

# Beyond Trilateration: GPS Positioning Geometry and Analytical Accuracy

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## 1. Introduction

Trilateration/multilateration is the fundamental basis for most GPS positioning algorithms. It begins by finding range estimates to known satellite positions which provides a spherical Locus of Position (LOP) for the receiver. Ideally four such spherical LOPs can be solved to precisely determine the receiver position. Thus, it is an analytical approach that finds receiver position by solving required number of linear/quadratic equations. This method can determine the receiver position precisely when the equations are perfectly formulated. However, determining the exact range is nearly impossible in real-life due to many external factors such as noise interference, signal fading, multi-path propagation, weather condition, clock synchronization problem etc (Strang & Borre, 1997). Hence, trilateration fails to achieve sufficient accuracy under real world conditions. It is also argued that GPS algorithms are not at all tri/multi-lateration rather they are difference of measurement (time-difference or second order difference of two ranges) based hyperbolic formulations (Chaffee & Abel, 1994). However, there are widely used useful range-based algorithms such as Bancroft (1985) method. Therefore, trilateration is still predominantly associated with positioning (Bajaj et al., 2002).

In this chapter, we first discuss about the analytical accuracy of trilateration based positioning algorithms. Subsequently, we show how noise can impact positioning accuracy in real world. In Section 3, we present existing analytical algorithms for GPS along with two new analytical approaches using Paired Measurement Localization (PML) of (Rahman & Kleman, 2009). PML approaches can cope up with practical improper range based equations and are computationally efficient for implementation by conventional and resource constraint GPS receivers. Section 4 draws some conclusions for this chapter.

## 2. Trilateration: its problems and alternative approaches

As alluded before, analytical approaches of positioning are based on accurate distance measurement from geo-stationary satellites. Trilateration is the basis of these techniques where the range measurements from  $n + 1$  satellites are used for an  $n$ -dimensional position estimation (Caffery, 2000).

In the ideal scenario when we can measure the precise range estimates of the GPS receiver, we can formulate a spherical locus of position for the receiver. The fundamental positioning geometry using three satellites placed in a hypothetical 2-Dimensional space is shown in Fig. 1(a).

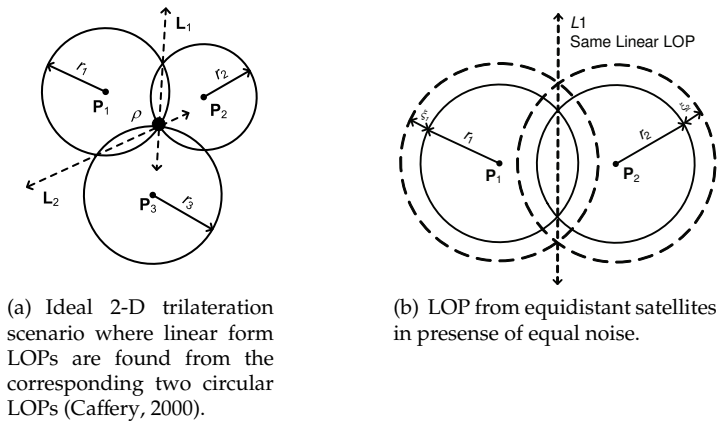


Fig. 1. Depiction of observation 1.

The circles surrounding the satellites with known positions  $\mathbf{p}_1(x_1, y_1)$ ,  $\mathbf{p}_2(x_2, y_2)$  and  $\mathbf{p}_3(x_3, y_3)$ , denote the LOPs obtained from the individual range measurements for each satellite. Ideally, the LOPs surrounding satellite  $i$  is given by,

$$r_i^2 = \|\mathbf{p}_i - \rho\|^2 = ((x - x_i)^2 + (y - y_i)^2) \tag{1}$$

In 2-D, it is feasible to calculate the exact receiver position using only three range measurements. Two range measurements can result in two solutions corresponding to the intersection of two circular LOPs. The third measurement resolves this ambiguity.

However, equating two circular LOPs will result in a straight line equation (in case of 3-D, it will be planar equation) passing through two intersecting points of the circular LOPs. This line does not represent the actual locus of the receiver position as it will be clarified later. However, following (Caffery, 2000) this line is referred as *Linear Form LOP* in the subsequent discussions. In Fig. 1,  $L_1$  and  $L_2$  are determined from the circular LOPs corresponding to satellite pairs  $(\mathbf{p}_1, \mathbf{p}_2)$  and  $(\mathbf{p}_1, \mathbf{p}_3)$  respectively, with the intersection point  $(x, y)$  of  $L_1$  and  $L_2$  denoting the actual position of the receiver.

As shown in Fig. 1, the positioning geometry works correctly for ideal case of exact range estimates being measured by the positioning devices. However, in reality it is quite difficult to measure the exact range both for external noise impact and internal errors such as receiver clock bias and satellite clock skews. However, we also showed the fact that accurate positioning can be obtained if the noise effect is exactly the same for two satellites. However, in case of variable noise presence for two satellite range estimates usual linear form LOP obtained from circular LOPs deviates significantly from the true position of the receiver and leads to a bad positioning geometry. This is further explained as follows.

As it is clarified before that the range equations are mostly not accurate in practical scenario. Though trilateration is a mathematical approach and ideally it can find the exact receiver position, however it cannot find the position very well when the range estimates are perturbed by noise. In this section we will specifically identify the problems of trilateration for inaccurate range equations. For the ease of understanding we still limit this discussion for 2-dimensions only.

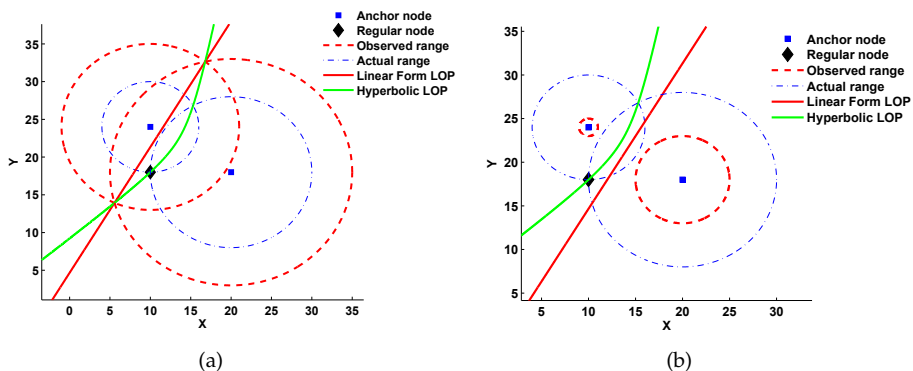


Fig. 2. The hyperbolic and linear form LOP of a receiver from range estimates by a pair of satellites under equal noise assumption. (a) The general case when two observed circular LOPs physically intersect. (b) The case when circular LOPs do not intersect due to noise and underestimation of the ranges.

At first, we present the following observation that identifies the case when the conventional trilateration works in consideration of noise.

**Observation 1.** Assuming a receiver uses range estimates from two satellites that are located at the same distance from the receiver and have equal noise components, it is shown below that the locus of positions for that receiver (as the error components vary) is a straight line whose equation is independent of range estimates.

Assume that due to noise, the range measurements for  $\mathbf{p}_1(x_1, y_1)$ ,  $\mathbf{p}_2(x_2, y_2)$  and  $\mathbf{p}_3(x_3, y_3)$  are corrupted to give respective LOPs of radii  $\tilde{r}_1 = r_1 + \xi_1$ ,  $\tilde{r}_2 = r_2 + \xi_2$  and  $\tilde{r}_3 = r_3 + \xi_3$ , where  $\tilde{r}_i, r_i$  represent the observed and actual distance (pseudorange and actual range respectively) between the  $i^{th}$  satellite and receiver respectively and  $\xi_i$  is the measurement noise at the receiver corresponding to the measurement. The circular LOP can then be expressed as:

$$(r_i + \xi_i)^2 = \|\mathbf{p}_i - \rho\|^2 \tag{2}$$

where  $\rho = (x, y)$  is the receiver position to be determined.

Equating the circular LOPs for  $\mathbf{p}_1$  and  $\mathbf{p}_2$  using (2),  $L_1$  becomes:

$$(x_2 - x_1)x + (y_2 - y_1)y = \frac{1}{2} (\|\mathbf{p}_2\|^2 - \|\mathbf{p}_1\|^2 + (r_1 + \xi_1)^2 - (r_2 + \xi_2)^2) \tag{3}$$

where the right hand side becomes independent of range parameters, *i.e.*, measurement values  $\tilde{r}_1$  and  $\tilde{r}_2$  whenever  $\tilde{r}_1 = \tilde{r}_2 \Rightarrow r_1 + \xi_1 = r_2 + \xi_2$ . One particular case is equidistant satellites and equal noise presence when the above condition is fulfilled. ■

The importance of this observation lies in the fact that it eliminates the signal propagation dependent parameters and receiver clock bias under assumed conditions completely. GPS measurements are mostly susceptible to these errors which are both device and environmentally dependent.

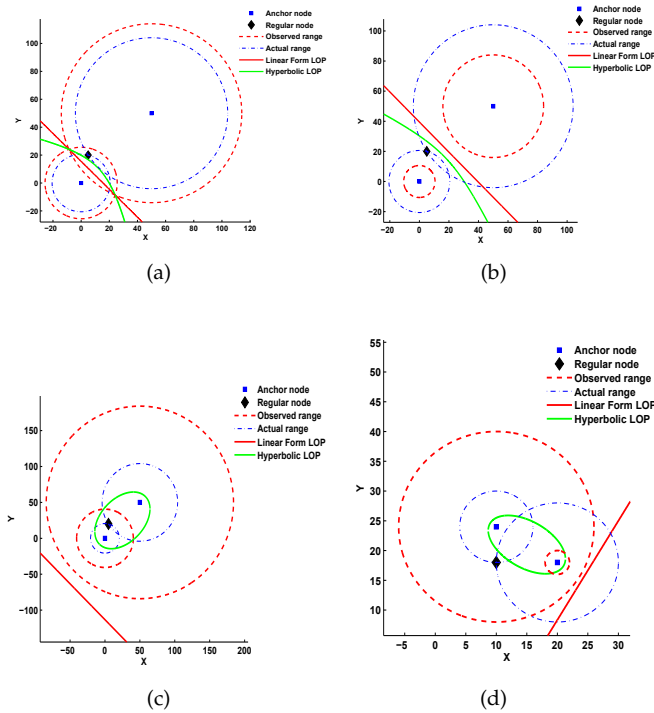


Fig. 3. The hyperbolic and linear form LOPs for unequal noise presence. (a) The general case when two observed circular LOPs physically intersect. (b) The case when observed circular LOPs do not intersect due to underestimation of the ranges. (c) The case when observed circular LOPs do not intersect but overlap completely due to overestimation of the ranges. (d) The case when ranging errors are of opposite signs.

Assuming equal noise presence, it is useful to explore paired measurements rather than individual ranges to mitigate the effect of noise. As the difference of the range estimates equate to actual difference for equal noise presence (e.g.,  $\tilde{r}_2 - \tilde{r}_1 = r_2 - r_1$ ), the LOP for the receiver position is found by the locus of positions maintaining constant difference from the pair of satellites. Hence, the hyperbolic LOP of the receiver can be found independent of the noise parameters as shown in Fig. 2 and formulated below:

$$\sqrt{(x - x_2)^2 + (y - y_2)^2} - \sqrt{(x - x_1)^2 + (y - y_1)^2} = (\tilde{r}_2 - \tilde{r}_1) \quad (4)$$

After algebraic manipulations, it takes the general hyperbolic form as follows for  $\mathbf{p}_1 = (0, 0)$ ,  $\mathbf{p}_2 = (a, 0)$ , and  $\tilde{r}_1 - \tilde{r}_2 = c$ .

$$\left(x - \frac{a}{2}\right)^2 - \frac{y^2}{\left(\frac{a^2}{c^2} - 1\right)} = \frac{c^2}{4} \quad (5)$$

The hyperbolic LOP represents the actual LOP for a pair of satellites under the equal noise assumption. The linear form LOP does not truly represent the locus of the receiver in presence of noise unless both ranges to the satellites are equal as clarified in Fig. 2. Two possible cases could arise due to equal noise presence: *a)* the circular ranges have a physical intersection and *b)* the circular ranges do not have any physical intersection. In both cases, the hyperbolic LOP is able to represent the original receiver position whereas linear form LOP deviates from receiver position significantly. As establishing the LOP is the first step in positioning, any error present at this step could aggravate the result significantly and hence finding a LOP closer to the original receiver position is fundamental to achieving high accuracy positioning.

It is also crucial to compare the hyperbolic and linear form LOPs for unequal noise components in individual measurements as in reality this assumption can be void. In these general situations three possible cases could arise. *a)* the observed circular ranges have a physical intersection; *b)* the observed circular ranges do not have any common intersection region; and *c)* One of the observed circular ranges overlap completely within the other circular region.

These three cases are shown in Fig. 3 where Fig. 3(a), (b) shows the hyperbolic and linear form LOPs for noise ratio ( $\xi_1/\xi_2$ ) of 2 while Fig. 3(c) shows the LOPs for noise ratio of 4. Fig. 3(c) also shows that for completely overlapped ranges the hyperbolic formulation turns into elliptic formulation. This is the case when coefficient of  $y^2$  in (5) changes sign as the range difference becomes greater than distance between the satellites ( $c > a$ ). The noise presence generally attenuates the signal more than that of ideal propagation scenario causing overestimation of the range. However, it is theoretically possible to imagine the case where range is underestimated due to noise. The simultaneous overestimation and underestimation of ranges is supposed to be the most detrimental for LOP estimation and hence this case is shown in Fig. 3(d). It is evident from the figures that for all the three cases of unequal noise presence as well as for noise having different signs, hyperbolic formulation is better suited than linear form and the impact of noise is less detrimental on hyperbolic LOPs than it is on linear form LOPs.

### 3. Analytical approaches for global positioning

We have discussed about the mathematical basis for positioning and presented the problems of regular trilateration from the viewpoint of noisy measurements. The positioning algorithms for GPS need greater care for noise and often augmented by filtering process to mitigate the effect of noise. However, they still largely depend on basic analytical positioning both for initial estimation and for error correcting/filtering phase. In this chapter, we present the different analytical algorithms for GPS.

We begin with the 3-D analogous formula for equation 2 which represents a sphere.

$$\tilde{r}_i^2 = (r_i + \xi_i)^2 = \|\mathbf{p}_i - \rho\|^2 = ((x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2) \quad (6)$$

A generally acceptable modeling of the ranging error  $\xi_i$  is described by the following equation (Strang & Borre, 1997).

$$\xi_i = I_i + T_i + c(dt_i(t - \tau_i) - dt(t)) - e_i \quad (7)$$

where  $I_i$  is the ionospheric error,  $T_i$  is the tropospheric error,  $c$  is the speed of light,  $dt_i$  is the satellite clock offset,  $dt$  is the receiver clock offset,  $t$  is the receiver time and  $\tau_i$  is the signal propagation time and  $e_i$  represents all other unmodelled error terms.

The equation 6 can be iteratively solved using Newton’s method. However, the iterative approach will be computationally expensive. Moreover, the positioning accuracy will be poor as there is no proper formalism to identify and mitigate the error components.

**3.1 Ordinary trilateration for positioning**

Let  $\mathbb{P}_i = (\mathbf{p}_1^i, \mathbf{p}_2^i)$  be an arbitrary satellite pair, where  $\mathbf{p}_1^i = (x_1^i, y_1^i, z_1^i)$  and  $\mathbf{p}_2^i = (x_2^i, y_2^i, z_2^i)$  represent satellite positions of the  $i^{th}$  pair. Analogous to 2-D linear form LOP of equation 3, a 3-D planar form LOP is found as follows.

$$\begin{aligned} &(x_2^i - x_1^i)x + (y_2^i - y_1^i)y + (z_2^i - z_1^i)z = \\ &\frac{1}{2} (\|\mathbf{p}_2^i\|^2 - \|\mathbf{p}_1^i\|^2 + (r_1^i)^2 - (r_2^i)^2 + 2\xi(r_1^i - r_2^i)) \end{aligned} \tag{8}$$

Where it is assumed that the noise are equal and constant for a particular satellite pair *i.e.*,  $\xi_1 = \xi_2 = \xi$ .

The equation becomes linear in terms of  $x, y, z$  and  $\xi$  if the noise is represented by a single parameter  $\xi$  for all pairs. In that case there are four unknowns in this equation and therefore four equations will be required to solve them. In practicality, the assumption is susceptible for large positioning error and hence iterative refinement approach of the following is rather adopted for real implementations.

**3.2 Iterative least squares estimate**

The iterative approach works by having a preliminary estimate of the receiver position ( $\rho^0 = [x^0 y^0 z^0]^T$ ). Let the rotation rate of the earth be  $\omega$ . The position vectors in the earth centered earth fixed (ECEF) system of the receiver be donated by  $\rho(t)_{ECEF}$  and geo-stationary position vector for satellite  $i$  be denoted by  $\mathbf{p}_i(t)_{geo}$  where the argument  $t$  denotes the dependence on time. The range equation can be written as:

$$r_i = \|R_3(\omega\tau_i)\mathbf{p}_i(t - \tau_i)_{geo} - \rho(t)_{ECEF}\| \tag{9}$$

Where  $R_3$  is the earth’s rotation matrix as defined below.

$$R_3(\omega\tau_i) = \begin{bmatrix} \cos(\omega\tau_i) & \sin(\omega\tau_i) & 0 \\ -\sin(\omega\tau_i) & \cos(\omega\tau_i) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = R_3(\omega\tau_i)\mathbf{p}_i(t - \tau_i)_{geo} \text{ and } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \rho(t)_{ECEF} \tag{10}$$

Now, omitting the refraction terms  $I_i$  and  $T_i$  and linearizing the equation 6, we get

$$-\frac{x_i - x^0}{(\tilde{r}_i)^0} \delta x - \frac{y_i - y^0}{(\tilde{r}_i)^0} \delta y - \frac{z_i - z^0}{(\tilde{r}_i)^0} \delta z + (c dt) = \tilde{r}_i - (\tilde{r}_i)^0 - \epsilon_i = b_i - \epsilon_i \tag{11}$$

where  $b_i$  denotes the correction to the preliminary range estimate.

When more than four observations are available we can compute the correction values  $\langle \delta x, \delta y, \delta z \rangle$  for the preliminary estimate. The least squares formulation can be concisely written as follows.

$$\mathbf{Ax} = \begin{bmatrix} -\frac{x_1-x^0}{r_1} & -\frac{y_1-y^0}{r_1} & -\frac{z_1-z^0}{r_1} & 1 \\ -\frac{x_2-x^0}{r_2} & -\frac{y_2-y^0}{r_2} & -\frac{z_2-z^0}{r_2} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{x_m-x^0}{r_m} & -\frac{y_m-y^0}{r_m} & -\frac{z_m-z^0}{r_m} & 1 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta c dt \end{bmatrix} = \mathbf{b} - \epsilon \tag{12}$$

The least squares solution is

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta c dt \end{bmatrix} = (A^T \Sigma^{-1} A)^{-1} A^T \Sigma^{-1} \mathbf{b} \tag{13}$$

If the code observations are independent and assumed to have equal variance, then the above can be simplified to

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta c dt \end{bmatrix} = (A^T A)^{-1} A^T \mathbf{b} \tag{14}$$

The final position vector can be estimated by  $\rho = [x^0 + \delta x, y^0 + \delta y, z^0 + \delta z]^T$ .

### 3.3 Bancroft's method (least squares solution)

We want to turn positioning into a linear algebra problem. Here is a clever method due to Bancroft (1985) that does some algebraic manipulations to reduce the equations to a least-squares problem. Multiplying things out in equation 6 and using the receiver clock bias  $-b = \xi_i^0 s$  as the only noise parameter, we get

$$x_i^2 - 2x_i x + x^2 + y_i^2 - 2y_i y + y^2 + z_i^2 - 2z_i z + z^2 = r_i^2 - 2r_i b + b^2 \tag{15}$$

Rearranging,

$$(x_i^2 + y_i^2 + z_i^2 - r_i^2) - 2(x_i x + y_i y + z_i z - r_i b) + (x^2 + y^2 + z^2 - r^2) = 0 \tag{16}$$

Let  $\rho = [x, y, z, r]^T$  denote the receiver position vector and  $\mathbf{p}_i = [x_i, y_i, z_i, r_i]^T$  denote the  $i^{th}$  satellite position and range vectors.

Using Lorentz inner product for 4-space defined by:

$$\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 - u_4 v_4$$

Equation 16 can be rewritten as:

$$\frac{1}{2} \langle \mathbf{p}_i, \mathbf{p}_i \rangle - \langle \mathbf{p}_i, \rho \rangle + \frac{1}{2} \langle \rho, \rho \rangle = 0; \tag{17}$$

In order to apply least squares estimation the equations for each satellite are organized as follows:

$$\mathbf{B} = \begin{bmatrix} x_1 & y_1 & z_1 & -r_1 \\ x_2 & y_2 & z_2 & -r_2 \\ \vdots & \vdots & \vdots & \vdots \\ x_m & y_m & z_m & -r_m \end{bmatrix},$$

$$\mathbf{a} = \frac{1}{2} \begin{bmatrix} \langle \mathbf{p}_1, \mathbf{p}_1 \rangle \\ \langle \mathbf{p}_2, \mathbf{p}_2 \rangle \\ \vdots \\ \langle \mathbf{p}_m, \mathbf{p}_m \rangle \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \text{and } \wedge = \frac{1}{2} \langle \rho, \rho \rangle$$

We can now rewrite equation 17 as:

$$\begin{aligned} \mathbf{a} - \mathbf{B}\rho + \wedge \mathbf{e} &= 0 \\ \Rightarrow \mathbf{B}\rho &= \mathbf{a} + \wedge \mathbf{e} \end{aligned} \quad (18)$$

For more than 4 satellites, we can have closed form least squares solution as follows:

$$\rho = \mathbf{B}^+ \mathbf{a} + \wedge \mathbf{e} \quad (19)$$

where  $\mathbf{B}^+ = (\mathbf{B}^T \mathbf{B})^{-1} \mathbf{B}^T$  is the pseudoinverse of Matrix  $\mathbf{B}$ .

However, the solution  $\rho$  involves  $\wedge$  which is defined in terms of unknown  $\rho$ . This problem is avoided by substituting  $\rho$  into the definition of the scalar  $\wedge$  and using the linearity of the Lorentz inner product as follows:

$$\wedge = \frac{1}{2} \langle \mathbf{B}^+ (\mathbf{a} + \wedge \mathbf{e}), \mathbf{B}^+ (\mathbf{a} + \wedge \mathbf{e}) \rangle$$

After rearranging,

$$\wedge^2 \langle \mathbf{B}^+ \mathbf{e}, \mathbf{B}^+ \mathbf{e} \rangle + 2\wedge (\langle \mathbf{B}^+ \mathbf{e}, \mathbf{B}^+ \mathbf{a} \rangle - 1) + \langle \mathbf{B}^+ \mathbf{a}, \mathbf{B}^+ \mathbf{a} \rangle = 0 \quad (20)$$

This is a quadratic equation in  $\wedge$  with coefficients  $\langle \mathbf{B}^+ \mathbf{e}, \mathbf{B}^+ \mathbf{e} \rangle$ ,  $2(\langle \mathbf{B}^+ \mathbf{e}, \mathbf{B}^+ \mathbf{a} \rangle - 1)$ , and  $\langle \mathbf{B}^+ \mathbf{a}, \mathbf{B}^+ \mathbf{a} \rangle$ . All these three values can be computed and we can solve for two possible values of  $\wedge$  using the quadratic equation. If we get the two solutions to this equation  $\wedge_1$  and  $\wedge_2$ , then we can solve for two possible solutions  $\rho_1$  and  $\rho_2$  in equation 19. One of these solutions will make sense, it will be on the surface of the earth (which has a radius of approximately 6371 km), and one will not.

The major advantage of the Bancroft's method is to have a closed form least squares solution for GPS equations. It has the same advantage of least squares approach of using all the available satellites for location estimation. On the contrary, it uses the fundamental equation of spherical ranging that in the course of solution leads to planar form LOPs which are than hyperboloid LOPs. Therefore as discussed before, this method cannot be used for high-accuracy positioning in presence of noise.



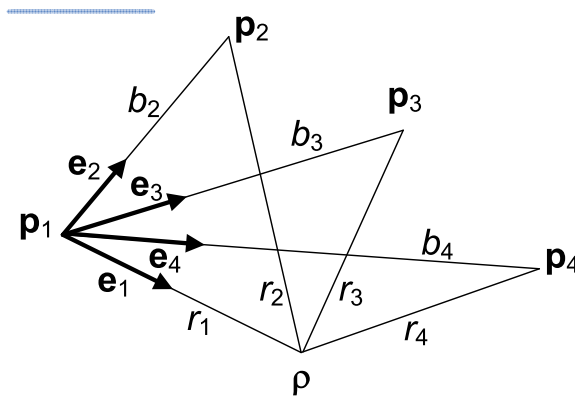


Fig. 4. Three dimensional vector representation for a receiver and four satellites.

**3.4 Kleusberg’s algorithm**

Kleusberg (1994) provided a vector algebraic solution for GPS. The geometry of the 3-D positioning is shown in figure 4. It begins with the fundamental equation 6 for range estimates. It also uses difference equation given below analogous to equation 4 between two satellite measurements.

$$\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2} - \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} = (\tilde{r}_i - \tilde{r}_1) = d_i \quad (21)$$

This represents a sheet of hyperboloid. We can find three such hyperboloids for  $i = 2, 3$  and  $4$  that can be solved for determining the receiver position. Mathematically, there will be two solutions though one of which can be discarded from the knowledge of the earth’s proximity.

Let  $b_2, b_3, b_4$  be the known distances from satellite 1 to satellites 2, 3, 4 along unit vectors  $e_2, e_3, e_4$ . From the cosine law for triangle 1 -  $i$  -  $\rho$ ,

$$\tilde{r}_i^2 = b_i^2 + \tilde{r}_1^2 - 2b_i\tilde{r}_1\mathbf{e}_1 \cdot \mathbf{e}_i \quad (22)$$

Squaring equation 21 and equating with  $\tilde{r}_i^2$  of equation 22, we get

$$2\tilde{r}_1 = \frac{b_i^2 - d_i^2}{d_i + b_i\mathbf{e}_1 \cdot \mathbf{e}_i} \quad (23)$$

Using satellite pairs (1, 2), (1, 3) and (1, 4); we can get three equations for  $\tilde{r}_1$  as follows:

$$\frac{b_2^2 - d_2^2}{d_2 + b_2\mathbf{e}_1 \cdot \mathbf{e}_2} = \frac{b_3^2 - d_3^2}{d_3 + b_3\mathbf{e}_1 \cdot \mathbf{e}_3} = \frac{b_4^2 - d_4^2}{d_4 + b_4\mathbf{e}_1 \cdot \mathbf{e}_4} \quad (24)$$

The only unknown in the above equation is the unit vector  $\mathbf{e}_1$ .

Some rewritings result in the two scalar equations as follows:

$$\begin{aligned} \mathbf{e}_1 \cdot \mathbf{f}_2 &= u_2 \quad \text{and} \\ \mathbf{e}_1 \cdot \mathbf{f}_3 &= u_3 \end{aligned} \quad (25)$$

Where for  $m = 2, 3$ ;

$$\mathbf{F}_m = \frac{b_m}{b_m^2 - d_m^2} \mathbf{e}_m - \frac{b_{m+1}}{b_{m+1}^2 - d_{m+1}^2} \mathbf{e}_{m+1}$$

$$\mathbf{f}_m = \frac{\mathbf{F}_m}{\|\mathbf{F}_m\|}$$

$$u_m = \frac{1}{\|\mathbf{F}_m\|} \left( \frac{d_{m+1}}{b_{m+1}^2 - d_{m+1}^2} - \frac{d_m}{b_m^2 - d_m^2} \right)$$

The unit vector  $\mathbf{f}_2$  lies in the plane through satellites 1, 2 and 3. This plane is spanned by  $\mathbf{e}_2$  and  $\mathbf{e}_3$ . Similarly  $\mathbf{f}_3$  is in the plane determined by satellites 1, 3 and 4.

Equation 25 determines the cosine of the two unit vectors  $\mathbf{f}_2$  and  $\mathbf{f}_3$  with the desired unit vector  $\mathbf{e}_1$ . It will have two solutions, one above and one below the plane spanned by  $\mathbf{f}_2$  and  $\mathbf{f}_3$ . In case these vectors are parallel their inner product is zero and there are infinitely many solutions and hence the position cannot be determined.

The algebraic solution to equation 25 can be derived using vector triple product identity,

$$\mathbf{e}_1 \times (\mathbf{f}_1 \times \mathbf{f}_2) = \mathbf{f}_1 (\mathbf{e}_1 \cdot \mathbf{f}_2) - \mathbf{f}_2 (\mathbf{e}_1 \cdot \mathbf{f}_1)$$

All the terms in the right hand of the above equation is readily computed using  $\mathbf{u}_2, \mathbf{u}_3$ . Substituting  $\mathbf{h}$  for the right hand side and  $\mathbf{g}$  for  $\mathbf{f}_1 \times \mathbf{f}_2$ , we get

$$\mathbf{e}_1 \times \mathbf{g} = \mathbf{h} \quad (26)$$

Multiplying both sides of the equation by  $\mathbf{g}$  and applying the vector triple product identity,

$$\mathbf{e}_1 (\mathbf{g} \cdot \mathbf{g}) - \mathbf{g} (\mathbf{g} \cdot \mathbf{e}_1) = \mathbf{g} \times \mathbf{h} \quad (27)$$

The scalar product in the second term of the left-hand side can be written in terms of the angle  $\theta$  between unit vector  $\mathbf{e}_1$  and  $\mathbf{g}$  as follows

$$\mathbf{g} \cdot \mathbf{e}_1 = [\mathbf{g} \cdot \mathbf{g}]^{\frac{1}{2}} \cos \theta$$

The sine value of the angle can be found from equation 26 as follows:

$$[\mathbf{h} \cdot \mathbf{h}]^{\frac{1}{2}} = [(\mathbf{e}_1 \times \mathbf{g}) \cdot (\mathbf{e}_1 \times \mathbf{g})]^{\frac{1}{2}} = [\mathbf{g} \cdot \mathbf{g}]^{\frac{1}{2}} \sin \theta$$

Using the sine value in the cosine formula above, we obtain,

$$\mathbf{g} \cdot \mathbf{e}_1 = \pm [\mathbf{g} \cdot \mathbf{g}]^{\frac{1}{2}} \left[ 1 - \frac{\mathbf{h} \cdot \mathbf{h}}{\mathbf{g} \cdot \mathbf{g}} \right]^{\frac{1}{2}} = \pm [\mathbf{g} \cdot \mathbf{g} - \mathbf{h} \cdot \mathbf{h}]^{\frac{1}{2}}$$

Substituting the above into equation 27, we obtain the desired solution:

$$\mathbf{e}_1 = \frac{1}{2} (\mathbf{g} \times \mathbf{h} \pm \mathbf{g} \sqrt{\mathbf{g} \cdot \mathbf{g} - \mathbf{h} \cdot \mathbf{h}}) \quad (28)$$

The two values can be put in equation 24 to check the correctness of the value. The correct parameter will result in a intersection point that lies on the earth's surface and hence must have

a distance of about 6371 km from the origin. We can eventually get the receiver coordinate using correct value of  $\mathbf{e}_1$  as follows:

$$\rho = \mathbf{p}_1 + \tilde{r}_1 \mathbf{e}_1 \quad (29)$$

The Kleusberg's method is geometrically oriented and uses a minimum number of satellites. On the other hand, it cannot utilize more number of satellites even when they are available. This method is also dependent on the proper geometrical orientation of the satellites. Moreover, it often gives different results for different set of satellites and depending on the order of the satellites in solving the equations.

### 3.5 Paired measurement localization

In trilateration, the positioning works by simultaneous solution of three spherical LOP equations. Similar to the 2-D steps, we can equate two spherical LOP equations to find equation for a 2-D plane representing the planar locus of position. Analogous to 2-D case, three planar equations can be solved to find the ultimate receiver position.

As shown in section 2, the effect of noise will have detrimental impact on the aforementioned simple solution. On the other hand, instead of equating the two imprecise range equations we can maintain an equi-distant locus of position from two satellites as formulated in equation 21 for a hyperboloid LOP. This will be more accurate than a traditional 2-D planar LOP based positioning.

Solving the nonlinear hyperbolic/hyperboloid equations is difficult. Moreover, existing hyperbolic positioning methods proceed by linearizing the system of equations using either Taylor-series approximation (Foy, 1976; Torrieri, 1984) or by linearizing with another additional variable (Chan & Ho, 1994; Friedlander, 1987; Smith & Abel, 1987). However, while linearizing works well for existing approaches it is not readily adaptable for the proposed paired approach as linearizing is indeed pairing with an arbitrarily chosen hyperbolic LOP. The assumption of equal noise cannot be held for any arbitrary selection of pairs and hence alternate ways to solve such LOPs for paired measurement is now formulated.

### 3.6 PML with single reference satellite

(Chan & Ho, 1994) provided closed form least squares solution for non-linear hyperbolic LOPs by linearizing with reference to a single satellite. Analogous to their approach a closed form solution is found for PML using pairs having a common reference satellite in them. The solution is simpler than (Chan & Ho, 1994)'s approach as the effect of noise is considered early in the paired measurements formulations.

Let  $r_{ij}^{\sim}$  represent the difference in the observed ranges for satellite pairs  $(i, j)$ . In case of equal noise presence it follows:

$$\begin{aligned} \tilde{r}_{ij} &= r_{ij}^{\sim} = r_i - r_j \\ \text{After squaring and rearranging,} \\ r_i^2 &= r_{ij}^2 + 2r_{ij}r_j + r_j^2 \end{aligned} \quad (30)$$

Hence, the actual spherical LOP can be transformed as follows:

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = (r_{ij}^{\sim})^2 + 2r_{ij}^{\sim}r_j + (r_j)^2 \quad (31)$$

Using (31) for pairs  $(\mathbf{p}_i, \mathbf{p}_j) = (\mathbf{p}_k, \mathbf{p}_1)$  and  $(\mathbf{p}_l, \mathbf{p}_1)$  and subtracting the second from the first,

$$-(x_k - x_l)x - (y_k - y_l)y - (z_k - z_l)z - (r_{\widehat{k1}} - r_{\widehat{l1}})r_1 = \frac{1}{2} \left( (r_{\widehat{k1}})^2 - (r_{\widehat{l1}})^2 - \|\mathbf{p}_k\|^2 + \|\mathbf{p}_l\|^2 \right) \quad (32)$$

where  $\|\mathbf{p}_k\|^2 = (x_k^2 + y_k^2)$ . The above formulation represents a set of linear equations with unknowns  $x, y, z$  and  $r_1$  for all combination of two pair of satellites having satellite 1 in common. Let  $x_{\widehat{ij}}, y_{\widehat{ij}}, z_{\widehat{ij}}$  represent the difference  $x_i - x_j, y_i - y_j, z_i - z_j$  respectively,  $C_i$  represent the  $i^{th}$  combination and  $m$  represent the total number of combinations with  $C_i = \{(\mathbf{p}_{k_i}, \mathbf{p}_1), (\mathbf{p}_{l_i}, \mathbf{p}_1)\}$ . The system of linear equations for these  $m$  combinations can be concisely written as follows:

$$\mathbf{A}\mathbf{X} = \mathbf{B} \quad (33)$$

where,

$$\mathbf{A} = - \begin{bmatrix} x_{\widehat{k_1l_1}} & y_{\widehat{k_1l_1}} & z_{\widehat{k_1l_1}} & -(r_{\widehat{k_11}} - r_{\widehat{l_11}}) \\ x_{\widehat{k_2l_2}} & y_{\widehat{k_2l_2}} & z_{\widehat{k_2l_2}} & -(r_{\widehat{k_21}} - r_{\widehat{l_21}}) \\ \cdot & \cdot & \cdot & \cdot \\ x_{\widehat{k_ml_m}} & y_{\widehat{k_ml_m}} & z_{\widehat{k_ml_m}} & -(r_{\widehat{k_m1}} - r_{\widehat{l_m1}}) \end{bmatrix},$$

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ r_1 \end{bmatrix}, \quad \mathbf{B} = \frac{1}{2} \begin{bmatrix} (r_{\widehat{k_11}})^2 - (r_{\widehat{l_11}})^2 - \|\mathbf{p}_{k_1}\|^2 + \|\mathbf{p}_{l_1}\|^2 \\ (r_{\widehat{k_21}})^2 - (r_{\widehat{l_21}})^2 - \|\mathbf{p}_{k_2}\|^2 + \|\mathbf{p}_{l_2}\|^2 \\ \cdot \\ (r_{\widehat{k_m1}})^2 - (r_{\widehat{l_m1}})^2 - \|\mathbf{p}_{k_m}\|^2 + \|\mathbf{p}_{l_m}\|^2 \end{bmatrix}$$

For  $m \geq 3$ , the system of equations can be solved. However,  $r_1$  is related to  $x, y, z$  by (6). For pairing and equivalence of  $\tilde{r}_i - \tilde{r}_1 = r_i - r_1$ , observed ranges are always used in the equations and thus the system of equations are essentially independent of relationship between  $(x, y, z)$  and  $r_1$ . This is also verified by the iterative refinement of  $r_1$  where  $\tilde{r}_1$  is modified by obtained  $r_1$  in successive runs. The results show no difference in position estimates  $(x, y, z)$  for successive iterations.

The equal noise assumption cannot be applied to any arbitrary selection of pairs while it is quite reasonable for satellites observing near equal ranges to have equal noise components. The selection of pairs with near equal ranges from a single reference satellite, may not be feasible for low visibility where only a very few satellites are available for positioning. This is the motivation for the next solution approach.

### 3.7 PML with refinement of the locus of positions

The linearization using one single reference satellite raises a performance issue and while it is superior to trilateration in most of the cases, occasionally it performs worse. In search for a positioning approach that can give consistently better estimates than basic trilateration, a locus refinement approach is now presented.

A refined and better approximation to planar form LOP is found from two imprecise planar form LOPs assuming equal noise presence due to receiver bias and ionospheric error in each pair and for specific instance of measurement as follows.

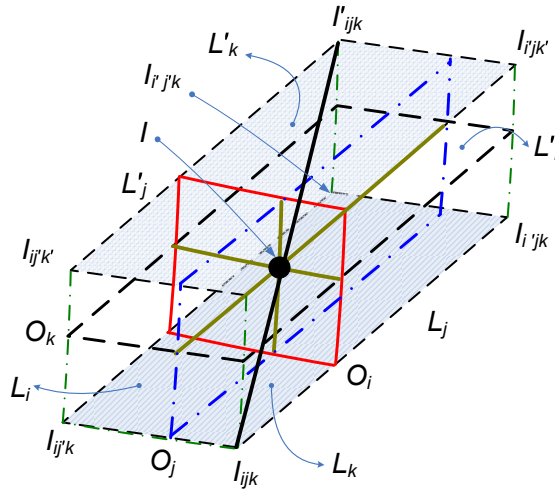


Fig. 5. Refining the locus of the receiver position under noisy measurement conditions.

Fig. 5 shows the ideal scenario where the position of the receiver to be determined,  $\rho$ , and the three respective planar form LOPs  $O_i, O_j$  and  $O_k$  are obtained from any three arbitrary satellite pairs  $\mathbb{P}_i, \mathbb{P}_j$  and  $\mathbb{P}_k$ .

The equation for  $L_i, L_j, L_k$  can be found using (8). For specific measurement instance  $\xi$  is constant due to identical receiver clock bias and exposure to similar atmospheric noise. Hence,  $L_i, L_j, L_k$  vary from the ideal noise free LOPs  $O_i, O_j, O_k$  by the extra constant terms of  $2\xi(r_1^i - r_2^i), 2\xi(r_1^j - r_2^j)$  and  $2\xi(r_1^k - r_2^k)$  respectively. Crucially their slopes remain unchanged (Left hand side of (8)), and these are shown by the solid planes  $L_i, L_j, L_k$  parallel to  $O_i, O_j$  and  $O_k$  in Fig. 5. For non co-planar satellite pairs,  $L_i, L_j$  and  $L_k$  will have a physical intersection point  $\mathbf{I}_{ijk} = (x_{ijk}, y_{ijk}, z_{ijk})$ .

Another plane  $L'_i$  parallel to  $L_i$  can be found as follows by modifying the term  $2\xi(r_1^i - r_2^i)$  with  $-q(r_1^i - r_2^i)$ , where  $q$  is an arbitrary positive constant.

$$\begin{aligned} (x_2^i - x_1^i)x + (y_2^i - y_1^i)y + (z_2^i - z_1^i)z = \\ \frac{1}{2} \left( \|\mathbf{p}_2^i\|^2 - \|\mathbf{p}_1^i\|^2 + (r_1^i)^2 - (r_2^i)^2 - q(r_1^i - r_2^i) \right) \end{aligned} \tag{34}$$

The original LOP  $O_i$  will then pass between the planes  $L'_i$  and  $L_i$  as the constants have opposite signs. A similar argument applies to  $L'_j, L'_k$  so that the parallelepiped bounded by the planes  $O_i, L_i, O_j, L_j, O_k, L_k$  will have an aspect ratio  $AR = (r_1^i - r_2^i : r_1^j - r_2^j : r_1^k - r_2^k)$  as  $L_i, L_j, L_k$  are  $2\xi(r_1^i - r_2^i), 2\xi(r_1^j - r_2^j)$  and  $2\xi(r_1^k - r_2^k)$  distances away from  $O_i, O_j$  and  $O_k$  respectively as they differ only by the constant terms in (8). The AR of the parallelepiped bounded by the planes  $O_i, L'_i, O_j, L'_j, O_k, L'_k$  will have exactly the same aspect ratio so indicating  $\mathbf{I}_{ijk}, \mathbf{I}'_{ijk}$  and  $\mathbf{I}$  to be the

diagonal points of the parallelepiped where  $I'_{ijk}$  denotes the intersection point of planes  $L'_i, L'_j$  and  $L'_k$ .

Hence, the equation of the actual LOP  $I_{ijk}I'_{ijk}$  passing through  $I$  is found from the three intersection points  $I_{ijk}, I'_{i'k}$  and  $I'_{j'k'}$  which are available from equations (8) and (34) and analogous equations for LOPs  $L_j, L_k, L'_j$  and  $L'_k$ .

As the LOPs obtained in this way are expressed by linear equations with unknowns  $x, y$  and  $z$ , they can be solved using simple algebraic or least squares methods.

The locus refinement formulation assumes noise to be present in the formulae. However, if the noise is absent the diagonal points  $I_{ijk}$  and  $I'_{ijk}$  would be very close and during the calculation process whenever pairs having distance  $< 2m$  are observed the estimated location is found as the mean of these two points.

The planar form LOP obtained from each satellite pair must be linearly independent so they do not represent either the same or a parallel planar LOP. Such satellite pairs are referred to as mutually independent, so a key objective is to identify such satellite pairs where each satellite has nearly similar distance from the receiver. PML may be intuitively viewed as positioning exploiting bearing measurements, as LOPs effectively denote a directional line. It is known that angular measurements are consistently more accurate compared to TOF range measurements and in (Chintalapudi et al., 2004) a combination of range and angular measurement has been shown to achieve better positioning results, providing a valuable insight as to why the LOP refinement furnishes better location estimation.

### 3.8 Selection of satellite pairs for PML

It is apparent from observation 1 that the existence of a pair of satellites having equal distance from the receiver position can have equal atmospheric noise exposure, with this prerequisite being relaxed and generalized by LOP refinement approach. Observation 1 highlights the significance of pairing the satellites for better noise cancellation and a better selection process can result in considerable improvement. With practical range estimations there is no explicit way to determine the best possible pairs following the observation. However, the range estimation ratios can be used as a rough measure for adhering to observation 1 which is the basis for the following empirically defined ranking criteria. The ranking criteria also considers the closeness of the satellites. If the two satellites are too close to each other they might have the best range estimation ratio while effectively they are like two satellites placed at the same place and hence providing no additional redundancy to help positioning. Utilizing, the above mentioned two principles the following empirical ranking criteria is introduced.

$$\mathfrak{R} = \frac{\tilde{r}_1}{\tilde{r}_2} \left( \frac{1}{\|\mathbf{p}_1\mathbf{p}_2\|} \right) \quad (35)$$

where  $\tilde{r}_1$  and  $\tilde{r}_2$  are the observed range estimates for satellite pair  $(\mathbf{p}_1, \mathbf{p}_2)$  such that  $\tilde{r}_1 \geq \tilde{r}_2$  and  $\|\mathbf{p}_1\mathbf{p}_2\|$  is the Euclidean distance between the two satellites. The pairs having lower ranks ( $\mathfrak{R}$ ) are preferred over ones with higher ranks. The complete satellite selection algorithm is given as follows.

Algorithm 1 searches all available satellites for a particular receiver so its computational complexity is  $O(\text{available satellites}^2)$  if an exhaustive search is applied. This selection process

**Algorithm 1** Satellite Selection for PML Refinement Approach

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for all pair of neighboring satellites do
  Calculate rank ( $\mathfrak{R}$ ) for the pair ( $\mathbf{p}_i, \mathbf{p}_j$ );
  if ( $\mathbf{p}_i, \mathbf{p}_j$ ) is co-planar with any previous selected pair and ( $\mathfrak{R}$ ) of present pair is lower
  than selected pair then
    Replace the previous co-planar pair with current pair
  else
    if Number of selected pair < Required number of pairs then
      Add the current pair to the selected pairs
    else
      Replace the worst ranking selected pair with the current pair
    end if
  end if
end for

```

---

can be run on-demand only when satellite positions are either changed or after considerable movement of the receiver. Given the small number of visible satellites in range, this will incur negligible cost.

Finally, as the new PML method itself is an analytical approach, the order of computational complexity is  $O(1)$  once satellite selection has been completed.

Summarizing, PML approaches are improvement over basic trilateration in that it considers noisy measurement conditions in its formulation. Thus, this new strategy performs significantly better for real time GPS and tracking performance.

#### 4. Conclusion

This chapter presented a detailed discussion on the analytical approaches for GPS positioning. Trilateration is the basis for most analytical positioning approaches and hence this chapter begins with fundamental discussion on trilateration. However, it performs poorly under noisy conditions which is analyzed in detail from theoretical and simulated scenarios. We also showed how difference of two range measurements can result in better positioning formulations. Subsequently, we present existing analytical approaches of Bancroft's method and Kleusberg's method that uses least squares and vector algebra respectively for solution of GPS equations. Later we present two newer approaches that are based on using better Locus Of Position (LOP) for the receiver than customary spherical locus in presence of noise. The first of these, called Paired Measurement Localization (PML) with single reference satellite uses hyperboloid planar locus of positions. The solution of these non-linear hyperboloids are found by linearizing with reference to a single satellite. The other PML approach obtains a better LOP from ordinary planar LOPs using a LOP refinement technique. Both of the PML based approaches have the advantage that they can utilize all the available satellites using least squares solution. If only four/three LOPs are used for PML single reference or PML LOP refinement respectively, the receiver position can be calculated by simple algebra. This has the advantage of avoiding matrix inversion for least squares solution and particularly suitable when the receiver has constraint computational support such as mobile embedded GPS receivers. Alternatively, when sufficient computational resources are available and better

precision is needed full fledged least squares solution and further filtering techniques could be applied.

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