An Algebraic Approach for Controlling Cascade of Reaches in Irrigation Canals

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1. Introduction

Due to the lack of water resources, the problem of water management and minimizing the losses becomes an attraction for many researchers. Although some problems have been already solved in the theoritical point of view, only few of the proposed solutions have been effectively tested in a real situation (Litrico et al., 2003). Limitations of water control technology have been discussed in (Gowing., 1999). However, there are problems that have not been solved yet, as reported in (Bastin et al., 2009). Those problems concern both technological applications and mathematical challenges. To solve water management problems, the so-called St. Venant equations (De Saint-Venant., 1871) are often used as a fondamental tool to describe the dynamics of canals and rivers. They are composed by a 2×2 system of hyperbolic partial differential equations.

For a long time, the matter of controlling water level and flow in open canals has been considered in the literature. Various methods have been used to design boundary controllers which satisfy farmers or navigability demands. Among those different methods, we have: LQ (Linear Quadratic) control that has been particularly developped and studied by (Balogun et al., 1988), and (Malaterre., 1998) (see also (Weyer., 2003) and (Chen et al., 2002)). (Weyer., 2003) has considered LQ control of an irrigation canal in which the water levels are controlled using overshot gates located along the canal. A LQ control problem for linear symmetric infinite-dimensional systems has been considered by (Chen et al., 2002). PI (proportional and integral) control method has been used by (Xu & Sallet., 1999) to propose an output feedback controller using a linear PDE model around a steady state. Such an approach has been considered by (Litrico et al., 2003), where the authors expose and validate a methodology to design efficient automatic controllers for irrigation canals. Riemann and Lyapunov approaches are also considered (Leugering & Schmidt., 2002), (De Halleux et al., 2003), and recently by (Cen & Xi., 2009) and (Bastin et al., 2009).

For networks of open canals, many results have been shown by researchers using some of the methods mentionned above. For example, (De Halleux et al., 2003) have used the Riemann approach to deduce a stabilization control, for a network made up by several interconnected reaches in cascade (also (Cen & Xi., 2009) and reference cited therein). (Bastin et al., 2009) have used the Lyapunov stability approach to study the exponential stability (in L2-norm) of the classical solutions of the linearised Saint-Venant equations for the same network with a sloping bottom. (Leugering & Schmidt., 2002) have studied stabilization and null controlability of pertubations around a steady state for a star configuration network. Star configuration network can also be found in (Li., 2005) and (Goudiaby et al., -). (Goudiaby et al., -), have used a new approach to design boundary feedback controllers which stabilize the water flow and level around a given steady state.

Concerning network made up by several interconnected reaches in cascade, we have noticed, in the theoritical point of view, two approaches that are the Riemann invariants (De Halleux et al., 2003) and Lyapunov Analysis approaches (Bastin et al., 2009), (Cen & Xi., 2009). The purpose of this paper is to apply the approach given in (Goudiaby et al., -) to that network. The approach is applied to a network of two reaches but it can be generalize. Choosing a different type of network requires different treatment of junction where canals met together. On the other hand, the Saint-Venant equations considered in the present paper are in the non-conservation form. We consider the velocity at the boundaries as the controllable quantities.

The approach consists in expressing the rate of change of energy of the linearized problem, as a second order polynomial in terms of the flow velocity at the boundaries. The polynomial is handled in such a way to construct boundary feedback controllers that result in the water flow and the height approaching a given steady state. The water levels at the boundaries and at the junction are used to build the controllers. After deriving the controllers, we numerically apply them to a real problem, which is nonlinear, in order to investigate the robustness and flexibility of the approach.

The paper is organized as follows. In section 2, we present the network and the equations. We discuss how to determine a steady state solution and derive the linearized system and corresponding characteristic variables, on which controllers are built. We also formulate the main result, stating controllers and corresponding energy decay rates. In section 3, we demonstrate the approach by proving a corresponding result for a single reach, while the case of the network is proven in section 4. Numerical results obtained by a high order finite volume method (Leveque., 2002; Toro., 1999) are presented in section 5.

2. Governing equations and main result

The network can be given by Figure 1 or by any type of network where several reaches are interconnected in cascade (see (Bastin et al., 2009; De Halleux et al., 2003)). In Figure 1, M is considered as the junction node. The network model is given by the 1D St. Venant equations in each reach ($i = 1, 2$) and a flow conservation condition at M. The following variables are used: h_i is the height of the fluid column (m), v_i is the flow velocity (ms^{-1}), L_i is the length of the reach (m). The one dimensional St. Venant equations considered in the present paper are the following:

$$\begin{cases} \dfrac{\partial h_i}{\partial t} + \dfrac{\partial (v_i h_i)}{\partial x} = 0, & \text{in} \quad [0, L_i] \\[2ex] \dfrac{\partial v_i}{\partial t} + \dfrac{1}{2}\dfrac{\partial v_i^2}{\partial x} + g\dfrac{\partial h_i}{\partial x} = 0, & \text{in} \quad [0, L_i] \end{cases} \tag{1}$$

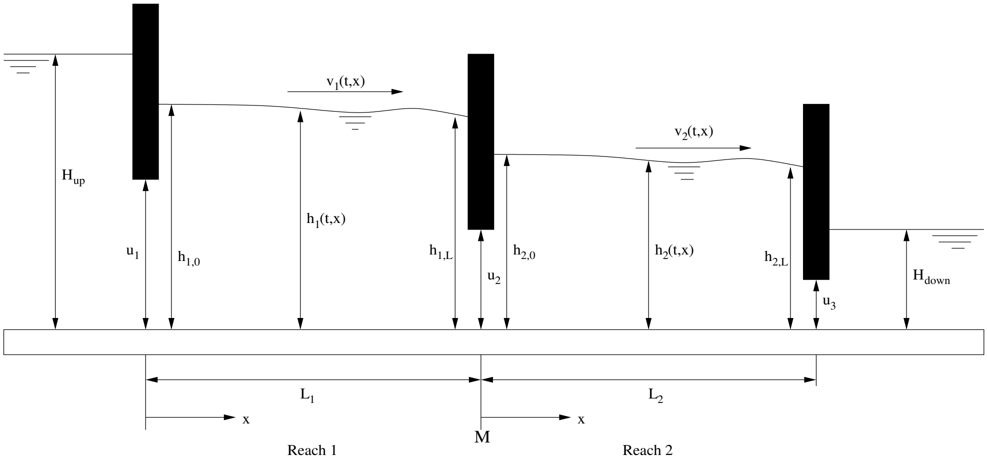


Fig. 1. The cascade network

together with a flow conservation condition at M,

$$h_1(t, L_1)v_1(t, L_1) = h_2(t, 0)v_2(t, 0), \tag{2}$$

initial conditions

$$h_i(0, x) = h_i^0(x), \quad v_i(0, x) = v_i^0(x), \tag{3}$$

and boundary conditions

$$v_1(t, 0) = v_{1,0}(t), \quad v_1(t, L_1) = v_{1,L_1}(t), \quad v_2(t, L_2) = v_{2,L_2}(t). \tag{4}$$

The results in the present paper concern a linearized system around a desired steady state. The controllers are built using that linear system and will be applied numerically to the above nonlinear model.

2.1 Steady state

The goal is to achieve a prescribed steady state $(\bar{h}_i, \bar{v}_i)$, with the help of the controllers, when time goes to infinity. From (1), the steady state solution $(\bar{h}_i, \bar{v}_i)$ satisfies:

$$\begin{cases} \dfrac{\partial \bar{v}_i}{\partial x} = 0, & \text{in} \quad [0, L_i], \\ \dfrac{\partial \bar{h}_i}{\partial x} = 0, & \text{in} \quad [0, L_i]. \\ \bar{h}_1(L_1)\bar{v}_1(L_1) = \bar{h}_2(0)\bar{v}_2(0) & \text{at} \quad M. \end{cases} \tag{5}$$

The steady state is such that

$$\bar{h}_2 < \bar{h}_1. \tag{6}$$

To determine the stady state (5), one gives $\bar{h}_1$, $\bar{v}_1$ and $\bar{h}_2$. On the other hand, using the flow direction (Figure 1) and the subcritical flow condition, one has

$$\bar{v}_i \geq 0 \quad \text{and} \quad \sqrt{g\bar{h}_i} > \bar{v}_i. \tag{7}$$

2.2 Linearized model

We introduce the residual state $(\check{h}_i, \check{v}_i)$ as the difference between the present state (h_i, v_i) and the steady state $(\bar{h}_i, \bar{v}_i)$: $\check{h}_i(t,x) = h_i(t,x) - \bar{h}_i(x)$, $\check{v}_i(t,x) = v_i(t,x) - \bar{v}_i(x)$. We use the assumptions $|\check{h}_i| \ll \bar{h}_i$ and $|\check{v}_i| \ll |\bar{v}_i|$ to linearize (1)-(4). Therefore, the solution $(\check{h}_i, \check{v}_i)$ satisfies

$$\begin{cases} (a)\ \dfrac{\partial \check{h}_i}{\partial t} + \bar{h}\dfrac{\partial \check{v}_i}{\partial x} + \bar{v}_i\dfrac{\partial \check{h}_i}{\partial x} = 0, \\ (b)\ \dfrac{\partial \check{v}_i}{\partial t} + \bar{v}_i\dfrac{\partial \check{v}_i}{\partial x} + g\dfrac{\partial \check{h}_i}{\partial x} = 0, \\ (c)\ \bar{v}_1\check{h}_1(t,L_1) + \bar{h}_1\check{v}_1(t,L_1) = \bar{v}_2\check{h}_2(t,0) + \bar{h}_2\check{v}_2(t,0) \quad \text{at} \quad M \end{cases} \tag{8}$$

together with the initial condition

$$\check{h}_i(0,x) = \check{h}_i^0(x), \quad \check{v}_i(0,x) = \check{v}_i^0(x), \tag{9}$$

and the boundary conditions as control laws

$$\check{v}_1(t,0) = \check{v}_{1,0}(t), \quad \check{v}_1(t,L_1) = \check{v}_{1,L_1}(t), \quad \check{v}_2(t,L_2) = \check{v}_{2,L_2}(t). \tag{10}$$

The functions $\check{v}_{1,0}(t)$, $\check{v}_{1,L_1}(t)$ and $\check{v}_{2,L_2}(t)$ are the feedback control laws to be prescribed in such a way to get an exponential convergence of $(\check{h}_i, \check{v}_i)$ to zero in time.

2.3 Eigenstructure and characteristic variables

The following characteristic variables are used to build the controllers:

$$\xi_{i1} = \check{v}_i - \check{h}_i\sqrt{\frac{g}{\bar{h}_i}} \qquad \text{and} \qquad \xi_{i2} = \check{v}_i + \check{h}_i\sqrt{\frac{g}{\bar{h}_i}}. \tag{11}$$

The characteristic velocities are

$$\lambda_{i1} = \bar{v}_i - \sqrt{g\bar{h}_i} \qquad \text{and} \qquad \lambda_{i2} = \bar{v}_i + \sqrt{g\bar{h}_i}.$$

The subcritical flow condition and the flow direction give

$$\lambda_{i1} < 0 < \lambda_{i2} \quad \text{and} \quad \lambda_{i1} + \lambda_{i2} \geq 0, \tag{12}$$

respectively. The characteristic variables satisfy

$$\frac{d\xi_{ij}}{dt} = \frac{\partial \xi_{ij}}{\partial t} + \lambda_{ij}\frac{\partial \xi_{ij}}{\partial x} = 0, \quad i,j = 1,2. \tag{13}$$

2.4 Main result

To build the feedback controllers, we express outgoing characteristic variables at the free endpoints and at the junction M in terms of initial data and the solution at the endpoints and at the junction M at earlier times. For reach 1, the outgoing characteristic variable at the endpoint $x = 0$ is ξ_{11}. For reach 2, the outgoing characteristic variable at the endpoint $x = L_2$ is ξ_{22}. Concerning the junction M, ξ_{12} and ξ_{21} are the outgoing characteristic variables. In section 4, we will see that

$$\begin{pmatrix} \xi_{11}(t,0) \\ \xi_{22}(t,L_2) \\ \xi_{12}(t,L_1) \\ \xi_{21}(t,0) \end{pmatrix} = \begin{pmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \\ b_4(t) \end{pmatrix}, \tag{14}$$

where $b_i, i = 1,2,3,4$ depend only on the initial condition and the solution at the endpoints and at the junction M at earlier times $\tau = t - \delta t$ with $\delta t \geq \min\left(\frac{L_1}{\lambda_{12}}, \frac{L_2}{\lambda_{22}}\right)$.

Let us consider $\theta_1 : \mathbb{R}^+ \longrightarrow]0,1]$ satisfying:

$$\theta_1(t) \geq \frac{2\bar{v}_1}{\lambda_{12}}. \tag{15}$$

and $\theta_2,\ \theta_3 : \mathbb{R}^+ \longrightarrow]0,1]$ two arbitrary functions. We choose the feedback controllers as follows:

$$\begin{aligned} \check{v}_{1,0}(t) &= -\frac{b_1(t)}{2}\left(\sqrt{1-\theta_1(t)}-1\right), \\ \check{v}_{2,L_2}(t) &= -\frac{b_2(t)}{2}\left(\sqrt{1-\theta_2(t)}-1\right), \\ \check{v}_{1,L_1}(t) &= \frac{\gamma(t)}{2\sigma}\left(\sqrt{1-\theta_3(t)}-1\right), \end{aligned} \tag{16}$$

where ,

$$\sigma = \bar{h}_1|\lambda_{11}|\left(1+\frac{|\lambda_{11}|}{\lambda_{22}}\right), \qquad \gamma(t) = \bar{h}_1|\lambda_{11}|\left(1-\frac{2\bar{v}_1}{\lambda_{22}}\right)b_3(t) + |\lambda_{11}|\sqrt{\bar{h}_1\bar{h}_2}\left(1-\frac{2\bar{v}_2}{\lambda_{22}}\right)b_4(t),$$

and $b_i,\ i = 1,2,3,4$, are given by (14). Therefore, defining

$$T = \max\left(\frac{L_1}{|\lambda_{11}|}, \frac{L_2}{|\lambda_{21}|}\right), \tag{17}$$

and the energy of the network by

$$E = \sum_{i=1}^{2} E_i, \quad E_i = \int_0^{L_i}\left(g\check{h}_i^2(t) + \bar{h}_i\check{v}_i^2(t)\right)dx, \tag{18}$$

we get the main result of this paper:

Theorem 1. *Let $t_k = kT$, $k \in \mathbb{N}$, where T is given by (17). Assume that the flow in the network is subcritical, the initial condition $(\check{h}_i^0, \check{q}_i^0)$ is continuous in $]0, L_i[$, $\check{v}_{1,0}, \check{v}_{1,L_1}, \check{v}_{2,L_2}$ satisfy (16), θ_1 satisfies (15) and $\lambda_{i1} + \lambda_{i2} \geq 0$. Then (8)-(10) has a unique solution $(\check{h}_i, \check{q}_i)$ continuous in $[t_k, t_{k+1}] \times]0, L_i[$ satisfying the following energy estimate:*

$$E(t_{k+1}) \leq (1 - \Theta^k) E(t_k), \tag{19}$$

where E is given by (18) and

$$\Theta^k = \min\left(\Gamma_1^k, \Gamma_2^k\right) \in [0, 1[,$$

$$\Gamma_1^k = \min\left(\inf_{x \in]0, L_1[}\left(\theta_1(t_k + \frac{x}{|\lambda_{11}|}) - \frac{2\bar{v}_1}{\lambda_{12}}\right), 4\bar{v}_1 \frac{(\bar{v}_2 - \bar{v}_1)}{\lambda_{22}\lambda_{12}}\right),$$

$$\Gamma_2^k = \min\left(\inf_{x \in]0, L_2[}\left(\frac{|\lambda_{21}|}{\lambda_{22}}\theta_2(t_k + \frac{L_2 - x}{\lambda_{22}}) + \frac{2\bar{v}_2}{\lambda_{22}}\right), 2\frac{(\bar{v}_2 - \bar{v}_1)}{\lambda_{22}}\left(1 - \frac{2\bar{v}_2}{\lambda_{22}}\right)\right).$$

Remark 1.

1. *In addition to (19), within the interval $]t_k, t_{k+1}[$, the energy is non-increasing.*
2. *The controllers (16) tend to zero when time goes to infinity. This is due to (19) and the fact that they are built on the solution at earlier times.*
3. *Estimation (19) can be written as*

$$E(t_k) \leq E(0) \exp\left(-\mu^k \, t_k\right).$$

where $\mu^k = \frac{1}{k}\sum_{j=0}^{k-1} \nu^j$ and $\nu^j = -\ln\left((1 - \Theta^j)^{\frac{1}{t_1}}\right)$. Thus, the functions θ can be viewed as stabilization rate for the exponential decrease.

3. Building the controller for a single reach

We construct a stabilization process for a single canal, which should drive the perturbations $\check{h}$ and $\check{v}$ to zero exponentially in time. We consider the 1D Saint-Venant equations (1) without the index i standing for the reach number:

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial (vh)}{\partial x} = 0, \\[2ex] \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\dfrac{\partial v^2}{\partial x} + g\dfrac{\partial h}{\partial x} = 0, \end{cases} \tag{20}$$

together with initial conditions

$$h(0, x) = h^0(x), \quad v(0, x) = v^0(x) \tag{21}$$

and boundary conditions

$$v(t, 0) = v_0(t), \quad v(t, L) = v_L(t). \tag{22}$$

The steady state solution $(\bar{h}, \bar{v})$ satisfies:

$$\frac{\partial \bar{v}}{\partial x} = 0, \quad \frac{\partial \bar{h}}{\partial x} = 0, \quad \text{in} \quad [0, L].$$

with

$$\bar{v} \geq 0 \quad \text{and} \quad \sqrt{g\bar{h}} > \bar{v}. \tag{23}$$

The linearized model is

$$\begin{cases} (a) \ \dfrac{\partial \check{h}}{\partial t} + \bar{h}\dfrac{\partial \check{v}}{\partial x} + \bar{v}\dfrac{\partial \check{h}}{\partial x} = 0, \\ (b) \ \dfrac{\partial \check{v}}{\partial t} + \bar{v}\dfrac{\partial \check{v}}{\partial x} + g\dfrac{\partial \check{h}}{\partial x} = 0, \end{cases} \tag{24}$$

together with initial conditions

$$\check{h}(0, x) = \check{h}^0(x), \quad \check{v}(0, x) = \check{v}^0(x), \tag{25}$$

and the boundary conditions

$$\check{v}(t, 0) = \check{v}_0(t), \quad \check{v}(t, L) = \check{v}_L(t). \tag{26}$$

The functions $\check{v}_L(t)$ and $\check{v}_0(t)$ are the feedback control laws to be prescribed in such a way to get an exponential convergence of $(\check{h}, \check{v})$ to zero in time.

The characteristic variables are:

$$\xi_1 = \check{v} - \check{h}\sqrt{\frac{g}{\bar{h}}} \qquad \text{and} \qquad \xi_2 = \check{v} + \check{h}\sqrt{\frac{g}{\bar{h}}}, \tag{27}$$

with the characteristic velocities

$$\lambda_1 = \bar{v} - \sqrt{g\bar{h}} \qquad \text{and} \qquad \lambda_2 = \bar{v} + \sqrt{g\bar{h}}.$$

The subcritical flow condition and the flow direction give

$$\lambda_1 < 0 < \lambda_2 \quad \text{and} \quad \lambda_1 + \lambda_2 \geq 0, \tag{28}$$

respectively. Considering the characteristic variables (27), system (24) is written as two independant equations:

$$\frac{d\xi_j}{dt} = \frac{\partial \xi_j}{\partial t} + \lambda_j \frac{\partial \xi_j}{\partial x} = 0, \quad j = 1, 2. \tag{29}$$

3.1 A priori energy estimation

Let E be the energy of (24) defined as

$$E(t) = \int_0^L \left(g\check{h}^2(t) + \bar{h}\check{v}^2(t) \right) dx. \tag{30}$$

We consider the following system as a weak formulation of (24)

$$\begin{cases} \forall(\psi,\phi) \in H^1(]0,L[), \\ \int_0^L g\psi \frac{\partial \check{h}}{\partial t}\,dx - g\bar{h}\int_0^L \check{v}\frac{\partial(\psi)}{\partial x}\,dx - g\bar{v}\int_0^L \check{h}\frac{\partial(\psi)}{\partial x}\,dx + \\ \qquad g\bar{h}\psi(L)\check{v}_L(t) - g\bar{h}\psi(0)\check{v}_0(t) + g\bar{v}\psi(L)\check{h}_L(t) - g\bar{v}\psi(0)\check{h}_0(t) = 0, \\ \int_0^L \bar{h}\phi \frac{\partial \check{v}}{\partial t}\,dx - \bar{h}\bar{v}\int_0^L \check{v}\frac{\partial(\phi)}{\partial x}\,dx - g\bar{h}\int_0^L \check{h}\frac{\partial(\phi)}{\partial x}\,dx \\ \qquad +\bar{h}\bar{v}\phi(L)\check{v}_L(t) - \bar{h}\bar{v}\phi(0)\check{v}_0(t) + g\bar{h}\phi(L)\check{h}_L(t) - g\bar{h}\phi(0)\check{h}_0(t) = 0, \end{cases} \tag{31}$$

together with boundary and initial conditions.

We estimate the variation of the energy E on the canal in order to define the controllers $\check{v}_L(t)$ on $\{x = L\}$ and $\check{v}_0(t)$ on $\{x = 0\}$. To this end, we let $(\psi,\phi) = (\check{h},\check{v})$ in (31) to get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}E(t) = &-\frac{\bar{h}\bar{v}}{2}\check{v}_L^2(t) - \frac{g\bar{v}}{2}\check{h}^2(t,L) - g\bar{h}\check{h}(t,L)\check{v}_L(t) \\ &+\frac{\bar{h}\bar{v}}{2}\check{v}_0^2(t) + \frac{g\bar{v}}{2}\check{h}^2(t,0) + g\bar{h}\check{h}(t,0)\check{v}_0(t). \end{aligned} \tag{32}$$

The difference among control methods depends on how the energy is defined and its variation handled to obtain a convergence of the perturbations $\check{h}$ and $\check{v}$ to zero in time (see (Bastin et al., 2009; De Halleux et al., 2003)).

3.2 Controllers and the stabilization process

The feedback control building relies on the fact that we can express the height at the boundaries in terms of the flow velocity and outgoing characteristic variables. Using (29)

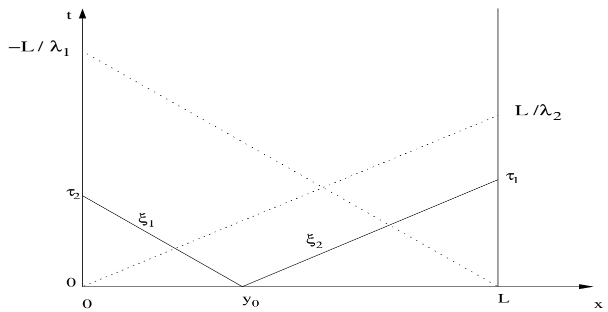


Fig. 2. Characteristic variables.

and refering on the characteristic variables indicated in figure 2, one has:

$$\begin{pmatrix} \xi_1(\tau_2,0) \\ \xi_2(\tau_1,L) \end{pmatrix} = \begin{pmatrix} b_1(\tau_2) \\ b_2(\tau_1) \end{pmatrix}, \tag{33}$$

where

$$b_1(\tau_2) = \begin{cases} \xi_1(0, |\lambda_1|\tau_2), & \tau_2 \le \frac{L}{|\lambda_1|}, \\ \xi_1(\tau_2 - \frac{L}{|\lambda_1|}, L), & \tau_2 \ge \frac{L_1}{|\lambda_1|}, \end{cases} \quad b_2(\tau_1) = \begin{cases} \xi_2(0, L - \lambda_2\tau_1), & \tau_1 \le \frac{L}{\lambda_2}, \\ \xi_2(\tau_1 - \frac{L}{\lambda_2}, 0), & \tau_1 \ge \frac{L}{\lambda_2}. \end{cases} \tag{34}$$

From (27), one derives

$$\check{h}(\tau_1, L) = \Big(\xi_2(\tau_1, L) - v_L(\tau_1)\Big)\sqrt{\frac{\bar{h}}{g}}, \tag{35}$$

$$\check{h}(\tau_2, 0) = \Big(-\xi_1(\tau_2, 0) + v_0(\tau_2)\Big)\sqrt{\frac{\bar{h}}{g}}. \tag{36}$$

Considering the energy equation (32), one deduces from (34)-(36) that

$$\frac{1}{2}\frac{dE}{dt}(t) = a_1\check{v}_0^2(t) - a_1 b_1(t)\check{v}_0(t) + c_1(t) + a_2\check{v}_L^2(t) - a_2 b_2(t)\check{v}_L(t) + c_2(t) \tag{37}$$

where

$$a_1 = \bar{h}\lambda_2, \quad a_2 = \bar{h}|\lambda_1|, \quad c_1(t) = \frac{\bar{h}\bar{v}}{2}b_1^2(t), \quad c_2(t) = -\frac{\bar{h}\bar{v}}{2}b_2^2(t), \tag{38}$$

$b_1(t)$ and $b_2(t)$ are given by (34).

The RHS of (37) is treated in such a way to get an exponential decrease of the energy. For this propose, the following observation for second order polynomials is used.

Lemma 1. *Consider a second order polynomial $P(q) = av^2 + bv$, where $a > 0$. For any $\theta \in [0,1]$*

$$P\left(\frac{b}{2a}(\sqrt{1-\theta} - 1)\right) = -\frac{b^2}{4a}\theta. \tag{39}$$

If the flow velocity at the boundary is prescribed as follows:

$$\check{v}_L(t) = -\frac{b_2(t)}{2}\left(\sqrt{1-\theta_2(t)} - 1\right) \quad \text{and} \quad \check{v}_0(t) = -\frac{b_1(t)}{2}\left(\sqrt{1-\theta_1(t)} - 1\right), \tag{40}$$

where $\theta_1, \theta_2 : \mathbb{R}^+ \longrightarrow [0,1]$, then by Lemma 1, (37) becomes

$$\begin{aligned} \frac{1}{2}\frac{dE}{dt}(t) &= -\frac{b_1^2(t)}{4a_1}\theta_1(t) + c_1 - \frac{b_2^2}{4a_2}\theta_2(t) + c_2, \\ &= -\frac{\bar{h}}{4}\big(\lambda_2\theta_1(t) - 2\bar{v}\big)b_1^2(t) - \frac{\bar{h}}{4}\big(|\lambda_1|\theta_2(t) + 2\bar{v}\big)b_2^2(t). \end{aligned} \tag{41}$$

In order to get an energy decrease, we chosse θ_1 such that the RHS of (41) is non-positive. In fact we choose θ_1 as follows:

$$\theta_1(t) \ge \frac{2\bar{v}}{\lambda_2}. \tag{42}$$

Note that this choice of θ_1 is always possible since $\frac{2\bar{v}}{\lambda_2} < 1$. Indeed $\frac{2\bar{v}}{\lambda_2} < 1$, because the subcritical flow condition (23) gives $\lambda_2 = \sqrt{g\bar{h}} + \bar{v} > 2\bar{v}$. Thus, we get the following result

Theorem 2. *Let $t_k = kL/|\lambda_1|$, $k \in \mathbb{N}$. Assume that (28) holds, the initial condition $(\check{h}^0, \check{v}^0)$ is continuous in $]0, L[$, $(\check{v}_0, \check{v}_L)$ satisfies (40) and θ_1 satisfies (42). Then (24)-(26) has a unique solution $(\check{h}, \check{v})$ continuous in $[t_k, t_{k+1}]\times]0, L[$ satisfying the following energy estimate:*

$$E(t_{k+1}) \le (1 - \Theta^k)E(t_k), \tag{43}$$

where E is given by (30) and

$$\Theta^k = \min\left(\inf_{x\in]0,L[}\left(\frac{|\lambda_1|}{\lambda_2}\theta_2(t_k + \frac{L-x}{\lambda_2}) + \frac{2\bar{v}}{\lambda_2}\right), \inf_{x\in]0,L[}\left(\theta_1\big(t_k + \frac{x}{|\lambda_1|}\big) - \frac{2\bar{v}}{\lambda_2}\right)\right) \in [0, 1[.$$

Proof: The existence and uniqueness of the solution follow by (27) and construction (33).

Integrating (41) from 0 to t_1, we have

$$\begin{aligned}
E(L/|\lambda_1|) &= E(0) - \frac{\bar{h}}{2}\int_0^{L/|\lambda_1|}(\lambda_2\theta_1(t) - 2\bar{v})b_1^2(t)\,dt - \frac{\bar{h}}{2}\int_0^{L/|\lambda_1|}(|\lambda_1|\theta_2(t) + 2\bar{v})b_2^2(t)\,dt,\\
&\le E(0) - \frac{\bar{h}}{2}\int_0^{L/|\lambda_1|}(\lambda_2\theta_1(t) - 2\bar{v})\xi_1^2(0, |\lambda_1|t)\,dt\\
&\qquad -\frac{\bar{h}}{2}\int_0^{L/\lambda_2}(|\lambda_1|\theta_2(t) + 2\bar{v})\xi_2^2(0, L-\lambda_2 t)\,dt,\\
&\le E(0) - \frac{\bar{h}}{2|\lambda_1|}\int_0^L\left(\lambda_2\theta_1(\frac{x}{|\lambda_1|}) - 2\bar{v}\right)\xi_1^2(0,x)\,dt\\
&\qquad -\frac{\bar{h}}{2\lambda_2}\int_0^L\left(|\lambda_1|\theta_2(\frac{L-x}{\lambda_2}) + 2\bar{v}\right)\xi_2^2(0,x)\,dt,\\
&\le E(0) - \frac{\bar{h}}{2\lambda_2}\int_0^L\left(\lambda_2\theta_1(\frac{x}{|\lambda_1|}) - 2\bar{v}\right)\xi_1^2(0,x)\,dt\\
&\qquad -\frac{\bar{h}}{2\lambda_2}\int_0^L\left(|\lambda_1|\theta_2(\frac{L-x}{\lambda_2}) + 2\bar{v}\right)\xi_2^2(0,x)\,dt,\\
&\le E(0) - \frac{\bar{h}}{2}\int_0^L\left(\theta_1(\frac{x}{|\lambda_1|}) - \frac{2\bar{v}}{\lambda_2}\right)\xi_1^2(0,x)\,dt\\
&\qquad -\frac{\bar{h}}{2}\int_0^L\left(\frac{|\lambda_1|}{\lambda_2}\theta_2(\frac{L-x}{\lambda_2}) + \frac{2\bar{v}}{\lambda_2}\right)\xi_2^2(0,x)\,dt,\\
&\le E(0) - \frac{\bar{h}}{2}\int_0^L\left[\xi_2^2(0,x) + \xi_1^2(0,x)\right]\Theta^0 dx,
\end{aligned} \tag{44}$$

where

$$\Theta^0 = \min\left(\inf_{x\in]0,L[}\left(\frac{|\lambda_1|}{\lambda_2}\theta_2(\frac{L-x}{\lambda_2}) + \frac{2\bar{v}}{\lambda_2}\right), \inf_{x\in]0,L[}\left(\theta_1(\frac{x}{|\lambda_1|}) - \frac{2\bar{v}}{\lambda_2}\right)\right).$$

We have $\Theta^0 \in [0, 1[$, since we get $0 < \theta_1(\frac{x}{|\lambda_1|}) - \frac{2\bar{v}}{\lambda_2} < 1$ from (42) and the fact that $\frac{2\bar{v}}{\lambda_2} < 1$.

On the other hand, one has the following estimation

$$\xi_1^2(0,x)+\xi_2^2(0,x) = \left(\check{v}^0(x)-\check{h}^0(x)\sqrt{\frac{g}{\bar{h}}}\right)^2+\left(\check{v}^0(x)+\check{h}^0(x)\sqrt{\frac{g}{\bar{h}}}\right)^2,$$
$$= 2(\check{v}^0(x))^2+\frac{2g}{\bar{h}}(\check{h}^0(x))^2, = \frac{2}{\bar{h}}\left(\bar{h}(\check{v}^0(x))^2+g(\check{h}^0(x))^2\right). \quad (45)$$

Therefore we deduce from (44)-(45) that

$$E(L/|\lambda_1|) \leq E(0) - \Theta^0\int_0^L \left(\bar{h}(\check{v}^0(x))^2+g(\check{h}^0(x))^2\right)dx \leq (1-\Theta^0)E(0). \quad (46)$$

In order to generalize (46) with respect to time, we consider the time $t_k = kL/|\lambda_1|$ as initial condition. Then, we let

$$b_1(t) = \xi_1(t_k, |\lambda_1|(t-t_k)), \quad if \quad t \in]t_k, t_k+L/|\lambda_1|[,$$
$$b_2(t) = \xi_2(t_k, L-\lambda_2(t-t_k)), \quad if \quad t \in]t_k, t_k+L/\lambda_2[,$$

and

$$\Theta^k = \min\left(\inf_{x\in]0,L[}\left(\frac{|\lambda_1|}{\lambda_2}\theta_2(t_k+\frac{L-x}{\lambda_2})+\frac{2\bar{v}}{\lambda_2}\right), \inf_{x\in]0,L[}\left(\theta_1(t_k+\frac{x}{|\lambda_1|})-\frac{2\bar{v}}{\lambda_2}\right)\right) \in [0,1[.$$

And, by integrating from t_k to t_{k+1} and using the same arguments as for the interval $[0, t_1]$, the proof of Theorem 2 is finished. □

Remark 2.

1. *Using the weak formulation (31) and the fact that $C^0(]0,L[)$ is dense in $L^2(]0,L[)$, it is possible (using the arguments of (Goudiaby et al., -)) to prove that for initial data $(\check{h}^0,\check{v}^0)$ in $(L^2(]0,L[))^2$, the solution $(\check{h},\check{v})$ of (24)-(26) satisfies (43) and the following regularity*

$$\begin{pmatrix}\check{h}\\ \bar{h}\check{v}+\bar{v}\check{h}\end{pmatrix}, \quad \begin{pmatrix}\check{v}\\ \bar{v}\check{v}+g\check{h}\end{pmatrix} \in H(div,Q), \quad (47)$$

where $Q =]t_k, t_{k+1}[\times]0,L[$,

$$div \equiv \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) \quad and \quad H(div,Q) = \left\{\mathcal{V} \in L^2(Q)^2 ; div\,\mathcal{V} \in L^2(Q)\right\}.$$

2. *It is also possible to stabilize the reach by acting only on one free endpoint as in (Goudiaby et al., -).*
3. *Only the initial condition and the on-line measurements of the water levels at the endpoints are required to implement the feedback control law (40).*
4. *For an application need, in order to implement the controllers (40), one can use two underflow gates located at the left end $(x=0)$ and the right end $(x=L)$ of the canal (see Fig 3). Denote by I_0 and I_L the gates opening. A relation between under flow gates opening and discharge is given as follows (see (De Halleux et al., 2003; Ndiaye & Bastin., 2004)).*

$$lh(t,0)v(t,0) = I_0(t)k_1\sqrt{2g(H_{up}-h(t,0))}, \quad (48)$$

$$lh(t,L)v(t,L) = I_L(t)k_2\sqrt{2g(h(t,L)-H_{down})}, \quad (49)$$

where, l is the width of the reach (m), k_1, k_2 are gate coefficients, $v(t,x) = \bar{v}(x) + \check{v}(t,x)$, $h(t,x) = \bar{h}(x) + \check{h}(t,x)$, H_{up} and H_{down} are the left and right water levels outside the canal, respectively. H_{up} and H_{down} are supposed to be constant and satisfy $H_{up} > h(t,0)$ and $h(t,L) > H_{down}$.

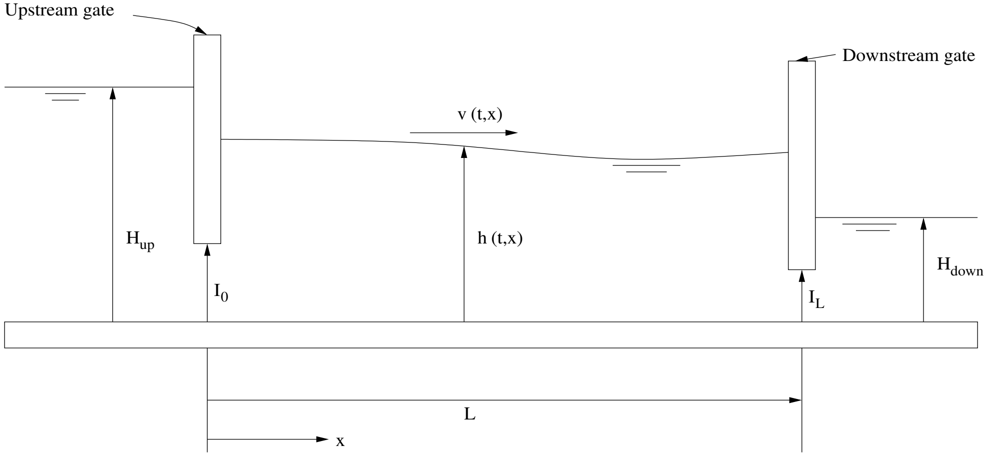


Fig. 3. A canal delimited by underflow gates

4. Building the controller for the cascade network

In this section, we use the idea of section 3.2, to build feedback control laws for the network.

4.1 Energy estimation and controllers building

Consider the energy of the network given by

$$E = \sum_{i=1}^{2} E_i, \quad E_i = \int_0^{L_i} \left(g\check{h}_i^2(t) + \bar{h}_i \check{v}_i^2(t) \right) dx. \tag{50}$$

Arguing as in section 3.1, from the weak formulation of (8), we deduce

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}E(t) = & -\frac{\bar{h}_1\bar{v}_1}{2}\check{v}_{1,L_1}^2(t) - \frac{g\bar{v}_1}{2}\check{h}_1^2(t,L_1) - g\bar{h}_1\check{h}_1(t,L_1)\check{v}_{1,L_1}(t) \\ & + \frac{\bar{h}_1\bar{v}_1}{2}\check{v}_{1,0}^2(t) + \frac{g\bar{v}_1}{2}\check{h}_1^2(t,0) + g\bar{h}_1\check{h}_1(t,0)\check{v}_{1,0}(t) \\ & -\frac{\bar{h}_2\bar{v}_2}{2}\check{v}_{2,L_2}^2(t) - \frac{g\bar{v}_2}{2}\check{h}_2^2(t,L_2) - g\bar{h}_2\check{h}_2(t,L_2)\check{v}_{2,L_2}(t) \\ & + \frac{\bar{h}_2\bar{v}_2}{2}\check{v}_{2,0}^2(t) + \frac{g\bar{v}_2}{2}\check{h}_2^2(t,0) + g\bar{h}_2\check{h}_2(t,0)\check{v}_{2,0}(t). \end{aligned} \tag{51}$$

Using (13) and refering to figure 4, we express outgoing characteristic variables in terms of initial data and the solution at the endpoints and at the junction M at earlier times, i.e (14) is

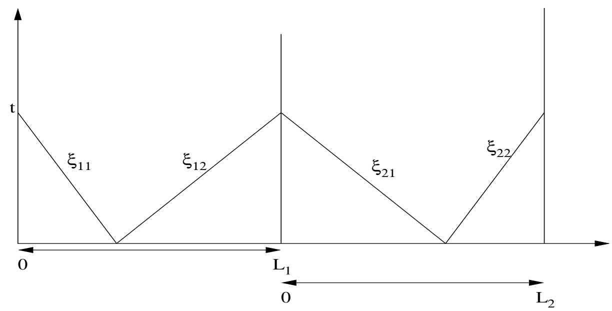


Fig. 4. Characteristic variables for the cascade network

satisfies with,

$$b_1(t) = \begin{cases} \xi_{11}(0, |\lambda_{11}|t), & t \le \frac{L_1}{|\lambda_{11}|}, \\ \xi_{11}(t - \frac{L_1}{|\lambda_{11}|}, L_1), & t \ge \frac{L_1}{|\lambda_{11}|}. \end{cases} \quad b_2(t) = \begin{cases} \xi_{22}(0, L_2 - \lambda_{22}t), & t \le \frac{L_2}{\lambda_{22}}, \\ \xi_{22}(t - \frac{L_2}{\lambda_{22}}, 0), & t \ge \frac{L_2}{\lambda_{12}}. \end{cases}$$
$$b_3(t) = \begin{cases} \xi_{12}(0, L_1 - \lambda_{12}t), & t \le \frac{L_1}{\lambda_{12}}, \\ \xi_{12}(t - \frac{L_1}{\lambda_{12}}, 0), & t \ge \frac{L_1}{\lambda_{12}}. \end{cases} \quad b_4(t) = \begin{cases} \xi_{21}(0, |\lambda_{21}|t), & t \le \frac{L_2}{|\lambda_{21}|}, \\ \xi_{21}(t - \frac{L_2}{|\lambda_{21}|}, L_2), & t \ge \frac{L_2}{|\lambda_{21}|}. \end{cases} \tag{52}$$

On the other hand, using (11) we also express the height at the boundaries and at the junction in terms of the flow velocity and outgoing characteristic variables:

$$\begin{aligned} \check{h}_i(t, L_i) &= \Big(\xi_{i2}(t, L_i) - v_{i,L_i}(t)\Big)\sqrt{\frac{\bar{h}_i}{g}}, \\ \check{h}_i(t, 0) &= \Big(-\xi_{i1}(t, 0) + v_{i,0}(t)\Big)\sqrt{\frac{\bar{h}_i}{g}}. \end{aligned} \tag{53}$$

Plugging (53) into (51), one gets

$$\begin{aligned} \frac{1}{2}\frac{dE}{dt}(t) =& a_1\check{v}_{1,0}^2(t) - a_1 b_1(t)\check{v}_{1,0}(t) + c_1(t) + a_2\check{v}_{2,L_2}^2(t) - a_2 b_2(t)\check{v}_{2,L_2}(t) + c_2(t) \\ &+ a_3\check{v}_{1,L_1}^2(t) - a_3 b_3(t)\check{v}_{1,L_1}(t) + c_3(t) + a_4\check{v}_{2,0}^2(t) - a_4 b_4(t)\check{v}_{2,0}(t) + c_4(t) \end{aligned} \tag{54}$$

where

$$\begin{aligned} &a_1 = \bar{h}_1\lambda_{12}, \quad a_2 = \bar{h}_2|\lambda_{21}|, \quad a_3 = \bar{h}_1|\lambda_{11}|, \quad a_4 = \bar{h}_2\lambda_{22}, \\ &c_1(t) = \frac{\bar{h}_1\bar{v}_1}{2}b_1^2(t),\ c_2(t) = -\frac{\bar{h}_2\bar{v}_2}{2}b_2^2(t),\ c_3(t) = -\frac{\bar{h}_1\bar{v}_1}{2}b_3^2(t),\ c_4(t) = \frac{\bar{h}_2\bar{v}_2}{2}b_4^2(t), \end{aligned} \tag{55}$$

$b_i, i = 1, 2, 3, 4$ are given by (52).

The flow conservation condition (8.c) is used to express $\check{v}_{2,0}$ in terms of $\check{v}_{1,L_1}$ and outgoing characteristic variables. From (8.c) and (53), one has

$$\check{v}_{2,0}(t) = \alpha \check{v}_{1,L_1}(t) + \beta b_3(t) + \delta b_4(t). \tag{56}$$

where

$$\alpha = \frac{|\lambda_{11}|}{\lambda_{22}}\sqrt{\frac{\bar{h}_1}{\bar{h}_2}}, \qquad \beta = \frac{\bar{v}_1}{\lambda_{22}}\sqrt{\frac{\bar{h}_1}{\bar{h}_2}}, \qquad \delta = \frac{\bar{v}_2}{\lambda_{22}}. \tag{57}$$

Thus, the last six terms of (54) can be expressed as follows:

$$\begin{aligned} a_3\check{v}^2_{1,L_1}(t) - a_3 b_3(t)\check{v}_{1,L_1}(t) + c_3(t) + a_4\check{v}^2_{2,0}(t) - a_4 b_4(t)\check{v}_{2,0}(t) + c_4(t) \\ = \sigma\check{v}^2_{1,L_1}(t) + \gamma(t)\check{v}_{1,L_1}(t) + \rho(t), \end{aligned} \tag{58}$$

where

$$\sigma = (a_3 + \alpha^2 a_4) = \bar{h}_1|\lambda_{11}|\left(1 + \frac{|\lambda_{11}|}{\lambda_{22}}\right) \tag{59}$$

$$\begin{aligned} \gamma(t) &= (2a_4\alpha\beta - \bar{h}_1|\lambda_{11}|)b_3(t) + \alpha(2a_4\delta - \bar{h}_2\lambda_{22})b_4(t) \\ &= \bar{h}_1|\lambda_{11}|\left(\frac{2\bar{v}_1}{\lambda_{22}} - 1\right)b_3(t) + |\lambda_{11}|\sqrt{\bar{h}_1\bar{h}_2}\left(\frac{2\bar{v}_2}{\lambda_{22}} - 1\right)b_4(t) \end{aligned} \tag{60}$$

$$\begin{aligned} \rho(t) &= \left(a_4\delta^2 - \bar{h}_2\lambda_{22}\delta + \frac{\bar{h}_2\bar{v}_2}{2}\right)b_4^2(t) + \left(a_4\beta^2 - \frac{\bar{h}_1\bar{v}_1}{2}\right)b_3^2(t) \\ &\quad + \beta(2a_4 - \bar{h}_2\lambda_{22})b_3(t)b_4(t) \\ &= \frac{\bar{h}_2\bar{v}_2}{2}\left(\frac{2\bar{v}_2}{\lambda_{22}} - 1\right)b_4^2(t) + \frac{\bar{h}_1\bar{v}_1}{2}\left(\frac{2\bar{v}_1}{\lambda_{22}} - 1\right)b_3^2(t) \\ &\quad + \bar{v}_1\sqrt{\bar{h}_1\bar{h}_2}\left(\frac{2\bar{v}_1}{\lambda_{22}} - 1\right)b_3(t)b_4(t). \end{aligned} \tag{61}$$

Taking into account (58), the energy law (54) becomes

$$\begin{aligned} \frac{1}{2}\frac{dE}{dt}(t) = {} & a_1\check{v}^2_{1,0}(t) - a_1 b_1(t)\check{v}_{1,0}(t) + c_1(t) + a_2\check{v}^2_{2,L_2}(t) - a_2 b_2(t)\check{v}_{2,L_2}(t) + c_2(t) \\ & + \sigma\check{v}^2_{1,L_1}(t) + \gamma(t)\check{v}_{1,L_1}(t) + \rho(t). \end{aligned} \tag{62}$$

If we prescribe the velocity at the boundaries as follows,

$$\begin{aligned} \check{v}_{1,0}(t) &= -\frac{b_1(t)}{2}\left(\sqrt{1 - \theta_1(t)} - 1\right), \\ \check{v}_{2,L_2}(t) &= -\frac{b_2(t)}{2}\left(\sqrt{1 - \theta_2(t)} - 1\right), \\ \check{v}_{1,L_1}(t) &= \frac{\gamma(t)}{2\sigma}\left(\sqrt{1 - \theta_3(t)} - 1\right), \end{aligned} \tag{63}$$

where $\theta_1,\ \theta_2,\ \theta_3:\ \mathbb{R}^+ \longrightarrow [0,1]$, it follows from Lemma 1 that

$$\frac{1}{2}\frac{dE}{dt}(t) = -\frac{b_1^2(t)}{4a_1}\theta_1(t) + c_1(t) - \frac{b_2^2(t)}{4a_2}\theta_2(t) + c_2(t) - \frac{\gamma^2(t)}{4\sigma}\theta_3(t) + \rho(t). \tag{64}$$

Let us calculate explicitely the RHS of (64). On the one hand, using (a_1, c_1) and (a_2, c_2) given in (55), we have

$$-\frac{b_1^2}{4a_1}\theta_1 + c_1 = -\frac{\bar{h}_1}{4}(\lambda_{12}\theta_1 - 2\bar{v}_1)b_1^2(t), \tag{65}$$

and

$$-\frac{b_2^2}{4a_2}\theta_2 + c_2 = -\frac{\bar{h}_2}{4}(|\lambda_{21}|\theta_2 + 2\bar{v}_2)b_2^2(t). \tag{66}$$

On the other hand, from (59)-(61) and using the fact that $\theta_3 \in]0,1]$ we have

$$\begin{aligned} -\frac{\gamma^2}{4\sigma}\theta_3 + \rho \le \rho = {} & \frac{\bar{h}_2\bar{v}_2}{2}\left(\frac{2\bar{v}_2}{\lambda_{22}} - 1\right)b_4^2(t) + \frac{\bar{h}_1\bar{v}_1}{2}\left(\frac{2\bar{v}_1}{\lambda_{22}} - 1\right)b_3^2(t) \\ & + \bar{v}_1\sqrt{\bar{h}_1\bar{h}_2}\left(\frac{2\bar{v}_1}{\lambda_{22}} - 1\right)b_3(t)b_4(t). \end{aligned} \tag{67}$$

Since $\frac{2\bar{v}_2}{\lambda_{22}} < 1$, we get

$$\bar{v}_1\sqrt{\bar{h}_1\bar{h}_2}\left(\frac{2\bar{v}_2}{\lambda_{22}} - 1\right)b_3(t)b_4(t) \le \frac{\bar{h}_1\bar{v}_1}{2}\left(1 - \frac{2\bar{v}_2}{\lambda_{22}}\right)b_3^2(t) + \frac{\bar{h}_2\bar{v}_1}{2}\left(1 - \frac{2\bar{v}_2}{\lambda_{22}}\right)b_4^2(t). \tag{68}$$

Combining (68) and (67), one has

$$-\frac{\gamma^2}{4\sigma}\theta_3 + \rho \le \bar{h}_1\bar{v}_1\frac{(\bar{v}_1 - \bar{v}_2)}{\lambda_{22}}b_3^2(t) + \frac{\bar{h}_2}{2}\left(\frac{2\bar{v}_2}{\lambda_{22}}(\bar{v}_2 - \bar{v}_1) - (\bar{v}_2 - \bar{v}_1)\right)b_4^2(t). \tag{69}$$

Using (65), (66) and (69), the energy law (64) becomes

$$\begin{aligned} \frac{1}{2}\frac{dE}{dt}(t) \le {} & -\frac{\bar{h}_1}{4}\left((\lambda_{12}\theta_1 - 2\bar{v}_1)b_1^2(t) + 4\bar{v}_1\frac{(\bar{v}_2 - \bar{v}_1)}{\lambda_{22}}b_3^2(t)\right) \\ & -\frac{\bar{h}_2}{4}\left(2(\bar{v}_2 - \bar{v}_1)\left(1 - \frac{2\bar{v}_2}{\lambda_{22}}\right)b_4^2(t) + (|\lambda_{21}|\theta_2 + 2\bar{v}_2)b_2^2(t)\right). \end{aligned} \tag{70}$$

The way the steady state $(\bar{h}_1, \bar{v}_1, \bar{h}_2, \bar{v}_2)$ is chosen (see (6)), yields that

$$\bar{v}_2 \ge \bar{v}_1. \tag{71}$$

The function θ_1 satisfies a condition similar to (42), i.e

$$\theta_1(t) \ge \frac{2\bar{v}_1}{\lambda_{12}}. \tag{72}$$

Using the fact that $\frac{2\bar{v}_2}{\lambda_{22}} < 1$, (71) and (72), the RHS of (70) is non-positive. Thus we give the proof of Theorem 1.

4.2 Proof of theoreme 1

The existence and uniqueness of the solution follow by (11) and constructions (14).

Integrating (70) from 0 to t_1, one has

$$
\begin{aligned}
E(t_1) \;\leq\; & E(0) - \frac{\bar{h}_1}{2}\int_0^{t_1}\left(\lambda_{12}\theta_1(t) - 2\bar{v}_1\right)b_1^2(t)dt - \frac{\bar{h}_1}{2}\int_0^{t_1} 4\bar{v}_1\frac{(\bar{v}_2-\bar{v}_1)}{\lambda_{22}}b_3^2(t)dt \\
& - \frac{\bar{h}_2}{2}\int_0^{t_1} 2(\bar{v}_2-\bar{v}_1)\left(1-\frac{2\bar{v}_2}{\lambda_{22}}\right)b_4^2(t)dt - \frac{\bar{h}_2}{2}\int_0^{t_1}\left(|\lambda_{21}|\theta_2 + 2\bar{v}_2\right)b_2^2(t)dt. \\
\overset{(52)}{\leq}\; & E(0) - \frac{\bar{h}_1}{2}\int_0^{\frac{L_1}{|\lambda_{11}|}}\left(\lambda_{12}\theta_1(t) - 2\bar{v}_1\right)\xi_{11}^2(0,|\lambda_{11}|t)dt \\
& - \frac{\bar{h}_1}{2}\int_0^{\frac{L_1}{\lambda_{12}}} 4\bar{v}_1\frac{(\bar{v}_2-\bar{v}_1)}{\lambda_{22}}\xi_{12}^2(t, L_1-\lambda_{12}t)dt \\
& - \frac{\bar{h}_2}{2}\int_0^{\frac{L_1}{|\lambda_{21}|}} 2(\bar{v}_2-\bar{v}_1)\left(1-\frac{2\bar{v}_2}{\lambda_{22}}\right)\xi_{21}^2(t,|\lambda_{21}|t)dt \\
& - \frac{\bar{h}_2}{2}\int_0^{\frac{L_2}{\lambda_{22}}}\left(|\lambda_{21}|\theta_2(t) + 2\bar{v}_2\right)\xi_{22}^2(0, L_2-\lambda_{22}t)dt, \\
\leq\; & E(0) - \frac{\bar{h}_1}{2}\int_0^{L_1}\left(\theta_1\left(\frac{x}{|\lambda_{11}|}\right) - \frac{2\bar{v}_1}{\lambda_{12}}\right)\xi_{11}^2(0,x)dx \\
& - \frac{\bar{h}_1}{2}\int_0^{L_1} 4\bar{v}_1\frac{(\bar{v}_2-\bar{v}_1)}{\lambda_{22}\lambda_{12}}\xi_{12}^2(0,x)dx \\
& - \frac{\bar{h}_2}{2}\int_0^{L_2}\left(\frac{|\lambda_{21}|}{\lambda_{22}}\theta_2\left(\frac{L_2-x}{\lambda_{22}}\right) + \frac{2\bar{v}_2}{\lambda_{22}}\right)\xi_{22}^2(0,x)dx \\
& - \frac{\bar{h}_2}{2}\int_0^{L_2} 2\frac{(\bar{v}_2-\bar{v}_1)}{\lambda_{22}}\left(1-\frac{2\bar{v}_2}{\lambda_{22}}\right)\xi_{21}^2(0,x)dx. \\
\leq\; & E(0) - \frac{\bar{h}_1}{2}\int_0^{L_1}\left[\xi_{11}^2(0,x) + \xi_{12}^2(0,x)\right]\Gamma_1^0 dx \\
& - \frac{\bar{h}_2}{2}\int_0^{L_2}\left[\xi_{22}^2(0,x) + \xi_{21}^2(0,x)\right]\Gamma_2^0 dx, \qquad (73)
\end{aligned}
$$

where

$$\Gamma_1^0 = \min\left(\inf_{x\in]0,L_1[}\left(\theta_1\left(\frac{x}{|\lambda_{11}|}\right) - \frac{2\bar{v}_1}{\lambda_{12}}\right), 4\bar{v}_1\frac{(\bar{v}_2-\bar{v}_1)}{\lambda_{22}\lambda_{12}}\right),$$

$$\Gamma_2^0 = \min\left(\inf_{x\in]0,L_2[}\left(\frac{|\lambda_{21}|}{\lambda_{22}}\theta_2\left(\frac{L_2-x}{\lambda_{22}}\right) + \frac{2\bar{v}_2}{\lambda_{22}}\right), 2\frac{(\bar{v}_2-\bar{v}_1)}{\lambda_{22}}\left(1-\frac{2\bar{v}_2}{\lambda_{22}}\right)\right).$$

Arguing as for (45), we get

$$\xi_{i1}^2(0,x) + \xi_{i2}^2(0,x) = \frac{2}{\bar{h}_i}\left(\bar{h}_i(\check{v}_i^0(x))^2 + g(\check{h}_i^0(x))^2\right). \tag{74}$$

Therefore, using (74) in (73), one has

$$E(t_1) \leq (1-\Theta^0)E(0) \tag{75}$$

where

$$\Theta^0 = \min\left(\Gamma_1^0, \Gamma_2^0\right) \in [0,1[, \quad \text{since} \quad 0 < \theta_1(\frac{x}{|\lambda_{11}|}) - \frac{2\bar{v}_1}{\lambda_{12}} < 1.$$

In order to generalize (75) with respect to time, we consider the time $t_k = kT$ as initial condition, with T given by (17). Then, we let

$$b_1(t) = \xi_{11}(t_k, |\lambda_{11}|(t-t_k)), \quad t \in]t_k, t_k + L_1/|\lambda_{11}|[,$$

$$b_2(t) = \xi_{22}(t_k, L_2 - \lambda_{22}(t-t_k)), \quad t \in]t_k, t_k + L_2/\lambda_{22}[,$$

$$b_3(t) = \xi_{12}(t_k, L_1 - \lambda_{12}(t-t_k)), \quad t \in]t_k, t_k + L_1/\lambda_{12}[,$$

$$b_3(t) = \xi_{21}(t_k, |\lambda_{21}|(t-t_k)), \quad t \in]t_k, t_k + L_2/|\lambda_{21}|[,$$

$$\Gamma_1^k = \min\left(\inf_{x\in]0,L_1[}\left(\theta_1(t_k + \frac{x}{|\lambda_{11}|}) - \frac{2\bar{v}_1}{\lambda_{12}}\right), 4\bar{v}_1\frac{(\bar{v}_2 - \bar{v}_1)}{\lambda_{22}\lambda_{12}}\right),$$

$$\Gamma_2^k(x) = \min\left(\inf_{x\in]0,L_2[}\left(\frac{|\lambda_{21}|}{\lambda_{22}}\theta_2(t_k + \frac{L_2 - x}{\lambda_{22}}) + \frac{2\bar{v}_2}{\lambda_{22}}\right), 2\frac{(\bar{v}_2 - \bar{v}_1)}{\lambda_{22}}\left(1 - \frac{2\bar{v}_2}{\lambda_{22}}\right)\right).$$

and

$$\Theta^k = \min\left(\Gamma_1^k, \Gamma_2^k\right) \in [0,1[.$$

Therefore, by integrating from t_k to t_{k+1} and using the same arguments as for the interval $[0, t_1]$, the proof of Theorem 1 is completed. □

5. Numerical results

Numerical results are obtained by using a high order finite volume method (see Leveque. (2002); Toro. (1999)).

5.1 A numerical example for a single reach

In this section, we illustrate the control design method on a canal with the following parameters. Lenght $L = 500m$, width $l = 1m$. The steady state is $\bar{q}(x) = 1m^3s^{-1}$ and $\bar{h}(x) = 1m$ and the initial condition is $h(0,x) = 2m$ and $q(0,x) = 3m^3s^{-1}$. The spatial step size is $\Delta x = 10m$ and the time step is $\Delta t = 1s$. We also set $H_{up} = 2.2m$ and $H_{down} = 0.5m$ and use relations (48)-(49) for gates opening.

We have tested a big perturbation in order to investigate the robusness and the flexibility of the control method. One sees that the bigger the θ's are, the faster the exponential deacrease is (Fig 5). Increasing θ's also produces some oscillations of the gates opening with heigh frequencies (Fig 6). We then notice that for the gates opening, choosing θ's between 0.5 and 0.7 gives a quite good behaviour of the gates opening (Fig 7-(b)). Generally, depending on the control action (gates, pumps etc) used, we can have a wide possibilities of choosing the θ's.

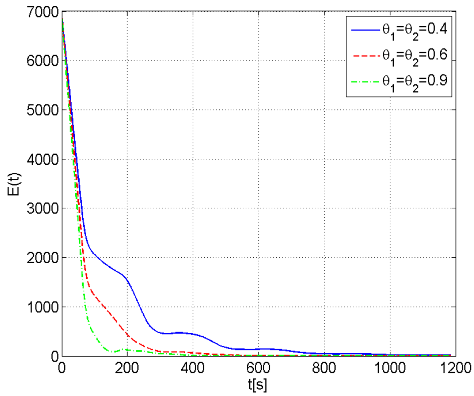
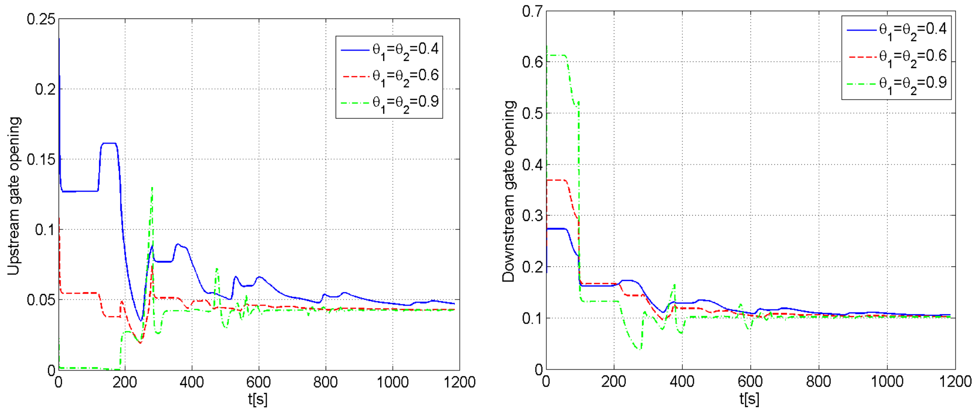


Fig. 5. Energy evolution for different values of θ_1 and θ_2.



(a) Downstream gate opening

(b) Upstream gate opening

Fig. 6. Gate openings for different values of θ_1 and θ_2.

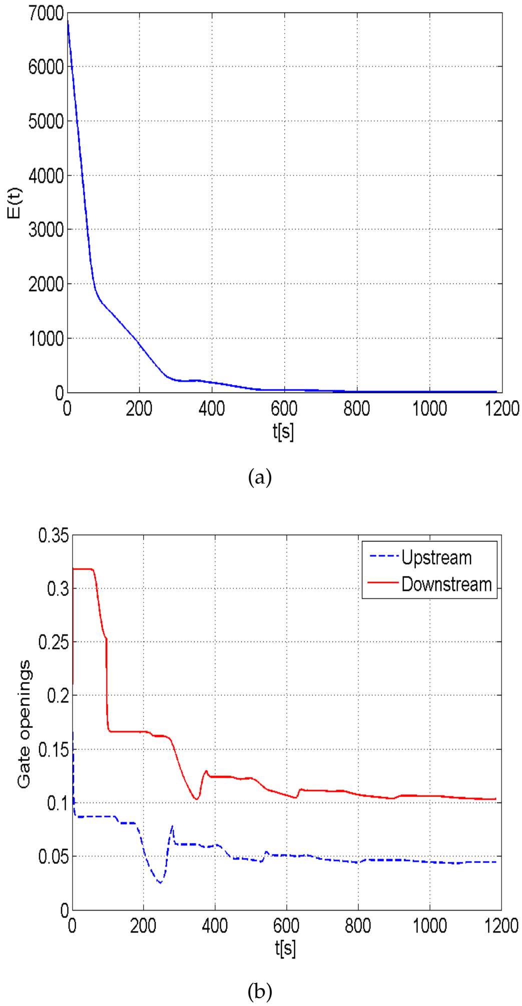
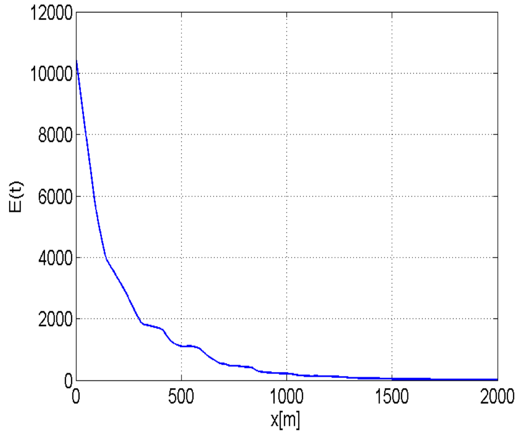


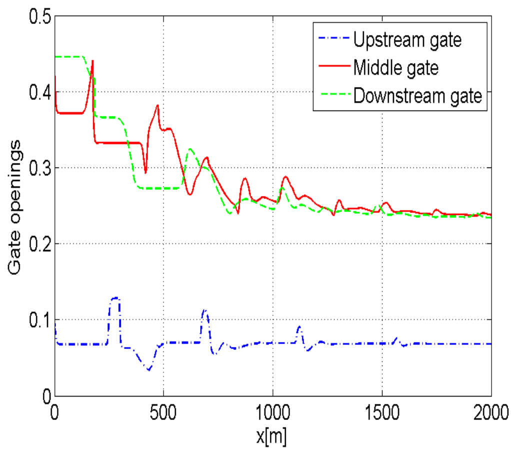
Fig. 7. Evolution of the energy (a) and gates opening (b) for $\theta_1 = \theta_2 = 0.5$.

5.2 A numerical example for the cascade network

We consider two reaches of Lenght $L_1 = L_2 = 1000m$, width $l = 1m$. The steady state is $\bar{q}_1(x) = 1.5m^3s^{-1}$, $\bar{h}_1(x) = 1.5m$ and $\bar{h}_2(x) = 1m$ and the initial condition is $h_1(0,x) = 2m$, $q_1(0,x) = 3m^3s^{-1}$, $h_2(0,x) = 1.5m$, $q_2(0,x) = 3m^3s^{-1}$. The spatial step size is $\Delta x = 10m$ and the time step is $\Delta t = 1s$. We also set $H_{up} = 3m$ and $H_{down} = 0.5m$. We have noticed as in the case of one single reach, that the bigger the θs are, the faster the exponential decrease is. In figure (8), we have plotted the energy decay and the gates opening for $\theta_1 = \theta_2 = \theta_3 = 0.7$. Although, the perturbations for reach 1 and 2 are different, the controllers act in such a way to drive the perturbations to zero simultaneously (Fig (9).



(a)



(b)

Fig. 8. Energy evolution (a) and gates opening (b) for $\theta_1 = \theta_2 = \theta_3 = 0.7$.

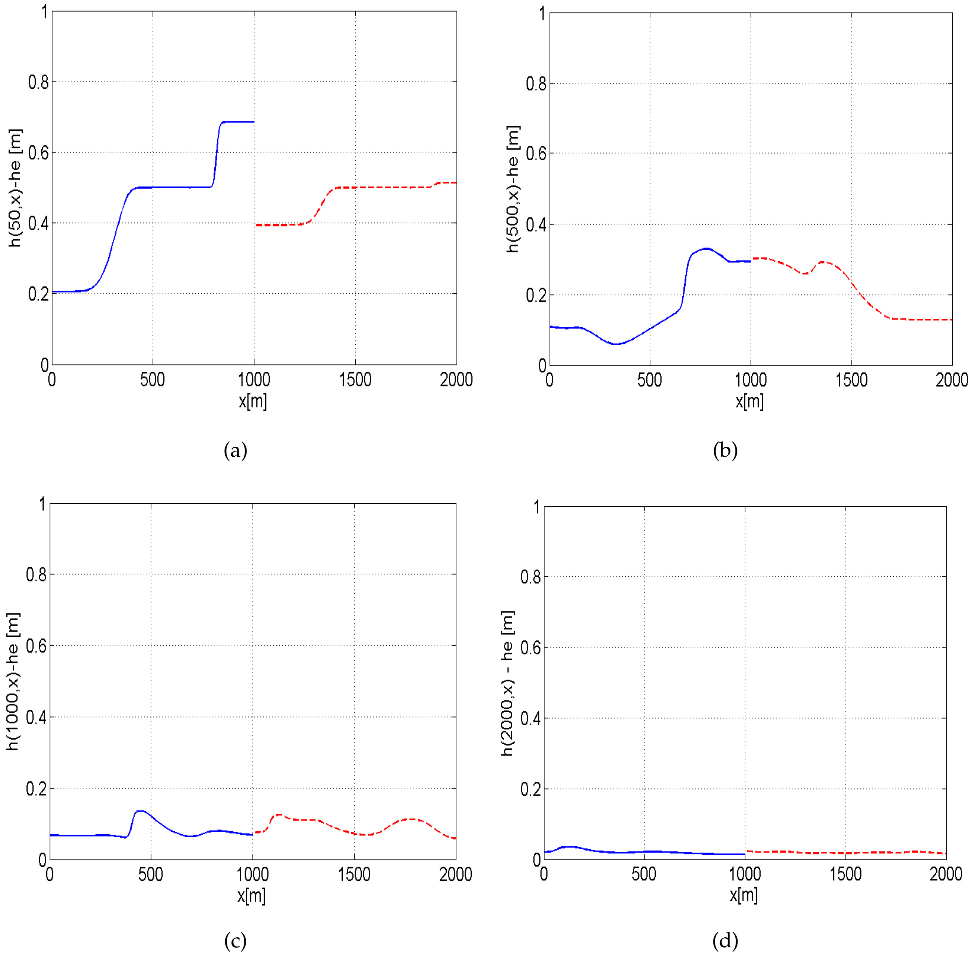


Fig. 9. Deviation of water height at instants $t = 50$ (a), $t = 500$ (b), $t = 1000$ (c) and $t = 2000$ (d), for $\theta_1 = \theta_2 = \theta_3 = 0.7$.

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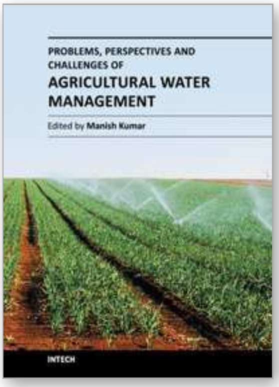

Problems, Perspectives and Challenges of Agricultural Water Management

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Food security emerged as an issue in the first decade of the 21st Century, questioning the sustainability of the human race, which is inevitably related directly to the agricultural water management that has multifaceted dimensions and requires interdisciplinary expertise in order to be dealt with. The purpose of this book is to bring together and integrate the subject matter that deals with the equity, profitability and irrigation water pricing; modelling, monitoring and assessment techniques; sustainable irrigation development and management, and strategies for irrigation water supply and conservation in a single text. The book is divided into four sections and is intended to be a comprehensive reference for students, professionals and researchers working on various aspects of agricultural water management. The book seeks its impact from the diverse nature of content revealing situations from different continents (Australia, USA, Asia, Europe and Africa). Various case studies have been discussed in the chapters to present a general scenario of the problem, perspective and challenges of irrigation water use.