^0 \frac{\widetilde{\mathbf{h}}_c^2 \cdot \mathbf{b}}{|\mathbf{h}_c|^2}\right\} \\ + \frac{[1 + e_c\cos f_c(t)]^2[\cos E(t) - e]}{p_c^2} &\times \left\{\frac{\sin f_c^0(t)}{|\mathbf{h}_c|}\widetilde{\mathbf{h}}_c^2\mathbf{a} - \cos f_c^0(t)\widetilde{\mathbf{h}}_c\mathbf{a}\right\} \\ + \frac{[1 + e_c\cos f_c(t)]^2[\sin E(t)]}{p_c^2} &\times \left\{\frac{\sin f_c^0(t)}{|\mathbf{h}_c|}\widetilde{\mathbf{h}}_c^2\mathbf{b} - \cos f_c^0(t)\widetilde{\mathbf{h}}_c\mathbf{b}\right\} \\ &- \frac{e_c|\mathbf{h}_c|\sin f_c(t)}{p_c}\frac{\mathbf{r}_c^0}{|\mathbf{r}_c^0|} \end{aligned}} \tag{49}$$

If the deputy trajectory is circular ($\mathbf{e} = 0$), Eqs. (43) are taken into account, together with:

$$p = |\mathbf{r}_c^0 + \Delta\mathbf{r}|; \qquad E(t) = \frac{|\mathbf{h}|}{|\mathbf{r}_c^0 + \Delta\mathbf{r}|^2}(t - t_0) \tag{50}$$

If the reference trajectory is circular, the closed-form Eqs. (48) and (49) change according to the following expressions:

$$e_c = 0; \quad f_c^0(t) = n_c(t - t_0) \tag{51}$$

It follows that in the situation when the chief spacecraft has an inertial circular trajectory, Eqs. (48) and (49) transform into

$$\boxed{\begin{aligned}\mathbf{r} = [\cos E(t) - e]\left\{\frac{\mathbf{h}_c \cdot \mathbf{a}}{|\mathbf{h}_c|^2}\mathbf{h}_c - \sin(n_c(t-t_0))\frac{\widetilde{\mathbf{h}}_c \cdot \mathbf{a}}{|\mathbf{h}_c|} - \cos(n_c(t-t_0))\frac{\widetilde{\mathbf{h}}_c^2 \cdot \mathbf{a}}{|\mathbf{h}_c|^2}\right\} \\ + \sin E(t)\left\{\frac{\mathbf{h}_c \cdot \mathbf{b}}{|\mathbf{h}_c|^2}\mathbf{h}_c - \sin(n_c(t-t_0))\frac{\widetilde{\mathbf{h}}_c \cdot \mathbf{b}}{|\mathbf{h}_c|} - \cos(n_c(t-t_0))\frac{\widetilde{\mathbf{h}}_c^2 \cdot \mathbf{b}}{|\mathbf{h}_c|^2}\right\} - \mathbf{r}_c^0\end{aligned}} \tag{52}$$

$$\boxed{\begin{aligned}\mathbf{v} = \frac{-n\sin E(t)}{1 - e\cos E(t)}\left\{\frac{\mathbf{h}_c \cdot \mathbf{a}}{|\mathbf{h}_c|^2}\mathbf{h}_c - \sin(n_c(t-t_0))\frac{\widetilde{\mathbf{h}}_c \cdot \mathbf{a}}{|\mathbf{h}_c|} - \cos(n_c(t-t_0))\frac{\widetilde{\mathbf{h}}_c^2 \cdot \mathbf{a}}{|\mathbf{h}_c|^2}\right\} \\ + \frac{n\cos E(t)}{1 - e\cos E(t)}\left\{\frac{\mathbf{h}_c \cdot \mathbf{b}}{|\mathbf{h}_c|^2}\mathbf{h}_c - \sin(n_c(t-t_0))\frac{\widetilde{\mathbf{h}}_c \cdot \mathbf{b}}{|\mathbf{h}_c|} - \cos(n_c(t-t_0))\frac{\widetilde{\mathbf{h}}_c^2 \cdot \mathbf{b}}{|\mathbf{h}_c|^2}\right\} \\ + \frac{\cos E(t) - e}{|\mathbf{r}_c^0|^2}\left\{\frac{1}{|\mathbf{h}_c|}\sin[n_c(t-t_0)]\widetilde{\mathbf{h}}_c^2\mathbf{a} - \cos[n_c(t-t_0)]\widetilde{\mathbf{h}}_c\mathbf{a}\right\} \\ + \frac{\sin E(t)}{|\mathbf{r}_c^0|^2}\left\{\frac{1}{|\mathbf{h}_c|}\sin[n_c(t-t_0)]\widetilde{\mathbf{h}}_c^2\mathbf{b} - \cos[n_c(t-t_0)]\widetilde{\mathbf{h}}_c\mathbf{b}\right\}\end{aligned}} \tag{53}$$

We make here the following remark: the equations (48) and (49) represent the generalization to the Tschauner-Hempel (TH) and Lawden solution. While TH and Lawden equations are the solution to the linearized model for the relative motion, the equations deduced here represent the solution to the nonlinear original model of the relative motion. They stand true for any elliptic targeted and reference trajectory. The Eqs. (52) and (53) generalize the Clohessy-Wiltshire model.

In the end of this subsection, we will present the closed-form exact expressions for the relative law of motion and velocity with respect to the eccentric anomalies in the situation when both chief and deputy are satellites (the ellipse-ellipse situation). From the Kepler equations written for both chief and deputy inertial motions

$$E_c - e_c \sin E_c = n_c(t - t_p^c) \tag{54}$$

$$E - e\sin E = n(t - t_p) \tag{55}$$

one may derive the implicit equation that links these anomalies by eliminating the time t from Eqs. (54) and (55):

$$\frac{E_c - e_c\sin E_c}{n_c} + t_p^c = \frac{E - e\sin E}{n} + t_p \tag{56}$$

As the motion of the chief satellite is known, so is function E_c. The eccentric anomaly of the Deputy satellite is then obtained by solving the implicit functional equation:

$$E - e\sin E = \frac{n}{n_c}(E_c - e_c\sin E_c) + n(t_p^c - t_p) \tag{57}$$

By taking into account the relations between the true anomaly and the eccentric anomaly of a Keplerian elliptic orbit

$$\begin{cases} \cos f = \dfrac{\cos E - e}{1 - e\cos E} \\ \sin f = \dfrac{\sqrt{1-e^2}\sin E}{1 - e\cos E} \end{cases}, \tag{58}$$

equations (49) and (50) are transformed into:

$$\boxed{\begin{aligned}
\mathbf{r} = [\cos E - e]&\left\{\frac{\mathbf{h}_c\cdot\mathbf{a}}{|\mathbf{h}_c|^2}\mathbf{h}_c - \sqrt{1-e_c^2}\frac{\sin(E_c^0+E_c) - e_c(\sin E_c + \sin E_c^0)}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}\frac{\widetilde{\mathbf{h}}_c\mathbf{a}}{|\mathbf{h}_c|}\right.\\
&\left.-\frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c\sin E_c^0}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}\frac{\widetilde{\mathbf{h}}_c^2\mathbf{a}}{|\mathbf{h}_c|^2}\right\}\\
+\sin E&\left\{\frac{\mathbf{h}_c\cdot\mathbf{b}}{|\mathbf{h}_c|^2}\mathbf{h}_c - \sqrt{1-e_c^2}\frac{\sin(E_c^0+E_c) - e_c(\sin E_c + \sin E_c^0)}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}\frac{\widetilde{\mathbf{h}}_c\mathbf{b}}{|\mathbf{h}_c|}\right.\\
&\left.-\frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c\sin E_c^0}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}\frac{\widetilde{\mathbf{h}}_c^2\mathbf{b}}{|\mathbf{h}_c|^2}\right\}\\
&-\frac{p_c(1-e_c\cos E_c)}{1-e_c^2}\frac{\mathbf{r}_c^0}{|\mathbf{r}_c^0|}
\end{aligned}} \tag{59}$$

$$\boxed{\begin{aligned}
\mathbf{v} = \frac{-n\sin E}{1-e\cos E}&\left\{\frac{\mathbf{h}_c\cdot\mathbf{a}}{|\mathbf{h}_c|^2}\mathbf{h}_c - \sqrt{1-e_c^2}\frac{\sin(E_c^0+E_c) - e_c(\sin E_c + \sin E_c^0)}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}\frac{\widetilde{\mathbf{h}}_c\mathbf{a}}{|\mathbf{h}_c|}\right.\\
&\left.-\frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c\sin E_c^0}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}\frac{\widetilde{\mathbf{h}}_c^2\mathbf{a}}{|\mathbf{h}_c|^2}\right\}\\
+\frac{n\cos E}{1-e\cos E}&\left\{\frac{\mathbf{h}_c\cdot\mathbf{b}}{|\mathbf{h}_c|^2}\mathbf{h}_c - \sqrt{1-e_c^2}\frac{\sin(E_c^0+E_c) - e_c(\sin E_c + \sin E_c^0)}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}\frac{\widetilde{\mathbf{h}}_c\mathbf{b}}{|\mathbf{h}_c|}\right.\\
&\left.-\frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c\sin E_c^0}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}\frac{\widetilde{\mathbf{h}}_c^2\mathbf{b}}{|\mathbf{h}_c|^2}\right\}\\
+\frac{(1-e_c^2)(\cos E - e)}{(1-e_c\cos E_c)p_c^2}\times&\left\{-\sqrt{1-e_c^2}\frac{\sin(E_c^0+E_c) - e_c(\sin E_c + \sin E_c^0)}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}\frac{\widetilde{\mathbf{h}}_c\mathbf{a}}{|\mathbf{h}_c|}\right.\\
&\left.-\frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c\sin E_c^0}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}\widetilde{\mathbf{h}}_c\mathbf{a}\right\}\\
+\frac{(1-e_c^2)(\sin E)}{(1-e_c\cos E_c)p_c^2}\times&\left\{-\sqrt{1-e_c^2}\frac{\sin(E_c^0+E_c) - e_c(\sin E_c + \sin E_c^0)}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}\frac{\widetilde{\mathbf{h}}_c\mathbf{b}}{|\mathbf{h}_c|}\right.\\
&\left.-\frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c\sin E_c^0}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}\widetilde{\mathbf{h}}_c\mathbf{b}\right\}\\
&-\frac{e_c|\mathbf{h}_c|(1-e_c^2)sinE_c}{(1-e_c\cos E_c)p_c}\frac{\mathbf{r}_c^0}{|\mathbf{r}_c^0|}
\end{aligned}} \tag{60}$$

where $E_c^0 = E_c(t_0)$.

4.1 Parametric Cartesian solution of relative orbital motion

In the following we present the scalar Cartesian expressions for the relative position and relative velocity as they are deduced from the expressions presented in this section. By denoting $\mathbf{r} = [x\ y\ z]^T$ the relative position vector, below we present the closed form expressions for $x, y, z, \dot{x}, \dot{y}, \dot{z}$. We denote $\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z$ the unit vectors that define the axes of the LVLH frame; their expressions are

$$\mathbf{u}_x = \frac{\mathbf{r}_c^0}{|\mathbf{r}_c^0|};\ \mathbf{u}_y = \frac{\widetilde{\mathbf{h}}_c \mathbf{r}_c^0}{|\mathbf{h}_c||\mathbf{r}_c^0|};\ \mathbf{u}_z = \frac{\mathbf{h}_c^0}{|\mathbf{h}_c^0|} \tag{61}$$

If $\zeta < 0, \mathbf{h} \neq \mathbf{0}$ then using (52) and (53) results:

$$\begin{aligned} x(t) = {} & [\cos E(t) - e]\left\{(\mathbf{u}_x \cdot \mathbf{a}) \cos f_c^0(t) + (\mathbf{u}_y \cdot \mathbf{a}) \sin f_c^0(t)\right\} \\ & + \sin E(t)\left\{(\mathbf{u}_x \cdot \mathbf{b}) \cos f_c^0(t) + (\mathbf{u}_y \cdot \mathbf{b}) \sin f_c^0(t)\right\} \\ & - \frac{p_c}{1 + e_c \cos f_c(t)} \end{aligned} \tag{62}$$

$$\begin{aligned} y(t) = {} & [\cos E(t) - e]\left\{(-\mathbf{u}_x \cdot \mathbf{a}) \sin f_c^0(t) + (\mathbf{u}_y \cdot \mathbf{a}) \cos f_c^0(t)\right\} \\ & + \sin E(t)\left\{(-\mathbf{u}_x \cdot \mathbf{b}) \sin f_c^0(t) + (\mathbf{u}_y \cdot \mathbf{b}) \cos f_c^0(t)\right\} \end{aligned} \tag{63}$$

$$z(t) = [\cos E(t) - e](\mathbf{u}_z \cdot \mathbf{a}) + \sin E(t)(\mathbf{u}_z \cdot \mathbf{b}) \tag{64}$$

$$\begin{aligned} \dot{x}(t) = {} & \frac{n \sin E(t)}{1 - e \cos E(t)}\left\{(\mathbf{u}_x \cdot \mathbf{a}) \cos f_c^0(t) + (\mathbf{u}_y \cdot \mathbf{a}) \sin f_c^0(t)\right\} \\ & + \frac{n \cos E(t)}{1 - e \cos E(t)}\left\{(\mathbf{u}_x \cdot \mathbf{b}) \cos f_c^0(t) + (\mathbf{u}_y \cdot \mathbf{b}) \sin f_c^0(t)\right\} \\ & - \frac{\mu[1 + e_c \cos f_c(t)]^2[\cos E(t) - e]}{|\mathbf{h}|_c}\left\{(-\mathbf{u}_x \cdot \mathbf{a}) \sin f_c^0(t) + (\mathbf{u}_y \cdot \mathbf{a}) \cos f_c^0(t)\right\} \\ & - \frac{\mu[1 + e_c \cos f_c(t)]^2 \sin E(t)}{|\mathbf{h}|_c}\left\{(-\mathbf{u}_x \cdot \mathbf{a}) \sin f_c^0(t) + (\mathbf{u}_y \cdot \mathbf{a}) \cos f_c^0(t)\right\} \\ & - \frac{e_c |\mathbf{h}_c| \sin f_c(t)}{p_c} \end{aligned} \tag{65}$$

$$\begin{aligned} \dot{y}(t) = {} & \frac{n \sin E(t)}{1 - e \cos E(t)}\left\{(\mathbf{u}_x \cdot \mathbf{a}) \sin f_c^0(t) - (\mathbf{u}_y \cdot \mathbf{a}) \cos f_c^0(t)\right\} \\ & - \frac{n \cos E(t)}{1 - e \cos E(t)}\left\{-(\mathbf{u}_x \cdot \mathbf{b}) \sin f_c^0(t) + (\mathbf{u}_y \cdot \mathbf{b}) \cos f_c^0(t)\right\} \\ & - \frac{|\mathbf{h}|_c[1 + e_c \cos f_c(t)]^2[\cos E(t) - e]}{p_c}\left\{(\mathbf{u}_y \cdot \mathbf{a}) \sin f_c^0(t) + (\mathbf{u}_x \cdot \mathbf{a}) \cos f_c^0(t)\right\} \\ & - \frac{|\mathbf{h}_c|[1 + e_c \cos f_c(t)]^2 \sin E(t)}{p_c}\left\{(\mathbf{u}_y \cdot \mathbf{b}) \sin f_c^0(t) + (\mathbf{u}_x \cdot \mathbf{b}) \cos f_c^0(t)\right\} \end{aligned} \tag{66}$$

$$\dot{z}(t) = \frac{n}{[1 - e \cos E(t)]}[-\sin E(t)(\mathbf{u}_z \cdot \mathbf{a}) + \cos E(t)(\mathbf{u}_z \cdot \mathbf{b})] \tag{67}$$

When the deputy trajectory is also an ellipse and one expresses the equations of the relative motion with respect to both eccentric anomalies, Eqs. (59-60) are transformed into:

$$\begin{aligned}
x(t) = [\cos E(t) - e] &\left\{ \frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c \sin E_c^0}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_x \cdot \mathbf{a}) \right.\\
&\left. - \sqrt{1-e_c^2}\frac{(\sin E_c^0 - e_c) - e_c(\sin E_c + \sin E_c^0)}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_y \cdot \mathbf{a}) \right\}\\
+ \sin E(t) &\left\{ \frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c \sin E_c^0}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_x \cdot \mathbf{a}) \right.\\
&\left. - \sqrt{1-e_c^2}\frac{(\sin E_c^0 - e_c) - e_c(\sin E_c + \sin E_c^0)}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_y \cdot \mathbf{b}) \right\} - \frac{p_c(1 - e_c\cos E_c)}{1-e_c^2}
\end{aligned} \tag{68}$$

$$\begin{aligned}
y(t) = -[\cos E(t) - e] &\left\{ (\mathbf{u}_x \cdot \mathbf{a}) \sin f_c^0(t) \right.\\
&\left. + \frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c \sin E_c^0}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_y \cdot \mathbf{a}) \right\}\\
- \sin E(t) &\left\{ - \sqrt{1-e_c^2}\frac{(\sin E_c^0 - e_c) - e_c(\sin E_c + \sin E_c^0)}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_x \cdot \mathbf{b}) \right.\\
&\left. + \frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c \sin E_c^0}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_y \cdot \mathbf{b}) \right\}
\end{aligned} \tag{69}$$

$$z(t) = [\cos E(t) - e](\mathbf{u}_z \cdot \mathbf{a}) + \sin E(t)(\mathbf{u}_z \cdot \mathbf{b}) \tag{70}$$

$$\begin{aligned}
\dot{x}(t) = \frac{n \sin E(t)}{1 - e\cos E(t)} &\left\{ \frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c \sin E_c^0}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_x \cdot \mathbf{a}) \right.\\
&\left. + \sqrt{1-e_c^2}\frac{(\sin E_c^0 - e_c) - e_c(\sin E_c + \sin E_c^0)}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_y \cdot \mathbf{a}) \right\}\\
+ \frac{n \cos E(t)}{1 - e\cos E(t)} &\left\{ \frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c \sin E_c^0}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_x \cdot \mathbf{b}) \right.\\
&\left. + \sqrt{1-e_c^2}\frac{(\sin E_c^0 - e_c) - e_c(\sin E_c + \sin E_c^0)}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_y \cdot \mathbf{b}) \right\}\\
- \frac{\mu[1 + e_c\cos f_c(t)]^2[\cos E(t) - e]}{|\mathbf{h}|_c} &\left\{ - \sqrt{1-e_c^2}\frac{(\sin E_c^0 - e_c) - e_c(\sin E_c + \sin E_c^0)}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_x \cdot \mathbf{a}) \right.\\
&\left. + \frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c \sin E_c^0}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_y \cdot \mathbf{a}) \right\}\\
- \frac{\mu[1 + e_c\cos f_c(t)]^2 \sin E(t)}{|\mathbf{h}|_c} &\left\{ - \sqrt{1-e_c^2}\frac{(\sin E_c^0 - e_c) - e_c(\sin E_c + \sin E_c^0)}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_x \cdot \mathbf{b}) \right.\\
&\left. + \frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c \sin E_c^0}{(1 - e_c\cos E_c)(1 - e_c\cos E_c^0)} (\mathbf{u}_y \cdot \mathbf{b}) \right\} - \frac{e_c|\mathbf{h}_c|\sin f_c(t)}{p_c}
\end{aligned} \tag{71}$$

$$\dot{y}(t) = -\frac{n\sin E(t)}{1-e\cos E(t)}\left\{-\frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c \sin E_c^0}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}(\mathbf{u}_y\cdot\mathbf{a})\right.$$
$$\left. -\sqrt{1-e_c^2}\frac{(\sin E_c^0 - e_c) - e_c(\sin E_c + \sin E_c^0)}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}(\mathbf{u}_x\cdot\mathbf{a})\right\}$$
$$+\frac{n\cos E(t)}{1-e\cos E(t)}\left\{-\frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c \sin E_c^0}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}(\mathbf{u}_y\cdot\mathbf{b})\right.$$
$$\left. +\sqrt{1-e_c^2}\frac{(\sin E_c^0 - e_c) - e_c(\sin E_c + \sin E_c^0)}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}(\mathbf{u}_x\cdot\mathbf{b})\right\}$$
$$-\frac{|\mathbf{h}_c|[1+e_c\cos f_c(t)]^2[\cos E(t) - e]}{|\mathbf{h}|_c}\left\{\sqrt{1-e_c^2}\frac{(\sin E_c^0 - e_c) - e_c(\sin E_c + \sin E_c^0)}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}(\mathbf{u}_y\cdot\mathbf{a})\right.$$
$$\left. +\frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c \sin E_c^0}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}(\mathbf{u}_x\cdot\mathbf{a})\right\}$$
$$-\frac{|\mathbf{h}_c|[1+e_c\cos f_c(t)]^2\sin E(t)}{|\mathbf{h}|_c}\left\{\sqrt{1-e_c^2}\frac{(\sin E_c^0 - e_c) - e_c(\sin E_c + \sin E_c^0)}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}(\mathbf{u}_y\cdot\mathbf{b})\right.$$
$$\left. +\frac{(\cos E_c - e_c)(\cos E_c^0 - e_c) - (1-e_c^2)\sin E_c \sin E_c^0}{(1-e_c\cos E_c)(1-e_c\cos E_c^0)}(\mathbf{u}_x\cdot\mathbf{b})\right\} - \frac{e_c|\mathbf{h}_c|\sin f_c(t)}{p_c} \tag{72}$$

$$\dot{z}(t) = \frac{n}{[1-e\cos E(t)]}[-\sin E(t)(\mathbf{u}_z\cdot\mathbf{a}) + \cos E(t)(\mathbf{u}_z\cdot\mathbf{b})] \tag{73}$$

An interesting remark is that the motion along the Oz axis of LVLH (the out-of-plane motion) is completely decoupled from the in-plane motion.

5. Periodicity conditions in relative orbital motion

An interesting geometric visualization of the relative motion is illustrated in Fig. 2.

It may be seen as the composition among:

- a classic Keplerian motion in a variable plane $\Pi(t), t \geq t_0$; plane $\Pi(t)$ is formed at moment $t = t_0$ if the inertial motion of the Deputy satellite is not rectilinear; this plane is determined by the initial position and initial velocity vectors of the Deputy;
- a precession of plane $\Pi(t)$ with angular velocity $-\boldsymbol{\omega}$ around the attraction center;
- a rectilinear translation of plane $\Pi(t)$ described by vector $-\mathbf{r}_c$.

This geometric interpretation shows that the relative orbital motion is in fact a Foucault pendulum like motion (Condurache & Martinusi (2008a)). Excluding the situation $\mathbf{h} = 0$, the case $\zeta \leq 0$ is equivalent with the Deputy elliptic inertial motion. If the Leader satellite also has an elliptic motion, then the motion of the Deputy with respect to the LVLH frame might be periodic. In fact, recall that:

$$\mathbf{r} = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{r}_i - \mathbf{r}_c \tag{74}$$

is the maps $\mathbf{R}_{-\boldsymbol{\omega}}$ and $\mathbf{r}_c$ have the same main period T_c, which is that of the Leader, and $\mathbf{r}_i$ has the main period of the Deputy inertial motion, denoted as T_d. The motion in LVLH is then

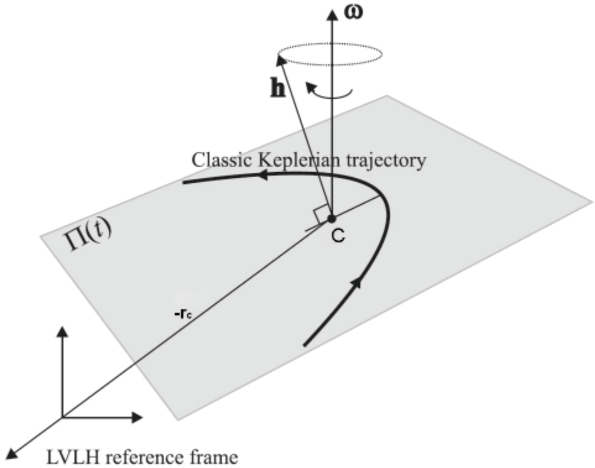

Fig. 2. Geometric interpretation of the relative orbital motion.

periodic if:

$$T_c / T_d \quad \text{is rational number} \tag{75}$$

This leads to a formula that involves the specific energies of the two satellites: the motion is periodic if

$$\left(\frac{\frac{1}{2}(\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}_0 \times \Delta\mathbf{r})^2 - \dfrac{\mu}{|\mathbf{r}_c^0 + \Delta\mathbf{r}|}}{\frac{1}{2}\mathbf{v}_c^{0\,2} - \dfrac{\mu}{r_c^0}} \right)^{\frac{3}{2}} = \frac{m}{n} \tag{76}$$

where m and n are relatively prime natural numbers. In spacecraft formations, it can be easily proven that a necessary condition for two or more satellites to remain at a reasonably small distance from one another is that their periods are equal, leading their specific energies to be equal:

$$\frac{1}{2}(\mathbf{v}_c^0 + \Delta\mathbf{v} + \boldsymbol{\omega}_0 \times \Delta\mathbf{r})^2 - \frac{\mu}{|\mathbf{r}_c^0 + \Delta\mathbf{r}|} = \frac{1}{2}\mathbf{v}_c^{0\,2} - \frac{\mu}{r_0} \tag{77}$$

Written with respect to the initial conditions Eq. (72) becomes:

$$\frac{1}{2}\Delta\mathbf{v}^2 + \mathbf{v}_c^0 \cdot \Delta\mathbf{v} + \frac{(\mathbf{v}_c^0 + \Delta\mathbf{v}, \mathbf{h}_c, \Delta\mathbf{r})}{r_c^{0\,2}} + \frac{1}{2}\frac{(\mathbf{h}_c \times \Delta\mathbf{r})^2}{r_c^{0\,4}} - \frac{\mu}{\sqrt{\mathbf{r}_c^{0\,2} + \Delta\mathbf{r}^2 + 2\mathbf{r}_c^0 \cdot \Delta\mathbf{r}}} + \frac{\mu}{r_c^0} = 0 \tag{78}$$

If the conditions from Eq. (78) are fulfilled the relative orbital motion trajectory is a closed curve.

6. A tensor invariant in the relative motion

In this Section, we will refrain to apply the state flow operator approach to the entire problem which models the relative motion in a gravitational field, but rather to apply it to a part of its solution. We will reveal a very interesting invariance relation, which relates the motion of the deputy and the motion of the attraction center, both referred to LVLH, as well as a very useful propagator for the state of the deputy spacecraft in the same frame.

Consider the relative motion in a gravitational field, where the relative state of the deputy spacecraft in the LVLH frame associated to the chief is expressed like:

$$\begin{cases} \mathbf{r}(t) = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{r}_*(t) - \mathbf{r}_c(t) \\ \dot{\mathbf{r}}(t) = \mathbf{R}_{-\boldsymbol{\omega}}[\dot{\mathbf{r}}_*(t) - \widetilde{\boldsymbol{\omega}}\mathbf{r}_*(t)] - \dot{\mathbf{r}}_c(t) \end{cases} \tag{79}$$

where $\mathbf{r}_* = \mathbf{r}_*(t)$ is the solution to the initial value problem and it models a Keplerian motion. Equation (79) can be written as:

$$\begin{bmatrix} \mathbf{r}_* \\ \dot{\mathbf{r}}_* \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi} & \mathbf{I}_3 \\ \mathbf{I}_3 & \boldsymbol{\Phi} \end{bmatrix} \begin{bmatrix} \mathbf{r}_c(t_0) + \Delta\mathbf{r} \\ \dot{\mathbf{r}}_c(t_0) + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times (\Delta\mathbf{r} + \mathbf{r}_c(t_0)) \end{bmatrix} \tag{80}$$

where $\boldsymbol{\Phi}$ is the unique tensor established by the conditions:

$$\begin{cases} \boldsymbol{\Phi}[\mathbf{r}_*(t_0)] = \mathbf{r}_*(t) \\ \boldsymbol{\Phi}[\dot{\mathbf{r}}_*(t_0)] = \dot{\mathbf{r}}_*(t) \\ \boldsymbol{\Phi}[\mathbf{h}(t_0)] = \mathbf{h}(t) \end{cases} \tag{81}$$

If the Deputy trajectory is elliptic ($\zeta \leq 0$ and $\mathbf{h} \neq 0$), $\boldsymbol{\Phi}$ can be computed as (Condurache & Martinusi (2011); Martinusi (2010)) :

$$\begin{aligned} \boldsymbol{\Phi}(E) = & \left[\frac{\cos E_0(\cos E - e)}{1 - e\cos E_0} + \frac{\sin E_0 \sin E}{1 - e\cos E}\right] \widehat{\mathbf{a}} \otimes \widehat{\mathbf{a}} \\ & + \left[\frac{\sin E_0(\cos E - e)}{1 - e\cos E_0} - \frac{(\cos E_0 - e)\sin E}{1 - e\cos E}\right] \widehat{\mathbf{a}} \otimes \widehat{\mathbf{b}} \\ & + \left[\frac{\cos E_0 \sin E}{1 - e\cos E_0} + \frac{\sin E_0 \cos E}{1 - e\cos E}\right] \widehat{\mathbf{b}} \otimes \widehat{\mathbf{a}} \\ & + \frac{n^3 a^3}{h^2}\left[\frac{\sin E_0 \sin E}{1 - e\cos E_0} - \frac{(\cos E_0 - e)\cos E}{1 - e\cos E}\right] \widehat{\mathbf{b}} \otimes \widehat{\mathbf{b}} + \widehat{\mathbf{h}} \otimes \widehat{\mathbf{h}} \end{aligned} \tag{82}$$

where $E_0 = E(t_0)$ and $\widehat{\mathbf{v}}$ is the unity vector attached to $\mathbf{v}$.

If we denote by $\mathbf{X}(t)$ the state vector attached to the Deputy

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{bmatrix}, \tag{83}$$

equation (79) may be rewritten like:

$$\mathbf{X}(t) = \boldsymbol{\Psi}(t)\mathbf{Y}_0 + \mathbf{X}_c(t) \tag{84}$$

where:

$$\begin{aligned} \boldsymbol{\Psi}(t) &= \begin{bmatrix} \mathbf{R}_{-\omega}\boldsymbol{\Phi} & \mathbf{0}_3 \\ -\widetilde{\boldsymbol{\omega}}\mathbf{R}_{-\omega}\boldsymbol{\Phi} & \mathbf{R}_{-\omega}\boldsymbol{\Phi} \end{bmatrix} \\ \mathbf{Y}_0 &= \begin{bmatrix} \mathbf{r}_c(t_0) + \Delta\mathbf{r} \\ \dot{\mathbf{r}}_c(t_0) + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta(\mathbf{r} + \mathbf{r}_c(t_0)) \end{bmatrix} \\ \mathbf{X}_c(t) &= \begin{bmatrix} -\mathbf{r}_c(t) \\ -\dot{\mathbf{r}}_c(t) \end{bmatrix} \end{aligned} \tag{85}$$

Notice that $\mathbf{X}_c(t)$ models the state of the attraction center with respect to the LVLH frame associated to the chief spacecraft. After some manipulations, it follows that the constant vector $\mathbf{Y}_0$ may be rewritten like:

$$\mathbf{Y}_0 = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 \\ \widetilde{\boldsymbol{\omega}} & \mathbf{I}_3 \end{bmatrix} \left\{ \begin{bmatrix} \Delta\mathbf{r} \\ \Delta\mathbf{v} \end{bmatrix} + \begin{bmatrix} \mathbf{r}_c(t_0) \\ \dot{\mathbf{r}}_c(t_0) \end{bmatrix} \right\} \tag{86}$$

Denote:

$$\boldsymbol{\Gamma}_0 = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 \\ \widetilde{\boldsymbol{\omega}} & \mathbf{I}_3 \end{bmatrix};\ \mathbf{X}_0 = \mathbf{X}(t_0);\ \mathbf{X}_c^0 = \mathbf{X}_c(t_0);\ \boldsymbol{\Lambda}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Gamma}_0 \tag{87}$$

From the above considerations, it follows that:

$$\mathbf{X}(t) - \boldsymbol{\Lambda}(t)\mathbf{X}_0 = \mathbf{X}_c(t) - \boldsymbol{\Lambda}(t)\mathbf{X}_c^0 \tag{88}$$

where the closed form expression of $\boldsymbol{\Lambda}(t)$ is determined by taking into account Equations (85) and (87):

$$\boldsymbol{\Lambda}(t) = \begin{bmatrix} \mathbf{R}_{-\boldsymbol{\omega}}\boldsymbol{\Phi} & \mathbf{0}_3 \\ \mathbf{R}_{-\boldsymbol{\omega}}[\boldsymbol{\Phi}, \widetilde{\boldsymbol{\omega}}] & \mathbf{R}_{-\boldsymbol{\omega}}\boldsymbol{\Phi} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{-\boldsymbol{\omega}} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{R}_{-\boldsymbol{\omega}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi} & \mathbf{0}_3 \\ [\boldsymbol{\Phi}, \widetilde{\boldsymbol{\omega}}] & \boldsymbol{\Phi} \end{bmatrix} \tag{89}$$

where [,] denotes the comutator brackets:

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}. \tag{90}$$

Note that Eq. (88) is very similar to the velocity invariant expression in rigid body kinematics (Condurache & Matcovschi (2001)). The relative state of the deputy spacecraft in LVLH is propagated by:

$$\mathbf{X}(t) = \mathbf{X}_c(t) + \boldsymbol{\Lambda}(t)(\mathbf{X}_0 - \mathbf{X}_c^0) \tag{91}$$

The above formula is the complete exact solution of relative orbital motion nonlinear problem (3).

7. Conclusions

The tensor approach used in this paper allows us to obtain closed-form exact expressions for the relative law of motion and the relative velocity. This instrument is only a catalyst, and it helps introduce a change of variable which transforms the relative orbital motion problem into the classic Kepler problem. Thus, the problem of the relative orbital motion is super-integrable. The shape of the chief inertial trajectory does not impose special problems, as it does in the linearized approaches. The deputy trajectory does not impose problems either, allowing us to derive exact equations of relative motion in any situation and for any initial conditions. The equations that describe the state of the deputy spacecraft in LVLH depend only on time and the initial conditions. Also all the computational stages needed by this solution are conducted on board in the LVLH frame. The long-term accuracy offered by this solution allows the study of the relative motion for indefinite time intervals, and with no restrictions on the magnitude of the relative distance. The solution may be used in the study of satellite constellations from the point of view of the relative motion. The solution offered in this paper gives a parameterization of the manifold associated to the relative motion. Perturbation techniques may be now used in order to derive more accurate equations of motion when assuming small perturbations on the relative trajectory, due to Earth oblateness, solar wind, moon attraction, and atmospheric drag. Based on this solution, a study of the full-body relative motion might be a subject for future work.

8. Nomenclature

$\mathbf{A}^T$ = transpose of tensor (matrix) $\mathbf{A}$

$\mathbf{r}$ = position vector

$\mathbf{r}_1 \cdot \mathbf{r}_2$ = dot product of vector $\mathbf{r}_1$ and $\mathbf{r}_2$

$\mathbf{r}_1 \times \mathbf{r}_2$ = cross product of vector $\mathbf{r}_1$ and $\mathbf{r}_2$

$\widehat{\mathbf{r}}$ = the unity vector attached to $\mathbf{r}$

a = semimajor axis

$\mathbf{a}$=vectorial semimajor axis

b = semimajor axis

$\mathbf{b}$=vectorial semimajor axis

e = eccentricity

$\mathbf{e}$ = vectorial eccentricity

$\mathbf{h}$=specific angular momentum

n=mean motion

p=semilatus rectum (conic parameter)

$\mathbf{R}_\omega$=rotation tensor with angular velocity ω

t= time

u = magnitude of vector $\mathbf{u}$

$\mathbf{v}$ = velocity vector

μ = gravitational parameter

ξ = specific energy

$\boldsymbol{\omega}$ = angular velocity of the rotating reference frame

$\tilde{\boldsymbol{\omega}}$ = skew-symmetric tensor associated with vector $\boldsymbol{\omega}$

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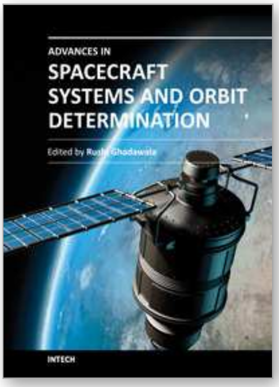

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"Advances in Spacecraft Systems and Orbit Determinations", discusses the development of new technologies and the limitations of the present technology, used for interplanetary missions. Various experts have contributed to develop the bridge between present limitations and technology growth to overcome the limitations. Key features of this book inform us about the orbit determination techniques based on a smooth research based on astrophysics. The book also provides a detailed overview on Spacecraft Systems including reliability of low-cost AOCS, sliding mode controlling and a new view on attitude controller design based on sliding mode, with thrusters. It also provides a technological roadmap for HVAC optimization. The book also gives an excellent overview of resolving the difficulties for interplanetary missions with the comparison of present technologies and new advancements. Overall, this will be very much interesting book to explore the roadmap of technological growth in spacecraft systems.

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