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Analytical Solution of Dynamic Response of Heat Exchanger

D. Gvozdenac
Faculty of Technical Sciences, Novi Sad
Serbia

1. Introduction

Two-fluid heat exchangers are widely used in almost every energy process such as those in power plants, gas turbines, air-conditioning systems, numerous chemical plants and home appliances. Every change of steady state or starting of a plant causes changes in the system which can considerably affect not only the observed process but also the safety of the plant's operations. In all above cases, it is important to know the dynamic behavior of a heat exchanger in order to choose the most suitable design, controls and operations. The traditional design based on stationary approach has become inadequate and nowadays, more attention is devoted to the analysis of the heat exchanger's dynamic behavior and its design is adjusted to such conditions of work. Although the process control technology has made considerable headway, its practical application requires the knowledge of the dynamic behavior of both the plant's components and the plant as a whole.

Ever since Profos (Profos, 1943) showed the first dynamic model of a simple heat exchanger and Takahashi (Takahashi, 1951) published the first transfer functions for ordinary heat exchangers, there have been numerous studies of the heat exchanger's dynamic behavior. The historic overview of dynamic modeling is given in (Kays & London, 1984) and (Roetzel & Xuan, 1999) thus, the attention of this paper will be directed exclusively towards the review of some significant works in this area and works which this paper has been influenced by.

The paper (Liapis & McAvoy, 1981) defines the conditions for obtaining analytical solutions of transient phenomena in the class of problems associated with heat and mass transfers in counter flow fluid streams. Their solutions take into account forced flow and the dependence of transient coefficient on the fluid's flow and do not involve the effect of wall finite heat capacity. The exact solution of dynamic behavior of a parallel heat exchanger in which wall heat capacity is negligible in relation to the fluid capacity was shown in (Li, 1986). These solutions are valid for both finite and nonfinite flow velocities. The paper (Romie, 1985) shows responses of outlet fluid temperatures for the equation of a step fluid inlet temperature change in a counter flow heat exchanger. The responses are determined by means of a finite difference method and involve the wall effect. The exact analytical solution for transient phenomena of a parallel flow heat exchanger for unit step change of inlet temperature of one of the fluids is given in (Romie, 1986). Although this solution includes the wall effect, it is limited to heat exchangers with equal fluid velocities or heat exchangers in which both fluids are gases. The paper (Gvozdenac, 1987) shows analytical solution for transient response of parallel and counter flow heat exchangers. However, these solutions are limited to the case in which heat capacities of both fluids are negligibly small in relation to the heat exchanger's separating wall capacity. Moreover, it is important to mention that papers (Romie, 1983), (Gvozdenac, 1986), (Spiga & Spiga, 1987) and (Spiga & Spiga, 1988) deal with two-dimensional problems of transition for cross flow heat exchangers with both fluids unmixed throughout. The last paper is the most general one and provides opportunities for calculating transient temperatures of wall temperatures and of both fluids by an analytical method for finite flow velocities and finite wall capacity. The paper (Gvozdenac, 1990) shows analytical solution of transient response of the parallel heat exchanger with finite heat capacity of the wall. The procedure presented in the above paper is also used for resolving dynamic response of the cross flow heat exchanger with the finite wall capacity (Gvozdenac, 1991).

A very important book is that of Roetzel W and Xuan Y (Roetzel & Xuan, 1999) which provides detailed analysis of all important aspects of the heat exchanger's dynamic behavior in general. It also gives detailed overview and analysis of literature.

This paper shows solutions for energy functions which describe convective heat transfer between the wall of a heat exchanger and fluid streams of constant velocities. The analysis refers to parallel, counter and cross flow heat exchangers. Initial fluids and wall temperatures are equal but at the starting moment, there is unit step change of inlet temperature of one of the fluids. The presented model is valid for finite fluid velocities and finite heat capacity of the wall. The mathematical model is comprised of three linear partial differential equations which are resolved by manifold Laplace transforms. To a certain extent, this paper presents a synthesis of the author's pervious papers with some simplified and improved final solutions.

The availability of such analytical solutions enables engineers and designers a much better insight into the nature of heat transfers in parallel, counter and cross flow heat exchangers.

For the purpose of easier practical application of these solutions, the potential users are offered MS Excel program at the web address: www.peec.uns.ac.rs. This program is open and can be not only adjusted to special requirements but also modified.

2. Mathematical formulation

Regardless of seeming similarity of partial differential equations arising from mathematical modeling, this paper analyzes parallel, cross and counter flow heat exchangers separately. However, simplifying assumptions in the derivation of differential equations are the same and are as follows:

a. Heat transfer characteristics and physical properties are independent of temperature, position and time;
b. The fluid velocity is constant in each flow passage;
c. Axial conduction is negligible in both fluids and the wall;
d. Overall heat losses are negligible;
e. The heat generation and viscous dissipation within the fluids are negligible;
f. Fluids are assumed to be finite-velocity liquids or gases. This means that the fluid transit or dwell times are not small compared to the duration of transience.

By respecting above assumptions, the energy balance for parallel, counter and cross flow heat exchangers will be mathematically formulated.

2.1 Parallel flow

On the basis of simplified assumptions and by applying energy equations to both fluids and the wall, one can obtain three simultaneous partial differential equations in the coordinate system as shown in Fig. 1. It is obvious that both fluids flow in the same direction but on different sides of the heat exchanger's separating wall. Heat transfer areas and heat transfer coefficients from both sides are known. The length of the heat exchanger is L.

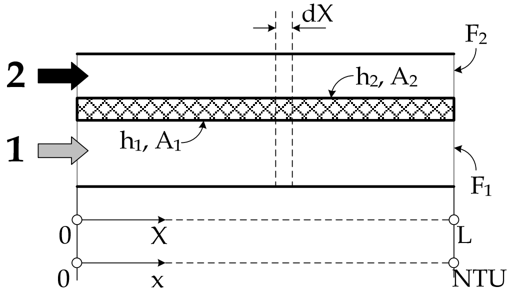


Fig. 1. Schematic Description of Parallel Flow Heat Exchanger.

Differential equations describing fluid-temperature fields in the heat exchanger core are statements of "micro" energy balances for an arbitrary differential control volume of that particular core. The following set of partial differential equations:

$$M_w \cdot c_w \cdot \frac{\partial T_w}{\partial t} = (hA)_1 \cdot (T_1 - T_w) - (hA)_2 \cdot (T_w - T_1)$$

$$\dot{m}_1 \cdot c_{p1} \cdot L \cdot \left(\frac{\partial T_1}{\partial X} + \frac{1}{U_1} \cdot \frac{\partial T_1}{\partial t} \right) = (hA)_1 \cdot (T_w - T_1)$$

$$\dot{m}_2 \cdot c_{p2} \cdot L \cdot \left(\frac{\partial T_2}{\partial X} + \frac{1}{U_2} \cdot \frac{\partial T_2}{\partial t} \right) = (hA)_2 \cdot (T_w - T_2) \quad (1)$$

represents the energy balance over the control volume shown in Fig. 1.

Due to simplified standard assumptions underlying the theory, the mathematical model is linear and tractable by available methods of calculus.

To define mathematical problem completely, inlet and initial conditions have to be prescribed:

$$T_1(0,t) = \begin{cases} T_r & for\ t < 0 \\ T^* & for\ t \geq 0 \end{cases}$$
$$T_2(0,t) = T_r = const. \quad (2)$$
$$T_1(X,0) = T_w(X,0) = T_2(X,0) = T_r = const$$

These conditions assume that only fluid 1 inlet condition is perturbed. The step change of inlet temperature of fluid 1 is certainly the most important from physical point of view. Other inlet temperature changes can be analyzed using described mathematical model and procedures for their analytical solution.

In equations 1 and 2, the convention of index 1 referring to weaker fluid flow and index 2 to stronger fluid flow is introduced. Fluid undergoing higher temperature changes because of smaller value of the thermal capacity $W = \dot{m} \cdot c_p$ is called "weaker"? The other flow is then "stronger" and it is less changed in the heat exchanger. The product of mass flow rate and isobaric specific heat of fluid is the indicator of fluid's flow "strength" and represents its essential characteristic. Therefore, it is necessary to make strict distinction between weaker and stronger flow. Only the weaker fluid flow can change the state for maximum temperature difference. Therefore, $\dot{Q}_{\max} = \left(\dot{m} \cdot c_p\right)_{\min} \cdot \left|T_1^{'} - T_2^{'}\right|$. This is valid in steady state conditions although flow designation convention is also applicable to unsteady state analysis.

Generally, the heat exchanger's effectiveness is defined in the relation of actually exchanged heat and maximum possible one and it is the measure of thermodynamic quality of the device. In this way, the effectiveness of all heat exchangers can be a number taken from a closed interval $\varepsilon = [0, 1]$

Another convention is useful for further analysis. If weaker and stronger fluid flows are designated with indices 1 and 2, respectively, then standardized relation between heat capacities of fluids is:

$$\omega = \frac{W_1}{W_2} \quad (0 \le \omega \le 1) \tag{3}$$

The value $\omega \Rightarrow 0$ always designates that the stronger fluid flow tends to isothermal change in the heat exchanger since $\left(\dot{m} \cdot c_p\right)_2 \Rightarrow \infty$. With final $\dot{Q}$, implying $\left|T_2^{''} - T_2^{'}\right| \Rightarrow 0$, this means that the flow 2 changes the phase (condensation or evaporation). On the contrary, $\omega = 1$ refers to well balanced flows, i.e. the temperatures from inlet to outlet change equally.

In order to define dimensionless temperatures, it is appropriate to choose reference temperature T_r and a characteristic temperature difference T^* - T_r so that:

$$\theta_i(X,t) = \frac{T_i(X,t) - T_r}{T^* - T_r} \quad (i = 1, 2, w) \tag{4}$$

It is suitable that reference temperatures are minimum and maximum ones, i.e. T^* and T_r, respectively. If the weaker flow is designated with index 1 and if $T^* = T_1^{'}$ and $T_r = T_2^{'}$ then, the weaker flow enters the heat exchanger with $\theta_1^{'} = 1$ and the stronger flow with $\theta_2^{'} = 0$.

For the purpose of simplifying the mathematical model the dimensionless distance and dimensionless time are introduced:

$$x = \frac{X}{L} \cdot NTU, \quad z = \frac{t}{t^*} \tag{5}$$

The number of heat transfer units is:

$$NTU = \frac{(hA)_1 \cdot (hA)_2}{(hA)_1 + (hA)_2} \cdot \frac{1}{W_1} \tag{6}$$

and time parameter

$$t^* = \frac{c_w \cdot M_w}{(hA)_1 + (hA)_2} \tag{7}$$

Further, the relation for the product of heat transfer coefficient and heat transfer area of each fluid and the sum of these products is as follows:

$$K_1 = \frac{(hA)_1}{(hA)_1 + (hA)_2}, \qquad K_2 = 1 - K_1 \tag{8}$$

Finally, complex dimensionless parameter is:

$$C_i = L \cdot \frac{W_i}{c_w \cdot M_w} \cdot \frac{1}{K_i \cdot U_i} \quad (i = 1, 2) \tag{9}$$

It is inversely proportional to the fluid speed in heat exchanger flow channels. The high fluid velocity with other unchanged values in the equation (9) means that $C_i \Rightarrow 0$ and that fluid dwell time in the heat exchanger is short. As the fluid velocity decreases, the value of parameters C_i increases and the time of fluid dwell time in the core of the heat exchanger is prolonged. Fluid velocity in heat exchangers is:

$$U_i = \frac{\dot{m}_i}{\rho_i \cdot F_i} \quad (\textit{fluid velocity}, i = 1, 2) \tag{10}$$

Now, the system of equations (1) can be written in the following form :

$$\begin{aligned}
&\frac{\partial \theta_w}{\partial z} + \theta_w = K_1 \cdot \theta_1 + K_2 \cdot \theta_2 \\
&C_1 \cdot \frac{\partial \theta_1}{\partial z} + K_2 \cdot \frac{\partial \theta_1}{\partial x} = \theta_w - \theta_1 \\
&C_2 \cdot \frac{\partial \theta_2}{\partial z} + \frac{K_1}{\omega} \cdot \frac{\partial \theta_2}{\partial x} = \theta_w - \theta_2
\end{aligned} \tag{11}$$

The initial and inlet conditions (Eqs. 2) become:

$$\begin{aligned}
&\theta_1(0,z) = \begin{cases} 0 & \textit{for } z < 0 \\ 1 & \textit{for } z \geq 0 \end{cases} \\
&\theta_2(0,z) = 0 \\
&\theta_1(x,0) = \theta_w(x,0) = \theta_2(x,0) = 0
\end{aligned} \tag{12}$$

The equation (11) and (12) define transient response of parallel flow heat exchanger with finite wall capacitance. Mathematical model is valid for the case when $W_1 \le W_2$ and temperature of fluid 1 is perturbed (unit step change).

Outlet temperatures of both fluids in steady state ($z \to \infty$) are:

$$\begin{aligned} \theta_1^{"}(NTU,\infty) &= 1-\varepsilon \\ \theta_2^{"}(NTU,\infty) &= \omega\cdot\varepsilon \end{aligned} \tag{13}$$

where ε is effectiveness of heat exchanger. Effectiveness of parallel heat exchanger is as follows:

$$\varepsilon = \frac{1-\exp\left[-NTU(1+\omega)\right]}{1+\omega} \quad for \;\; 0<\omega\le 1 \tag{14}$$

For the case $\omega = 0$ the effectiveness is

$$\varepsilon = 1-\exp\left(-NTU\right) \tag{15}$$

and is valid for all types of heat exchangers.

For the case when stronger fluid (fluid 2) is perturbed, the inlet condition of the mathematical problem is changed and is as follows:

$$\begin{aligned} &\theta_1(0,z)=0 \\ &\theta_2(0,z)=\begin{cases} 0 & for \; z<0 \\ 1 & for \; z\ge 0 \end{cases} \\ &\theta_1(x,0)=\theta_w(x,0)=\theta_2(x,0)=0 \end{aligned} \tag{16}$$

In this case, outlet temperatures in the conditions of steady state are equal:

$$\begin{aligned} \theta_1^{"}(NTU,\infty) &= \omega\cdot\varepsilon \\ \theta_2^{"}(NTU,\infty) &= 1-\varepsilon \end{aligned} \tag{17}$$

In this way, resolving of this mathematical problem for two inlet conditions includes all possible cases of fluid flow strength, i.e. $W_1 \le W_2 \;\; and \;\; W_1 \ge W_2$. Only the case $W_1 \le W_2$ is analyzed in this paper because of limited space. However, the presented procedure for resolving mathematical model for all types of heat exchangers gives opportunities to get easily to the solution in case when $W_1 \ge W_2$.

2.2 Counter flow

In the same way as in the case of parallel flow heat exchanger, it is possible to set up mathematical model of counter flow heat exchanger (Fig. 2). The essential difference between these two heat exchangers is in inlet conditions.

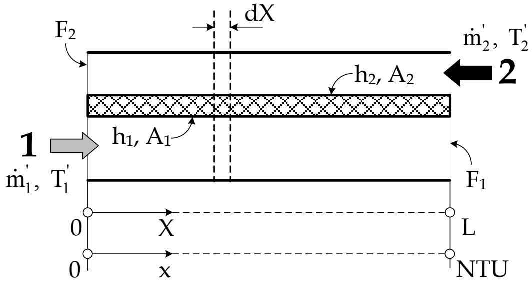


Fig. 2. Schematic Description of Counter Flow Heat Exchanger

Procedure similar to the above for parallel flow delivers the following mathematical formulation for the transient behavior of counter flow heat exchanger:

$$\frac{\partial \theta_w}{\partial z} + \theta_w = K_1 \cdot \theta_1 + K_2 \cdot \theta_2$$

$$C_1 \cdot \frac{\partial \theta_1}{\partial z} + K_2 \cdot \frac{\partial \theta_1}{\partial x} = \theta_w - \theta_1$$

$$C_2 \cdot \frac{\partial \theta_2}{\partial z} - \frac{K_1}{\omega} \cdot \frac{\partial \theta_2}{\partial x} = \theta_w - \theta_2 \tag{18}$$

The initial and inlet conditions are:

$$\begin{aligned} &\theta_1(0,z) = \begin{cases} 0 & for\ z < 0 \\ 1 & for\ z \geq 0 \end{cases} \\ &\theta_2(NTU,z) = 0 \\ &\theta_1(x,0) = \theta_w(x,0) = \theta_2(x,0) = 0 \end{aligned} \tag{19}$$

If the system of equations (11) and (18) is compared, it can be observed that the difference is only in the sign before the second member on the right side of the third equation. If we compare equations (12) and (19) (inlet and initial conditions), the difference is only in the second equation. However, these seemingly small differences make substantial differences in the solution of the problem which will be shown later on.

Outlet temperatures of both fluids in steady state ($z \rightarrow \infty$) are as in the case of parallel flow heat exchanger but the effectiveness is in case of counter flow heat exchanger designed as follows:

$$\varepsilon = \frac{1 - \exp\left[-NTU(1-\omega)\right]}{1 - \omega \cdot \exp\left[-NTU(1-\omega)\right]} \quad for\ \ 0 \leq \omega < 1 \tag{20}$$

and

$$\varepsilon = \frac{NTU}{1 + NTU} \quad for\ \ \omega = 1 \tag{21}$$

When stronger fluid (fluid 2) is perturbed, the inlet condition of the mathematical problem is changed and is as follows:

$$\begin{aligned}
&\theta_1(0,z)=0\\
&\theta_2(NTU,z)=\begin{cases}0 & for\ z<0\\ 1 & for\ z\geq 0\end{cases}\\
&\theta_1(x,0)=\theta_w(x,0)=\theta_2(x,0)=0
\end{aligned} \tag{22}$$

The problem formulated in this way is valid for $W_1 \leq W_2$. For the case $W_1 \geq W_2$, the problem is very similar and because of that it will not be elaborated in details.

2.3 Cross flow (both fluids unmixed)

The drawing of cross flow heat exchanger which is used for mathematical analysis is shown in Fig. 3. It contains the necessary system of designation and coordinates which will be used in this paper. The fluid 1 flows in the X direction and the fluid 2 in the Y direction. The fluid flows are not mixed perpendicularly to their flow.

Based on these assumptions and by applying energy equations to both fluids, three simultaneous partial differential equations can be obtained in the coordinate system as shown in Fig. 3.

$$M_w c_w \frac{\partial T_w}{\partial t} = (h\cdot A)_1 (T_1 - T_w) - (h\cdot A)_2 (T_w - T_2)$$

$$m_1 c_{p1} X_o \left(\frac{\partial T_1}{\partial X} + \frac{1}{U_1}\frac{\partial T_1}{\partial t}\right) = (h\cdot A)_1 (T_w - T_1)$$

$$m_2 c_{p2} Y_o \left(\frac{\partial T_2}{\partial Y} + \frac{1}{U_2}\frac{\partial T_2}{\partial t}\right) = (h\cdot A)_2 (T_w - T_2) \tag{23}$$

Independent variables in space and time (X, Y and t) vary from 0 to the length of heat exchangers X_o and Y_o, i.e from 0 to ∞. By comparing the system of equations (1), it can be noticed that there is the presence of the space coordinate (Y) and the existence of two dimensions of heat exchangers (X_o and Y_o).

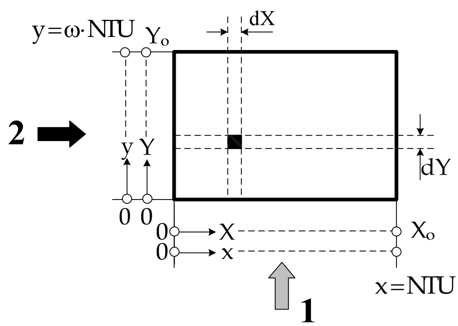


Fig. 3. Schematic Description of Cross Flow Heat Exchanger.

Initial and inlet conditions of analyzed problem are as follows:

$$T_1(0,Y,t)=\begin{cases} T & for \quad t<0 \\ T^* & for \quad t>0 \end{cases}$$

$$T_2(X,0,t)=T=const$$

$$T_1(X,Y,0)=T_w(X,Y,0)=T_2(X,Y,0)=T=const \tag{24}$$

By introducing new dimensionless variable:

$$x=\frac{X}{X_o}\cdot NTU,\ x=\frac{Y}{Y_o}\cdot NTU,\ z=\frac{t}{t^*} \tag{25}$$

the set of equations (23) is as follows:

$$\frac{\partial \theta_w}{\partial z}+\theta_w=K_1\cdot\theta_1+K_2\cdot\theta_2$$

$$C_1\cdot\frac{\partial \theta_1}{\partial z}+K_2\cdot\frac{\partial \theta_1}{\partial x}=\theta_w-\theta_1$$

$$C_2\cdot\frac{\partial \theta_2}{\partial z}+K_1\cdot\frac{\partial \theta_2}{\partial y}=\theta_w-\theta_2 \tag{26}$$

and initial and inlet conditions (Eq. 24) as:

$$\theta_1(0,y,z)=\begin{cases} 0 & for \quad z<0 \\ 1 & for \quad z>0 \end{cases}$$

$$\theta_2(x,0,z)=0$$

$$\theta_1(x,y,0)=\theta_w(x,y,0)=\theta_2(x,y,0)=0 \tag{27}$$

Outlet temperatures of both fluids in steady state ($z\to\infty$) are defined by Eq. (13) but the effectiveness in the case of cross flow heat exchanger is defined as follows (Bačlić, 1978):

$$\varepsilon=1-\exp\left[-(1+\omega)NTU\right]\cdot$$
$$\left[I_0\left(2\cdot NTU\cdot\sqrt{\omega}\right)+\sqrt{\omega}\cdot I_1\left(2\cdot NTU\cdot\sqrt{\omega}\right)-\frac{1-\omega}{\omega}\sum_{n=2}^{\infty}\omega^{n/2}\cdot I_n\left(2\cdot NTU\cdot\sqrt{\omega}\right)\right] \tag{28}$$

and

$$\varepsilon=1-\exp\left[-2\cdot NTU\right]\cdot\left[I_0\left(2\cdot NTU\right)+I_1\left(2\cdot NTU\right)\right] \quad for \ \ \omega=1 \tag{29}$$

In Eqs. (28 and 29), the $I_n(\cdot)$ is modified Bessel function.

For the case when stronger fluid (fluid 2) is perturbed, the inlet condition of the mathematical problem is changed and it is as follows:

$$\theta_1(x,0,z)=0$$

$$\theta_2(0,y,z)=\begin{cases}0 & for \quad z<0\\ 1 & for \quad z>0\end{cases}$$

$$\theta_1(x,y,0)=\theta_w(x,y,0)=\theta_2(x,y,0)=0 \tag{30}$$

As opposed to parallel and counter flow heat exchangers where outlet fluid temperatures are constant over the whole length of outlet edges, it is not the case for cross flow heat exchangers. Then, outlet temperature from the heat exchanger is obtained as mean temperature at the outlet edge of the heat exchanger.

Special cases of cross flow heat exchangers when one or both fluid flows are mixed throughout will not be elaborated in this paper.

In the Section that follows, defined mathematical problems for determining temperature fields and outlet temperatures will be resolved for three basic types: parallel, counter and cross flow heat exchangers.

3. General solution

The set of three partial differential equations for all types of heat exchanger are linear (Eqs. 11, 18 and 26). These systems can be solved by using multifold Laplace transform. In the case of parallel and counter flow heat exchangers, it is double-fold and in the case of cross flow it is three-fold Laplace transform.

3.1 Parallel flow

By applying this transform over the equations (11) and initial and inlet condition (Eq.16), the following algebraic equations are obtained:

$$\tilde{\theta}_w=\frac{K_1\cdot\tilde{\theta}_1+K_2\cdot\tilde{\theta}_2}{p+1}$$

$$\left(s+\frac{C_1\cdot p+1}{K_2}\right)\cdot\tilde{\theta}_1=\frac{1}{K_2}\cdot\tilde{\theta}_w+\frac{1}{p}$$

$$\left(s+\frac{\omega\cdot(C_2\cdot p+1)}{K_1}\right)\cdot\tilde{\theta}_2=\frac{\omega}{K_1}\cdot\tilde{\theta}_w \tag{31}$$

From this set of equations, the outlet and wall temperatures are as follows:

$$\tilde{\theta}_w=\frac{\dfrac{K_1\cdot K_2}{K_2\cdot s+C_1\cdot p+1}}{p+1-\dfrac{K_1}{K_2\cdot s+C_1\cdot p+1}-\dfrac{K_2}{\dfrac{K_1}{\omega}\cdot s+C_1\cdot p+1}}\cdot\frac{1}{p} \tag{32}$$

$$\tilde{\theta}_1 = \frac{\tilde{\theta}_w}{K_2 \cdot s + C_1 \cdot p + 1} + \frac{K_2}{K_2 \cdot s + C_1 \cdot p + 1} \cdot \frac{1}{p} \tag{33}$$

$$\tilde{\theta}_2 = \frac{\tilde{\theta}_w}{\frac{K_1}{\omega} \cdot s + C_1 \cdot p + 1} \tag{34}$$

After performing some mathematical transformations and by using some well known relations:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{x^{n+1}} \;; \qquad (a+b)^n = \sum_{m=0}^{n} \binom{n}{m} \cdot a^m \cdot b^{n-m} \tag{35}$$

the temperatures can be expressed in the following form which is convenient for developing the inverse Laplace transform:

$$\begin{aligned}\tilde{\theta}_w = {} & K_1 \cdot \sum_{n=0}^{\infty} \left(\frac{K_1}{K_2}\right)^{n+1} \cdot \frac{1}{p \cdot (p+1)^{n+1}} \cdot \frac{1}{\left(s + \frac{C_1}{K_2} \cdot p + \frac{1}{K_2}\right)^{n+1}} + \\ & K_1 \cdot \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \binom{n}{m} \cdot \left(\frac{K_1}{K_2}\right)^{m+1} \cdot \left(\frac{\omega \cdot K_2}{K_1}\right)^{n-m} \cdot \\ & \cdot \frac{1}{p \cdot (p+1)^{n+1}} \cdot \frac{1}{\left(s + \frac{C_1}{K_2} \cdot p + \frac{1}{K_2}\right)^{m+1}} \cdot \frac{1}{\left(s + \frac{\omega \cdot C_2}{K_1} \cdot p + \frac{\omega}{K_1}\right)^{n-m}}\end{aligned} \tag{36}$$

$$\begin{aligned}\tilde{\theta}_1 = {} & \frac{1}{p} \cdot \frac{1}{s + \frac{C_1}{K_1} \cdot p + \frac{1}{K_1}} + \sum_{n=0}^{\infty} \left(\frac{K_1}{K_2}\right)^{n+1} \cdot \frac{1}{p \cdot (p+1)^{n+1}} \cdot \frac{1}{\left(s + \frac{C_1}{K_2} \cdot p + \frac{1}{K_2}\right)^{n+2}} + \\ & \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \binom{n}{m} \cdot \left(\frac{K_1}{K_2}\right)^{m+1} \cdot \left(\frac{\omega \cdot K_2}{K_1}\right)^{n-m} \cdot \\ & \cdot \frac{1}{p \cdot (p+1)^{n+1}} \cdot \frac{1}{\left(s + \frac{C_1}{K_2} \cdot p + \frac{1}{K_2}\right)^{m+2}} \cdot \frac{1}{\left(s + \frac{\omega \cdot C_2}{K_1} \cdot p + \frac{\omega}{K_1}\right)^{n-m}}\end{aligned} \tag{37}$$

$$\begin{aligned}\tilde{\theta}_2 = {} & \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \cdot \left(\frac{K_1}{K_2}\right)^{m+1} \cdot \left(\frac{\omega \cdot K_2}{K_1}\right)^{n-m+1} \cdot \\ & \cdot \frac{1}{p \cdot (p+1)^{n+1}} \cdot \frac{1}{\left(s + \frac{C_1}{K_2} \cdot p + \frac{1}{K_2}\right)^{m+1}} \cdot \frac{1}{\left(s + \frac{\omega \cdot C_2}{K_1} \cdot p + \frac{\omega}{K_1}\right)^{n-m+1}}\end{aligned} \tag{38}$$

From the techniques of Laplace transformation (convolution and translation theorems) and using the Laplace transforms of special functions $F_n(x, c)$ and $I_{n,m}(x, c, d)$, defined in the Appendix, one can obtain the inverse Laplace transformation of Eqs. 36-38, and the transient temperature distributions for the parallel flow heat exchanger:

$$\begin{aligned}\theta_w(x,z) &= K_2 \cdot \sum_{n=0}^{\infty}\left(\frac{K_1}{K_2}\right)^{n+1} \cdot F_{n+1}\left(x,\frac{1}{K_2}\right)\cdot I_{1,n+1}\left(z-\frac{C_1}{K_2}\cdot x,0,-1\right)+\\ &+K_2\cdot\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\binom{n}{m}\cdot\left(\frac{K_1}{K_2}\right)^{m+1}\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n-m}\cdot\\ &\cdot\int_0^x F_{m+1}\left(x-u,\frac{1}{K_2}\right)\cdot F_{n-m}\left(u,\frac{\omega}{K_1}\right)\cdot I_{1,n+1}\left[z-\left(\frac{C_1}{K_2}(x-u)+\frac{\omega\cdot C_2}{K_1}\cdot u\right),0,-1\right]\cdot du\end{aligned} \tag{39}$$

$$\begin{aligned}\theta_1(x,z) &= \kappa\cdot\left(z-\frac{C_1}{K_2}\cdot x\right)\cdot F_1\left(x,\frac{1}{K_2}\right)+\\ &+\sum_{n=0}^{\infty}\left(\frac{K_1}{K_2}\right)^{n+1}\cdot F_{n+2}\left(x,\frac{1}{K_2}\right)\cdot I_{1,n+1}\left(z-\frac{C_1}{K_2}\cdot x,0,-1\right)+\\ &+\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\binom{n}{m}\cdot\left(\frac{K_1}{K_2}\right)^{m+1}\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n-m}\cdot\\ &\cdot\int_0^x F_{m+2}\left(x-u,\frac{1}{K_2}\right)\cdot F_{n-m}\left(u,\frac{\omega}{K_1}\right)\cdot I_{1,n+1}\left[z-\left(\frac{C_1}{K_2}(x-u)+\frac{\omega\cdot C_2}{K_1}\cdot u\right),0,-1\right]\cdot du\end{aligned} \tag{40}$$

$$\begin{aligned}\theta_2(x,z) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\cdot\left(\frac{K_1}{K_2}\right)^{m+1}\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n-m+1}\cdot\\ &\cdot\int_0^x F_{m+1}\left(x-u,\frac{1}{K_2}\right)\cdot F_{n-m+1}\left(u,\frac{\omega}{K_1}\right)\cdot I_{1,n+1}\left[z-\left(\frac{C_1}{K_2}(x-u)+\frac{\omega\cdot C_2}{K_1}\cdot u\right),0,-1\right]\end{aligned} \tag{41}$$

Outlet temperatures of both fluids are obtained for x = NTU.

In the practical use of solutions, the computation of integrals in this paper is done through collocation at nine Chebishev's points: 0.0000000000; ± 0.1679061842; ± 0.5287617831; ± 0.6010186554; ± 0.9115893077, for the given integration interval.

Special case ω = 0

In this case, $\theta_2(x,z)=0$ resulting in reduced Eq. (31):

$$\tilde{\theta}_w = \frac{K_1}{p}\cdot\frac{1}{(p+1)\cdot\left(s+\frac{C_1}{K_2}\cdot p+\frac{1}{K_2}\right)-\frac{K_1}{K_2}} \tag{42}$$

After some mathematical manipulations, using already mentioned techniques, this equation can be transformed into:

$$\tilde{\theta}_w = K_2 \cdot \sum_{n=0}^{\infty} \left(\frac{K_1}{K_2} \right)^{n+1} \cdot \frac{1}{p \cdot (p+1)^{n+1} \cdot \left(s + \frac{C_1}{K_2} \cdot p + \frac{1}{K_2} \right)^{n+1}} \tag{43}$$

The inverse two-fold Laplace transform of Eq. 43 gives:

$$\theta_w(x,z) = K_2 \cdot \sum_{n=0}^{\infty} \left(\frac{K_1}{K_2} \right)^{n+1} \cdot F_{n+1}\left(x, \frac{1}{K_2} \right) \cdot I_{n+1,1}\left(z - \frac{C_1 \cdot x}{K_2}, 1, 1 \right) \tag{44}$$

and Eq. 32 gives:

$$\theta_1(x,z) = \kappa \left(z - \frac{C_1 \cdot x}{K_2} \right) \cdot F_1\left(x, \frac{1}{K_2} \right) + \sum_{n=0}^{\infty} \left(\frac{K_1}{K_2} \right)^{n+1} \cdot F_{n+2}\left(x, \frac{1}{K_2} \right) \cdot I_{n+1,1}\left(z - \frac{C_1 \cdot x}{K_2}, 1, 1 \right) \tag{45}$$

This solution is valid for all types of heat exchangers with $\omega = 0$.

3.2 Counter flow

A very similar procedure can be applied for resolving the mathematical model of counter flow heat exchanger. The set of algebraic equations obtained after two-fold Laplace transform of Eqs. (18) and initial and inlet conditions (Eq (19)) is as follows:

$$(p+1) \cdot \tilde{\theta}_w = K_1 \cdot \tilde{\theta}_1 + K_2 \cdot \tilde{\theta}_2 \tag{46}$$

$$\left(s + \frac{C_1}{K_2} \cdot p + \frac{1}{K_2} \right) \cdot \tilde{\theta}_1 = \frac{1}{K_2} \cdot \tilde{\theta}_w + \frac{1}{p} \tag{47}$$

$$-\left(s - \frac{\omega \cdot C_2}{K_1} \cdot p - \frac{1}{K_1} \right) \cdot \tilde{\theta}_2 = \frac{\omega}{K_1} \cdot \tilde{\theta}_w - \tilde{\theta}_2(0,p) \tag{48}$$

The procedure will be explained in more details here since this case is much more complex than the previous one. By introducing designations:

$$\alpha(s,p) = K_2 \cdot s + C_1 \cdot p + 1 = K_2 \cdot \left(s + \frac{C_1}{K_2} \cdot p + \frac{1}{K_2} \right), \tag{49}$$

$$\beta(s,p) = -\frac{K_1}{\omega} \cdot s + C_2 \cdot p + 1 = -\frac{K_1}{\omega} \cdot \left(s - \frac{\omega \cdot C_2}{K_1} \cdot p - \frac{\omega}{K_1} \right), \tag{50}$$

$$A(s,p) = p + 1 - \frac{K_1}{\alpha(s,p)} - \frac{K_2}{\beta(s,p)}, \tag{51}$$

the both fluids and wall temperatures of the counter flow heat exchanger are as follows:

$$\tilde{\theta}_w = \frac{K_1 \cdot K_2}{p \cdot \alpha \cdot A} - \frac{K_1 \cdot K_2}{\omega \cdot \beta \cdot A} \cdot \tilde{\theta}_2(0,p) \tag{52}$$

$$\tilde{\theta}_1 = \frac{K_2}{p \cdot \alpha} + \frac{K_1 \cdot K_2}{p \cdot \alpha^2 \cdot A} - \frac{K_1 \cdot K_2}{\omega \cdot \alpha \cdot \beta \cdot A} \cdot \tilde{\theta}_2(0,p) \tag{53}$$

$$\tilde{\theta}_2 = \frac{K_1 \cdot K_2}{p \cdot \alpha \cdot \beta \cdot A} - \frac{K_1}{\omega \cdot \beta} \cdot \left(1 + \frac{K_2}{\beta \cdot A}\right) \cdot \tilde{\theta}_2(0,p) \tag{54}$$

It is very simple to prove that:

$$\frac{1}{A} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} K_1^m \cdot K_2^{n-m} \cdot \binom{n}{m} \cdot \frac{1}{(p+1)^{n+1}} \cdot \frac{1}{\alpha^m \cdot \beta^{n-m}}, \tag{55}$$

and that inverse Laplace transformations of the functions $1/\alpha^{m+1}(s,p)$ and $1/\beta^{m+1}(s,p)$ (m=1,2,3,…) with respect to the complex parameter **s** are:

$$L_{s \to x}^{-1}\left\{\frac{1}{\alpha^{m+1}}\right\} = \frac{1}{K_2^{m+1}} \cdot F_{m+1}\left(x, \frac{1}{K_2}\right) \cdot \exp\left(-\frac{C_1}{K_2} \cdot x \cdot p\right), \tag{56}$$

$$L_{s \to x}^{-1}\left\{\frac{1}{\beta^{m+1}}\right\} = (-1)^{m+1} \cdot \left(\frac{\omega}{K_1}\right)^{m+1} \cdot F_{m+1}\left(x, -\frac{\omega}{K_1}\right) \cdot \exp\left(\frac{\omega \cdot C_2}{K_1} \cdot x \cdot p\right). \tag{57}$$

The essential problem in resolving dynamic behavior of the counter flow heat exchanger is in the use of other inlet conditions (Eq. 19).

If the Eq. 54 is collocated into x=NTU then, INLET temperature of the fluid 2 is obtained which is according to given inlet conditions $\theta_2(NTU, z) = 0$, therefore:

$$L_{s \to NTU}^{-1}\left\{\left(\frac{K_1}{\omega \cdot \beta} + \frac{K_1 \cdot K_2}{\omega \cdot \beta^2 \cdot A}\right) \cdot \tilde{\theta}_2(0,p)\right\} = L_{s \to NTU}^{-1}\left\{\frac{K_1 \cdot K_2}{p \cdot \alpha \cdot \beta \cdot A}\right\} \tag{58}$$

This is Fredholm's integral equation of the second order. The problem is reduced to its solving.

The collocation method is used for solving this equation. Perhaps, it is the simplest one. The trial function is:

$$\theta_2(0,z) = \theta_2(0,\infty) \cdot \left[1 - \exp(-z) - \sum_{k=1}^{NCP} a_k \cdot \frac{z^k}{k!} \cdot \exp(-z)\right] \tag{59}$$

In equations (58) and further on, $\theta_2(0,\infty)$ is the steady-state fluid 2 outlet temperature for the counter flow heat exchanger. It can be calculated by using the second of Eq. 13 and effectiveness of counter flow heat exchanger (Eqs. 20 and 21). It follows that:

$$\tilde{\theta}_2(0,\infty)=\begin{cases}\dfrac{NTU}{1+NTU} & for\ \omega=1\\ \omega\cdot\dfrac{1-\exp[-(1-\omega)\cdot NTU]}{1-\omega\cdot\exp[-(1-\omega)\cdot NTU]} & for\ 0\le\omega<1\end{cases} \tag{60}$$

Laplace transform of trial function (Eq. 59) is:

$$\tilde{\theta}_2(0,p)=\theta_2(0,\infty)\cdot\left[\frac{1}{(p+1)\cdot p}-\sum_{k=1}^{9}a_k\cdot\frac{1}{(p+1)^{k+1}}\right] \tag{61}$$

The trial function chosen in this way satisfies completely the equation (58) in points z = 0 and $z\to\infty$. Within the interval $0<z<\infty$, it is necessary to determine collocation points and coefficients a_k (k = 1, 2, 3, ... , NCP). Here, the NCP is the number of collocation points. The accuracy in which the outlet temperatures of fluid 2 versus time are determined depends directly on NCP. In this model of heat exchanger, there are many influential factors and determination of the number of collocation points for the given accuracy of outlet temperature is simplest through practical testing of the solution. For the heat exchanger's parameters appearing in practice, it can be said that NCP varying from 5 to 7 is sufficient for the accuracy of four significant figures and for $z\le 15$.

Substituting the equation (61) in the equation (58) and collocating resulting equation in the NCP point, a set of linear algebraic equations is obtained and their solving generates unknown constants a_k. The set of algebraic solutions can also be written in the following form :

$$\sum_{k=1}^{NCP}a_k\cdot\Delta_k=\Delta_R \tag{62}$$

Substituting the equation (61) in (58) and using Eqs. (55), (56) and (57), it is obtained:

$$\begin{aligned}\Delta_k=\theta_2(0,\infty)\cdot\Bigg\{&F_{k+1}(z(r),1)-\sum_{n=0}^{\infty}(-1)^n\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n+1}\cdot F_{n+2}(NTU,0)\cdot F_{n+k+2}(z(r),1)-\\&-\sum_{n=1}^{\infty}\sum_{m=1}^{n}(-1)^{n-m}\cdot\left(\frac{K_1}{K_2}\right)^{m}\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n-m+1}\cdot\binom{n}{m}\cdot\int_0^{NTU}F_m\left(u,\frac{1}{K_2}\right)\cdot F_{n-m+2}(NTU-u,0)\cdot\\&\cdot F_{n+k+2}\left[z(r)-\left(\frac{C_1}{K_2}+\frac{\omega\cdot C_2}{K_1}\right)\cdot u,1\right]\cdot\exp\left(-\frac{\omega}{K_1}\cdot u\right)\cdot du\Bigg\}\qquad(k,r=1,2,...NCP)\end{aligned} \tag{63}$$

$$\begin{aligned}\Delta_R=\theta_2(0,\infty)\cdot\Bigg\{&I_{1,1}(z(r),1,1)-\sum_{n=0}^{\infty}(-1)^n\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n+1}\cdot F_{n+2}(NTU,0)\cdot I_{n+2,1}(z(r),1,1)-\\&-\sum_{n=1}^{\infty}\sum_{m=1}^{n}(-1)^{n-m}\cdot\left(\frac{K_1}{K_2}\right)^{m}\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n-m+1}\cdot\binom{n}{m}\cdot\int_0^{NTU}F_m\left(u,\frac{1}{K_2}\right)\cdot F_{n-m+2}(NTU-u,0)\cdot ...\end{aligned}$$

$$\ldots \cdot I_{n+2,1}\left[z(r)-\left(\frac{C_1}{K_2}+\frac{\omega\cdot C_2}{K_1}\right)\cdot u,1,1\right]\cdot \exp\left(-\frac{\omega}{K_1}\cdot u\right)\cdot du-$$
$$-\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^{n-m+1}\cdot\left(\frac{K_1}{K_2}\right)^{m+1}\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n-m+1}\cdot\binom{n}{m}\cdot\int_0^{NTU}F_{m+1}\left(u,\frac{1}{K_2}\right)\cdot F_{n-m+1}\left(NTU-u,0\right)\cdot \quad (64)$$
$$\cdot I_{n+1,1}\left[z(r)-\left(\frac{C_1}{K_2}+\frac{\omega\cdot C_2}{K_1}\right)\cdot u,1,1\right]\cdot \exp\left(-\frac{\omega}{K_1}\cdot u\right)\cdot du\Bigg\}$$

The equations (63) and (64) define members in the set of algebraic equations (62). For determining constants $a_{k,}$ it is possible to use any of the well known methods.

The temperature distribution of both fluids and the separating wall can be calculated by using Eqs. (52-54) and by substituting the Laplace transform of fluid 2 outlet temperature given by Eq. (59). Constants a_k are now known and are valid for all values of z within the close interval where the collocation is performed.

Temperatures of fluid and wall are as follows:

$$\theta_1(x,z)=\kappa\left(z-\frac{C_1}{K_2}\cdot x\right)\cdot F_1\left(x,\frac{1}{K_2}\right)+$$
$$\sum_{n=0}^{\infty}\left(\frac{K_1}{K_2}\right)^{n+1}\cdot F_{n+2}\left(x,\frac{1}{K_2}\right)\cdot I_{n+1,1}\left(z-\frac{C_1}{K_2}\cdot x,1,1\right)+$$
$$\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}(-1)^{n-m}\cdot\left(\frac{K_1}{K_2}\right)^{m+1}\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n-m}\cdot\binom{n}{m}\cdot$$
$$\int_0^{x}F_{m+2}\left(u,\frac{1}{K_2}\right)\cdot F_{n-m}\left(x-u,-\frac{\omega}{K_1}\right)\cdot I_{n+1,1}\left(z-\left[\frac{C_1}{K_2}\cdot u-\frac{\omega\cdot C_2}{K_1}\cdot(x-u)\right],1,1\right)\cdot du- \quad (65)$$
$$\frac{\theta_2(0,\infty)}{\omega}\cdot\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^{n-m+1}\cdot\left(\frac{K_1}{K_2}\right)^{m+1}\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n-m+1}\cdot\binom{n}{m}\cdot$$
$$\int_0^{x}F_{m+1}\left(u,\frac{1}{K_2}\right)\cdot F_{n-m+1}\left(x-u,-\frac{\omega}{K_1}\right)\cdot I_{n+2,1}\left[z-\left(\frac{C_1}{K_2}\cdot u-\frac{\omega\cdot C_2}{K_1}\cdot(x-u)\right),1,1\right]-$$
$$\sum_{k=1}^{9}a_k\cdot F_{n+k+2}\left[z-\left(\frac{C_1}{K_2}\cdot u-\frac{\omega\cdot C_2}{K_1}\cdot(x-u)\right),1\right]\cdot du$$

$$\theta_w(x,z)=K_2\left\{\sum_{n=0}^{\infty}\left(\frac{K_1}{K_2}\right)^{n+1}\cdot F_{n+1}\left(x,\frac{1}{K_2}\right)\cdot I_{n+1,1}\left(z-\frac{C_1}{K_2}\cdot x,1,1\right)+\right.$$
$$\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}(-1)^{n-m}\cdot\left(\frac{K_1}{K_2}\right)^{m+1}\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n-m}\cdot\binom{n}{m}\cdot$$
$$\left.\int_0^{x}F_{m+1}\left(u,\frac{1}{K_2}\right)\cdot F_{n-m}\left(x-u,-\frac{\omega}{K_1}\right)\cdot I_{n+1,1}\left(z-\left[\frac{C_1}{K_2}\cdot u-\frac{\omega\cdot C_2}{K_1}\cdot(x-u)\right],1,1\right)\cdot du\right\}-\ldots$$

$$\ldots \frac{K_1}{\omega}\sum_{n=0}^{\infty}(-1)^{n+1}\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n+1}\cdot F_{n+1}\left(x,-\frac{\omega}{K_1}\right)\cdot\left[I_{n+2,1}\left(z+\frac{\omega\cdot C_2}{K_1}\cdot x,1,1\right)-\right.$$
$$\left.\sum_{k=1}^{K}a_k\cdot F_{n+k+2}\left(z+\frac{\omega\cdot C_2}{K_1}\cdot x,1\right)\right]+\frac{1}{\omega}\sum_{n=1}^{\infty}\sum_{m=1}^{n}(-1)^{n-m}\cdot\left(\frac{K_1}{K_2}\right)^{m+1}\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n-m+2}\cdot\binom{n}{m}\cdot$$
$$\int_0^x F_m\left(u,\frac{1}{K_2}\right)\cdot F_{n-m+1}\left(x-u,-\frac{\omega}{K_1}\right)\cdot\left\{I_{n+2,1}\left[z-\left(\frac{C_1}{K_2}\cdot u-\frac{\omega\cdot C_2}{K_1}\cdot(x-u)\right),1,1\right]-\right.$$
$$\left.\sum_{k=1}^{K}a_k\cdot F_{n+k+2}\left[z-\left(\frac{C_1}{K_2}\cdot u-\frac{\omega\cdot C_2}{K_1}\cdot(x-u)\right),1\right]\right\}\cdot du \tag{66}$$

$$\theta_2(x,z)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^{n-m+1}\cdot\left(\frac{K_1}{K_2}\right)^{m+1}\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n-m+1}\cdot\binom{n}{m}\cdot$$
$$\int_0^x F_{m+1}\left(u,\frac{1}{K_2}\right)\cdot F_{n-m+1}\left(x-u,-\frac{\omega}{K_1}\right)\cdot I_{n+1,1}\left\{z-\left[\frac{C_1}{K_2}\cdot u-\frac{\omega\cdot C_2}{K_1}\cdot(x-u)\right],1,1\right\}\cdot du+$$
$$\theta_2(0,\infty)\cdot F_1\left(x,-\frac{\omega}{K_1}\right)\cdot\left[I_{1,1}\left(z+\frac{\omega\cdot C_2}{K_1}\cdot x,1,1\right)-\sum_{k=1}^{9}a_k\cdot F_{n+k+2}\left(z+\frac{\omega\cdot C_2}{K_1}\cdot x,1\right)\right]-$$
$$\theta_2(0,\infty)\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n+1}\cdot F_{n+2}\left(x,-\frac{\omega}{K_1}\right)\cdot$$
$$\left[I_{n+2,1}\left(z+\frac{\omega\cdot C_2}{K_1}\cdot x,1,1\right)-\sum_{k=1}^{9}a_k\cdot F_{n+k+2}\left(z+\frac{\omega\cdot C_2}{K_1}\cdot x,1\right)\right]-$$
$$\theta_2(0,\infty)\cdot\sum_{n=1}^{\infty}\sum_{m=1}^{n}(-1)^{n-m}\cdot\left(\frac{K_1}{K_2}\right)^{m}\cdot\left(\frac{\omega\cdot K_2}{K_1}\right)^{n-m+1}\cdot\binom{n}{m}\cdot$$
$$\int_0^x F_m\left(u,\frac{1}{K_2}\right)\cdot F_{n-m+2}\left(x-u,-\frac{\omega}{K_1}\right)\cdot\left\{I_{n+2,1}\left[z-\left(\frac{C_1}{K_2}\cdot u-\frac{\omega\cdot C_2}{K_1}\cdot(x-u)\right),1,1\right]-\right.$$
$$\left.\sum_{k=1}^{9}a_k\cdot F_{n+k+2}\left[z-\left(\frac{C_1}{K_2}\cdot u-\frac{\omega\cdot C_2}{K_1}\cdot(x-u)\right),1\right]\right\}\cdot du \tag{67}$$

3.3 Cross flow

The equations (25) are linear per $\theta_1(x, y, z)$, $\theta_w(x, y, z)$, and $\theta_2(x, y, z)$. If three-fold Laplace transform of above equations is taken in relation to x, y and z with complex parameters s, q, and p, respectively, and if inlet and initial conditions are used (equation 15), a set of algebraic equations is generated :

$$(p+1)\cdot\tilde{\theta}_w=K_1\cdot\tilde{\theta}_1+K_2\cdot\tilde{\theta}_2 \tag{68}$$

$$\left(K_2\cdot s+C_1\cdot p+1\right)\cdot\tilde{\theta}_1=\tilde{\theta}_w+\frac{K_2}{p\cdot q} \tag{69}$$

$$\left(K_1 \cdot q + C_2 \cdot p + 1\right) \cdot \tilde{\theta}_2 = \tilde{\theta}_w \tag{70}$$

Solving the set of algebraic set (equations (16)-(18)) is as follows:

$$\tilde{\theta}_w = \frac{\dfrac{K_1 \cdot K_2}{p \cdot q \cdot \left(K_2 \cdot s + C_1 \cdot p + 1\right)}}{p + 1 - \dfrac{K_1}{K_2 \cdot s + C_1 \cdot p + 1} - \dfrac{K_2}{K_1 \cdot q + C_2 \cdot p + 1}} \tag{71}$$

$$\tilde{\theta}_1 = \frac{\tilde{\theta}_w}{\left(K_2 \cdot s + C_1 \cdot p + 1\right)} + \frac{K_2}{p \cdot q \cdot \left(K_2 \cdot s + C_1 \cdot p + 1\right)} \tag{72}$$

$$\tilde{\theta}_2 = \frac{\tilde{\theta}_w}{\left(K_1 \cdot q + C_2 \cdot p + 1\right)} \tag{73}$$

After performing certain mathematical transformations as done in previous cases, the algebraic equation (71) can be expressed in the following form:

$$\tilde{\theta}_w = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \cdot \frac{K_1^{m+1} \cdot K_2^{n-m+1}}{p \cdot (p+1)^{n+1} \cdot \left(K_2 \cdot s + C_1 \cdot p + 1\right)^{m+1} \cdot q \cdot \left(K_1 \cdot q + C_2 \cdot p + 1\right)^{n-m}} \tag{74}$$

which is very suitable for inverse Laplace transforms by means of functions $F_n(x,c)$ and $I_{n,m}(x,c,d)$ defined in the Annex. However, for the case n = m in the equation (74) and later on, the twofold sum will be separated into two (single and double) sums so that:

$$\begin{aligned} \tilde{\theta}_w = {} & \sum_{n=0}^{\infty} \frac{K_1^{n+1} \cdot K_2}{p \cdot (p+1)^{n+1} \cdot q \cdot \left(K_2 \cdot s + C_1 \cdot p + 1\right)^{n+1}} + \\ & \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \binom{n}{m} \cdot \frac{K_1^{m+1} \cdot K_2^{n-m+1}}{p \cdot (p+1)^{n+1} \cdot \left(K_2 \cdot s + C_1 \cdot p + 1\right)^{m+1} \cdot q \cdot \left(K_1 \cdot q + C_2 \cdot p + 1\right)^{n-m}} \end{aligned} \tag{75}$$

The insertion of the equation (74) in equations (72) and (73) generates the following algebraic equations:

$$\begin{aligned} \tilde{\theta}_1 = {} & \frac{K_2}{p \cdot q \cdot \left(K_2 \cdot s + C_1 \cdot p + 1\right)} + \sum_{n=0}^{\infty} \frac{K_1^{n+1} \cdot K_2}{p \cdot (p+1)^{n+1} \cdot q \cdot \left(K_2 \cdot s + C_1 \cdot p + 1\right)^{n+2}} + \\ & \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \binom{n}{m} \cdot \frac{K_1^{m+1} \cdot K_2^{n-m+1}}{p \cdot (p+1)^{n+1} \cdot \left(K_2 \cdot s + C_1 \cdot p + 1\right)^{m+2} \cdot q \cdot \left(K_1 \cdot q + C_2 \cdot p + 1\right)^{n-m}} \end{aligned} \tag{76}$$

$$\tilde{\theta}_2 = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \cdot \frac{K_1^{m+1} \cdot K_2^{n-m+1}}{p \cdot (p+1)^{n+1} \cdot \left(K_2 \cdot s + C_1 \cdot p + 1\right)^{m+1} \cdot q \cdot \left(K_1 \cdot q + C_2 \cdot p + 1\right)^{n-m+1}} \tag{77}$$

Now it is possible to get the inverse Laplace transform equation (75)-(77), so that:

$$\theta_w(x,y,z)=\sum_{n=0}^{\infty}K_1^{n+1}\cdot F_{n+1}\left(\frac{x}{K_2},1\right)\cdot I_{n+1,1}\left(z-C_1\cdot\frac{x}{K_2},1,1\right)+$$
$$\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\binom{n}{m}\cdot K_1^{m+1}\cdot K_2^{n-m}\cdot F_{m+1}\left(\frac{x}{K_2},1\right)\cdot \tag{78}$$
$$\int_0^{y/K_1}F_{n-m}\left(\frac{v}{K_1},1\right)\cdot I_{n+1,1}\left(z-C_1\cdot\frac{x}{K_2}-C_2\cdot\frac{v}{K_1},1,1\right)\cdot\frac{dv}{K_1}$$

$$\theta_1(x,y,z)=\kappa\left(z-C_1\frac{x}{K_2}\right)\cdot F_1\left(\frac{x}{K_2},1\right)+$$
$$\sum_{n=0}^{\infty}K_1^{n+1}\cdot F_{n+2}\left(\frac{x}{K_2},1\right)\cdot I_{n+1,1}\left(z-C_1\cdot\frac{x}{K_2},1,1\right)+$$
$$\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\binom{n}{m}\cdot K_1^{m+1}\cdot K_2^{n-m}\cdot F_{m+2}\left(\frac{x}{K_2},1\right)\cdot \tag{79}$$
$$\int_0^{y/K_1}F_{n-m}\left(\frac{v}{K_1},1\right)\cdot I_{n+1,1}\left(z-C_1\cdot\frac{x}{K_2}-C_2\cdot\frac{v}{K_1},1,1\right)\cdot\frac{dv}{K_1}$$

$$\theta_2(x,y,z)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\cdot K_1^{m+1}\cdot K_2^{n-m}\cdot F_{m+1}\left(\frac{x}{K_2},1\right)\cdot \tag{80}$$
$$\int_0^{y/K_1}F_{n-m+1}\left(\frac{v}{K_1},1\right)\cdot I_{n+1,1}\left(z-C_1\cdot\frac{x}{K_2}-C_2\cdot\frac{v}{K_1},1,1\right)\cdot\frac{dv}{K_1}$$

The equations (78)-(80) are analytical expressions for temperature fields of fluids 1 and 2 and separating wall of cross heat exchanger dependant on time. At the beginning, the inlet temperature of fluid 1 is instantly raised from 0 to 1, and flow velocities of both fluids are constant.

Outlet temperatures of both fluids are obtained by integrating temperatures along outlet edges of the heat exchanger. This is how outlet temperatures become equal;

$$\bar{\theta}_1''(z)=\frac{1}{b}\int_0^b\theta_1(a,y,z)\cdot dy \tag{81}$$

$$\bar{\theta}_2''(z)=\frac{1}{a}\int_0^a\theta_2(x,b,z)\cdot dx\ , \tag{82}$$

where a = NTU and b = ω ·NTU.

Substituting equations (79) and (80) in equations (81) and (82) generates accurate explicit expressions for mean outlet temperatures:

$$\begin{aligned}\bar{\theta}_1''(z) = &\kappa\left(z - C_1\frac{a}{K_2}\right)\cdot F_1\left(\frac{a}{K_2},1\right) + \\ &\sum_{n=0}^{\infty} K_1^{n+1}\cdot F_{n+2}\left(\frac{a}{K_2},1\right)\cdot I_{n+1,1}\left(z - C_1\cdot\frac{a}{K_2},1,1\right) + \\ &\frac{1}{b}\cdot\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\binom{n}{m}\cdot K_1^{m+2}\cdot K_2^{n-m}\cdot F_{m+2}\left(\frac{a}{K_2},1\right)\cdot \\ &\int_0^{b/K_1}\frac{b-v}{K_1}\cdot F_{n-m}\left(\frac{v}{K_1},1\right)\cdot I_{n+1,1}\left(z - C_1\cdot\frac{a}{K_2} - C_2\cdot\frac{v}{K_1},1,1\right)\cdot\frac{dv}{K_1}\end{aligned} \quad (83)$$

$$\begin{aligned}\bar{\theta}_2''(z) = &\frac{1}{a}\cdot\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\cdot K_1^{m+1}\cdot K_2^{n-m}\cdot \\ &\int_0^{a/K_2}\int_0^{y/K_1} F_{m+1}\left(\frac{u}{K_2},1\right)\cdot F_{n-m+1}\left(\frac{v}{K_1},1\right)\cdot I_{n+1,1}\left(z - C_1\cdot\frac{u}{K_2} - C_2\cdot\frac{v}{K_1},1,1\right)\cdot\frac{du}{K_2}\frac{dv}{K_1}\end{aligned} \quad (84)$$

Above solutions are also valid for the case of indefinite fluid velocities ($C_1 = C_2 = 0$).

4. Calculation results

The main purpose of this paper is to provide exact analytical solutions by which performances of parallel, counter and cross flow heat exchangers can be calculated and compared. Many parameters are involved in temperature distributions of both fluids and the wall and, therefore, it is not possible to present quantitative influences of all these parameters in this paper. However, there is enough space to give particular results showing main characteristics of solutions.

Programming of equations expressing temperature fields and outlet temperatures for considered types of heat exchangers can be very tiresome. Therefore, the website www.peec.uns.ac.rs presents programs in MS EXCEL for calculations. Programs can be modified and improved as required.

The example of a heat exchanger where NTU = 1, $\omega = 0.5$, $K_1 = 0.25$ ($K_2 = 1 - K_1 = 0.75$), $C_1 = 4.0$ and $C_2 = 0.5$ will be discussed below. The temperature distributions of both fluids and the wall of PARALLEL flow heat exchanger are plotted versus dimensionless heat exchanger length (distance x) for z = 2 and 4 in Figure 4.

The occurrence of heating up of separating wall and fluid 1 by fluid 2 is typical for parallel flow heat exchanger. This can happen at the beginning of a non-steady state process when the velocity of the fluid 2 flow is higher than the velocity of fluid 1. This will be explained somewhat later when comparing outlet temperatures for all three types of heat exchangers.

The Figure 5 shows temperature distribution for the COUNTER flow heat exchanger. The parameters of this heat exchanger are the same as for the parallel one. Differences of temperature distribution between parallel and counter flow heat exchangers are evident.

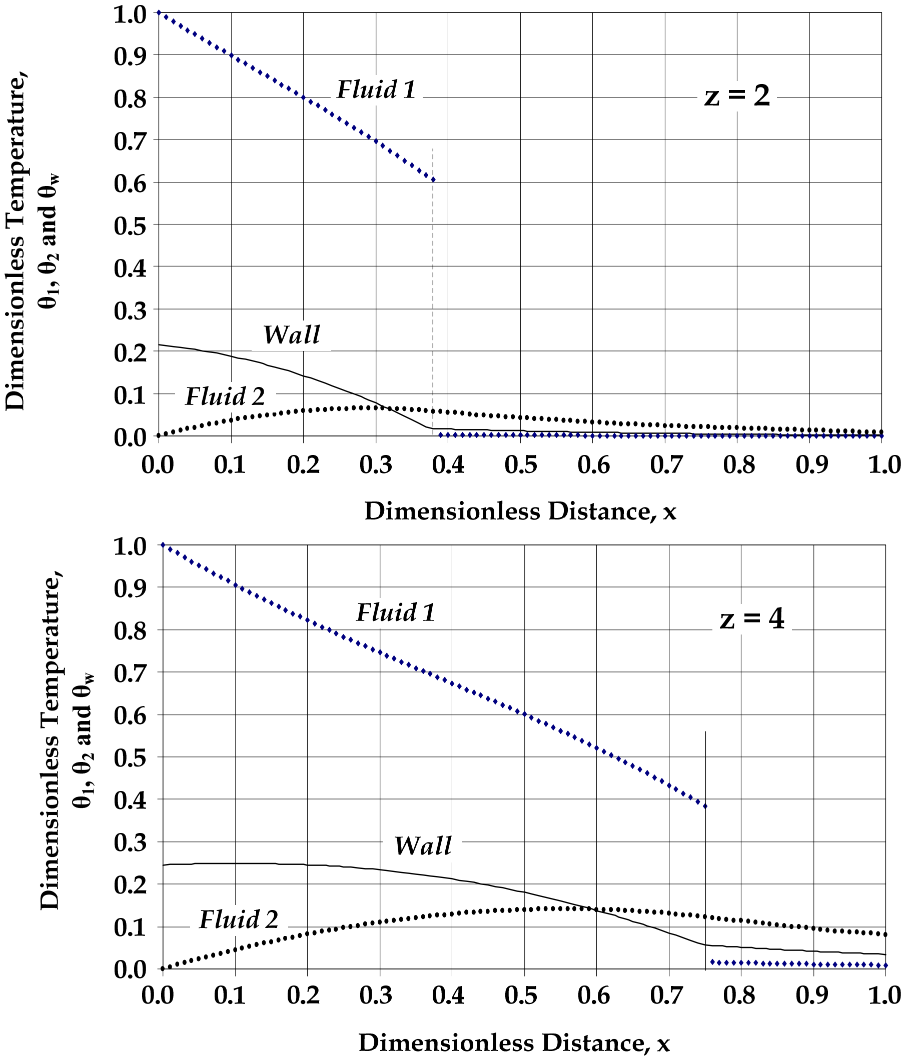
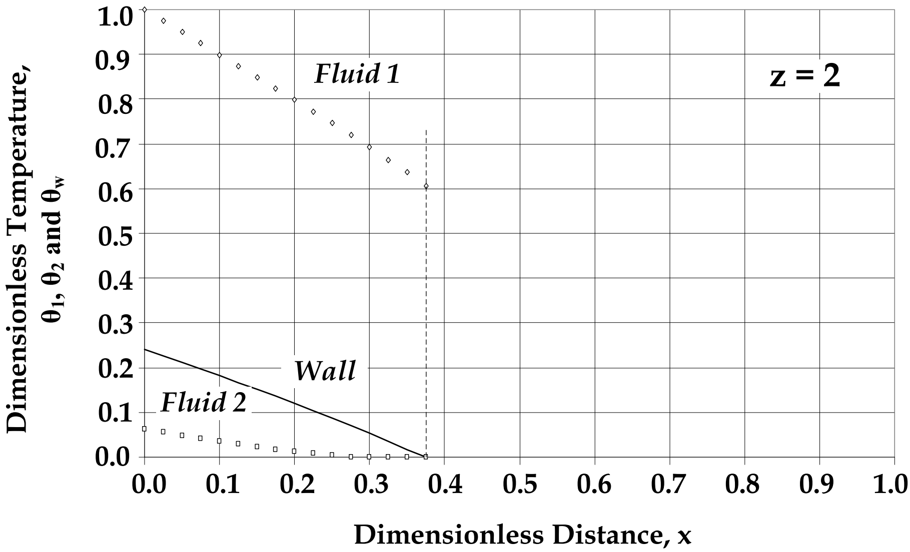


Fig. 4. Temperature Distribution of Both Fluids and the Wall of Parallel Flow Heat Exchanger for z = 2 and 4.



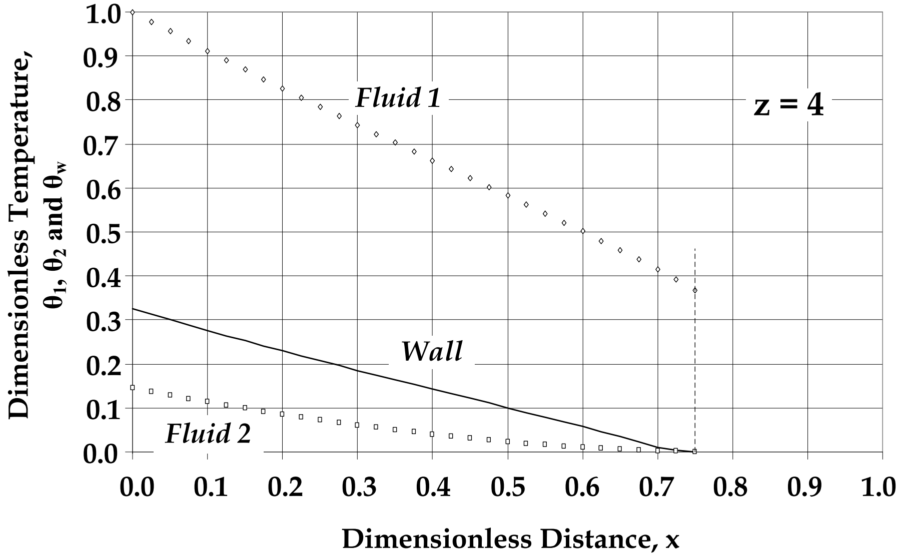


Fig. 5. Temperature Distribution of Both fluids and the Wall of Counter Flow Heat Exchanger for z = 2 and 4.

As an example of the use of presented solutions for cross flow heat exchanger, temperature fields for both fluids and separating wall are given for the same case (NTU = 1, ω = 0.5, K_1 = 0.25, C_1 = 4, and C_2 = 0.5). Temperature fields of both fluids and the wall are shown for dimensionless lengths of heat exchangers at dimensionless time z = 6 (Figure 6). At the time z = 6, the front of both fluids has left boundaries of the heat exchanger. Along the outlet fluid edge, wall temperature has been significantly raised but wall temperature along the outlet edge of the fluid 1 is very modest. The perturbation of the fluid 1 has just left the outlet edge of the heat exchanger. For the fluid 2, the perturbation has moved far away from the outlet edge. Since the the perturbation front of the fluid 1 has just left the outlet edge of the heat exchanger, wall temperature at this edge are low. The same conclusion is also valid for fluid 2 temperature. However, it should be noted that the strength of the fluid 2 flow is two times higher that the strength of the fluid 1.

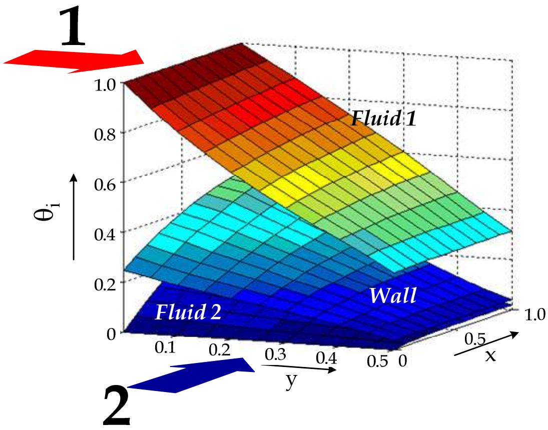


Fig. 6. Temperature Fields of Both Fluids and The wall of CROSS Flow Heat Exchanger for z = 2 and 4.

Fig. 7 shows outlet temperatures of both fluid flows for all three types of heat exchangers. The size of these three heat exchangers is NTU = 1.0 and ω = 0.5. The characteristics of transient heat are also equal for all three types of heat exchangers and they are defined by $K_1 = 0.25$, i.e., $K_2 = 1 - K_1 = 0.75$. The velocity of fluid flow 1 ($C_1 = 4.0$, i.e., $U_1 \sim 1/C_1$) is less than the velocity of the flow 2 ($C_2 = 0.5$, i.e., $U_2 \sim 1/C_2$). This means that the fluid 1 flows longer through the flow channels than fluid 2. In the analyzed case, the ratio of fluid velocities is $U_1/U_2 = 0.04167$. For the fluid 2, the time from z=0 to 1 is necessary to pass the whole length of the heat exchanger at its side of the separating wall. The time z = 5.33 is required for the fluid 1.

The change curve of outlet temperature of fluid 2 is continuous for all three cases (Fig. 7). It is logical that the highest outlet temperature is achieved in the counter flow heat exchanger for which the effectiveness (steady-state) is also the highest for the same values of NTU and ω. It is followed by the cross flow and then by the parallel flow heat exchanger as the worst among the three. In all cases, the final outlet temperature $(z \to \infty)$ is equal to $= 1 - \varepsilon(NTU, \omega, \mathit{flow\ arrangement})$. Also, in transient regime, differences regarding the quality of exchangers are retained.

It is opposite for the fluid 1. The lowest temperature is obtained for the counter flow heat exchanger and the highest for the parallel one. Final outlet temperatures are equal

$= \omega \cdot \varepsilon(NTU, \omega, flow\ arrangement)$. It is logical that the outlet temperature of the fluid 1 is a discontinued function. After the step unit increase of the temperature of the fluid 1 at z = 0, the temperature of the fluid 1 falls due to heating of the wall of the heat exchanger and then heating of the fluid 2. However, in the case of the parallel flow heat exchanger, in the beginning after perturbation, the outlet temperature of the fluid 1 grows even before the perturbation reaches the outlet of the exchanger. This means that at one time of the non-steady state part of the process, the fluid 2 heats up the flow of the fluid 1, as well as the wall instead of vice versa. Namely, ahead of the front, there is the fluid flow 2 heated up by the fluid flow 1. Since the velocity of the fluid flow 2 is higher than the velocity of the fluid flow 1 therefore, it heats up later non-perturbed part of the flow 1 which is ahead of the moving front of the perturbation. By all means, this indicates that before the occurrence of the perturbation all non-dimensionless temperatures are equal to zero (initial condition). After the time z = 5.33, the perturbation of the fluid 1 has reached the outlet edge of the exchanger which is registered by the step change of the outlet temperature. In case of the cross and counter flow heat exchangers, there is not heating up of the fluid flow 1 ahead of the perturbation front (Fig. 7). The fluid flow 1 cools down in the beginning by heating up the wall of the heat exchanger and the part of the fluid flow 2 in case of the cross flow heat exchanger and the whole fluid flow 1 in the case of counter flow but, it cannot happen that the fluid flow 2 gets ahead of the perturbation front and causes a reversal process of the heat transfer which is possible in case of the parallel flow heat exchanger.

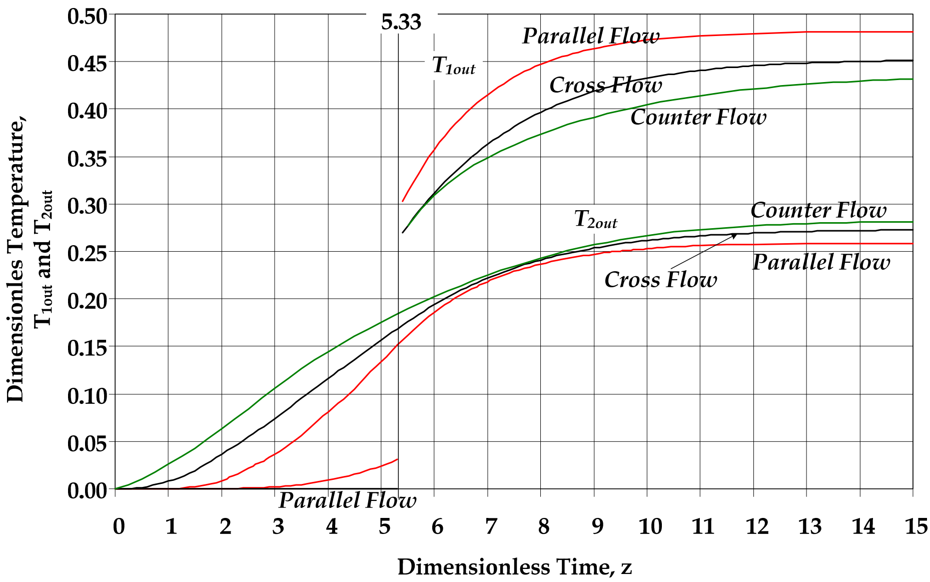


Fig. 7. Outlet Temperature of Both Fluids for Parallel, Counter and Cross Flow Heat Exchangers.

5. Conclusion

A method providing exact analytical solutions for transient response of parallel, counter and cross flow heat exchangers with finite wall capacitance is presented. Solutions are valid in the case where velocities are different or equal. These solutions procedure provides necessary basis for the study of parameters estimated, model discriminations and control of all analyzed heat exchangers.

Generally speaking, the analytical method is superior to numerical techniques because the final solution also preserves physical essence of the problem. Testing of solutions given in this paper indicates that they can be used in practice efficiently when designing and managing processes with heat exchangers.

6. Appendix

Functions $F_n(x,c)$ and $I_{n,m}(x,c,d)$ and their Laplace transforms are given as described below (x ≥ 0, $-\infty < c,d < \infty$, and n, m = 1, 2, 3,....). For x < 0, both functions are equal to zero.

$$F_n(x,c) = \frac{x^{n-1}}{(n-1)!} \cdot \exp(-c \cdot x) \Leftrightarrow \frac{1}{(s+c)^n} \tag{A.1}$$

$$I_{n,m}(x,c,d) = \sum_{j=1}^{\infty} \binom{m+j-1}{j} \cdot d^j \cdot F_{n+m+j}(x,c,d) \Leftrightarrow \frac{1}{(s+c)^n \cdot (s+c-d)^m} \tag{A.2}$$

Some additional details about these functions can be found in an earlier paper (Gvozdenac, 1986).

7. Nomenclature

A_1, A_2	total heat transfer area on sides 1 and 2 of a heat exchanger, respectively, [m^2]
F_1, F_2	cross-section area of flow passages 1 and 2, respectively, [m^2]
c_p	isobaric specific heat of fluid, [J/(kg K)]
c_w	specific heat of core material, [J/(kg K)]
h	heat transfer coefficient between fluid and the heat exchanger wall, [W/(K m^2)]
M_w	mass of heat exchanger core, [kg]
$\dot{m}$	mass flow rate, [kg/s]
NTU	number of heat transfer units, [-] (Eq.)
T	temperature, [K]
t	time, [s]
W	thermal capacity rate of fluid, $= \dot{m} \cdot c_p$, [W/K]
W_{min}	lesser of W_1 and W_2, [W/K]
X, Y	distance from fluid entrances, [m]
U	fluid velocity, [m/s]
ρ	density, [kg/m^3]
κ	unit step function
θ	dimensionless temperature
x, y, z	dimensionless independent variables, (Eqs.)

Subscripts:

1 fluid 1
2 fluid 2
w wall

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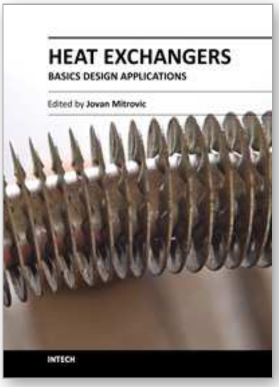

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Selecting and bringing together matter provided by specialists, this project offers comprehensive information on particular cases of heat exchangers. The selection was guided by actual and future demands of applied research and industry, mainly focusing on the efficient use and conversion energy in changing environment. Beside the questions of thermodynamic basics, the book addresses several important issues, such as conceptions, design, operations, fouling and cleaning of heat exchangers. It includes also storage of thermal energy and geothermal energy use, directly or by application of heat pumps. The contributions are thematically grouped in sections and the content of each section is introduced by summarising the main objectives of the encompassed chapters. The book is not necessarily intended to be an elementary source of the knowledge in the area it covers, but rather a mentor while pursuing detailed solutions of specific technical problems which face engineers and technicians engaged in research and development in the fields of heat transfer and heat exchangers.