

# Strength of a Polycrystalline Material

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## 1. Introduction

There are numerous polycrystalline materials, including polycrystals whose crystals have a cubic symmetry. Polycrystals with cubic symmetry comprise minerals and metals such as cubic pyrites ( $\text{FeS}_2$ ), fluorite ( $\text{CaF}_2$ ), rock salt ( $\text{NaCl}$ ), sylvite ( $\text{KCl}$ ), iron (Fe), aluminum (Al), copper (Cu), and tungsten (W) (Love, 1927; Vainstein et al., 1981).

It is assumed that many materials can be treated as a homogeneous and isotropic medium independently of the specific characteristics of their microstructure. It is clear that, in fact, this is impossible already because of the molecular structure of materials. For example, materials with polycrystalline structure, which consist of numerous chaotically located small crystals of different size and different orientation, cannot actually be homogeneous and isotropic. Each separate crystal of the metal is anisotropic. But if the volume contains very many chaotically located crystals, then the material as a whole can be treated as an isotropic material. Just in a similar way, if the geometric dimensions of a body are large compared with the dimensions of a single crystal, then, with a high degree of accuracy, one can assume that the material is homogeneous (Feodos'ev, 1979; Timoshenko & Goodyear, 1951).

On the other hand, if the problem is considered in more detail, then the anisotropy both of the material and of separate crystals must be taken into account. For a body under the action of external forces, it is impossible to determine the stress-strain state theoretically with its polycrystalline structure taken into account.

Assume that a body consists of crystals of the same material. Moreover, in general, the principal directions of elasticity of neighboring crystals do not coincide and are oriented arbitrarily. The following question arises: Can stress concentration exist near a corner point of the interface between neighboring crystals and near and edge of the interface?

To answer this question, it is convenient to replace the problem under study by several simplified problems each of which can reflect separate situations in which several neighboring crystals may occur.

A similar problem for two orthotropic crystals having the shape of wedges rigidly connected along their jointing plane was considered in (Belubekyan, 2000). They have a common vertex, and their external faces are free. Both of the wedges consist of the same material. The wedges have common principal direction of elasticity of the same name, and the other elastic-equivalent principal directions form a nonzero angle. We consider longitudinal shear (out-of-plane strain) along the common principal direction.

In (Belubekyan, 2000), it is shown that if the joined wedges consist of the same orthotropic material but have different orientations of the principal directions of elasticity with respect to their interface, then the compound wedge behaves as a homogeneous wedge.

The behavior of the stress field near the corner point of the contour of the transverse cross-section of the compound body formed by two prismatic bodies with different characteristics which are welded along their lateral surfaces was studied in the case of plane strain in (Chobanyan, 1987). It was assumed there that the compound parts of the body are homogeneous and isotropic and the corner point of the contour of the prism transverse cross-section lies at the edge of the contact surface of the two bodies.

In (Chobanyan, 1987; Chobanyan & Gevorkyan 1971), the character of the stress distribution near the corner point of the contact surface is also studied for two prismatic bodies welded along part of their lateral surfaces. The plane strain of the compound prism is considered.

There are numerous papers dealing with the mechanics of contact interaction between strained rigid bodies. The contact problems of elasticity are considered in the monographs (Alexandrov & Romalis, 1986; Alexandrov & Pozharskii 1998). In (Alexandrov & Romalis, 1986), exact or approximate analytic solutions are obtained in the form convenient to be used directly to verify the contact strength and rigidity of machinery elements. The monograph (Alexandrov & Pozharskii 1998) presents numerical-analytical methods and the results of solving many nonclassical spatial problems of mechanics of contact interaction between elastic bodies. Isotropic bodies of semibounded dimensions (including the wedge and the cone) and the bodies of bounded dimensions were considered. The monograph presents a vast material developed in numerous publications. There are also many studies in this field, which were published in recent years (Ulitko & Kochalovskaya, 1995; Pozharskii & Chebakov, 1998; Alexandrov & Pozharskii, 1998, 2004; Alexandrov et al., 2000; Osrtik & Ulitko, 2000; Alexandrov & Klindukhov, 2000, 2005; Pozharskii, 2000, 2004; Aleksandrov, 2002, 2006; Alexandrov & Kalyakin, 2005).

In the present paper, we study the problem of existence of stress concentrations near the corner point of the interface between two joined crystals with cubic symmetry made of the same material.

## 2. Statement of the problem

We assume that there are two crystals with rectilinear anisotropy and cubic symmetry, which are rigidly connected along their contact surface (Fig. 1). The crystal contact surface forms a dihedral angle with linear angle  $\alpha$  whose trace is shown in the plane of the drawing. The contact surface edge passes through point  $O$ . The  $z$ -axis of the cylindrical coordinate system  $(r, \varphi, z)$  coincides with the edge of the dihedral angle. The coordinate surfaces and  $\varphi = 0$  and  $\varphi = \alpha$  ( $\varphi = \alpha - 2\pi$ ) coincide with the faces of the dihedral angle. Thus, the first crystal (1) occupies the domain  $\varphi \in [0; \alpha]$  and the second crystal (2) occupies the domain  $\varphi \in [\alpha - 2\pi; 0]$ . In this case  $0 < \alpha < 2\pi$  and  $0 < r < \infty$ .

For simplicity, we assume that the crystals have a single common principal direction of elasticity coinciding with the  $z$ -axis. The other two principal directions  $x_1$  and  $y_1$  of the first crystal make some nonzero angles with the principal directions  $x_2$  and  $y_2$  of the

second crystal. By  $\theta_1$  we denote the angle between  $x_1$  and the polar axis  $\varphi=0$ , and by  $\theta_2$ , the angle between  $x_2$  and the axis  $\varphi=0$ . In this case,  $\theta_1, \theta_2 \in [\alpha-2\pi, \alpha]$ . If  $\theta_1 = \theta_2 = 0$ , then we have a homogeneous medium, i.e., a monocrystal with cubic symmetry, one of whose principal directions  $x_1 = x_2 = x$  coincides with the polar axis  $\varphi=0$ . In this case, the equations of generalized Hooke's law written in the principal axes of elasticity  $x, y, z$  have the form

$$\begin{aligned} \varepsilon_x &= a_{11}\sigma_x + a_{12}(\sigma_y + \sigma_z), & \gamma_{yz} &= a_{44}\tau_{yz}, \\ \varepsilon_y &= a_{11}\sigma_y + a_{12}(\sigma_z + \sigma_x), & \gamma_{zx} &= a_{44}\tau_{zx}, \\ \varepsilon_z &= a_{11}\sigma_z + a_{12}(\sigma_x + \sigma_y), & \gamma_{xy} &= a_{44}\tau_{xy}, \end{aligned} \tag{1}$$

where  $\varepsilon_x, \varepsilon_y, \dots, \gamma_{xy}$  are the strain components,  $\sigma_x, \sigma_y, \dots, \tau_{xy}$  are the stress components, and  $a_{11}, a_{12}, a_{44}$  are the strain coefficients.

Equations (1) can be obtained from the equations of generalized Hooke's law for an orthotropic body written in the principal axes of elasticity  $x, y, z$ , using the method described in (Lekhnitskii, 1981).

Rotating the coordinate system  $(x, y, z)$  about the common axis  $z = z'$  by the angle  $\varphi = 90^\circ$ , we obtain a symmetric coordinate system  $(x', y', z')$ . Since the directions of the axes  $x, y, z$  and  $x', y', z'$  of the same name are equivalent with respect to their elastic properties, the equations of generalized

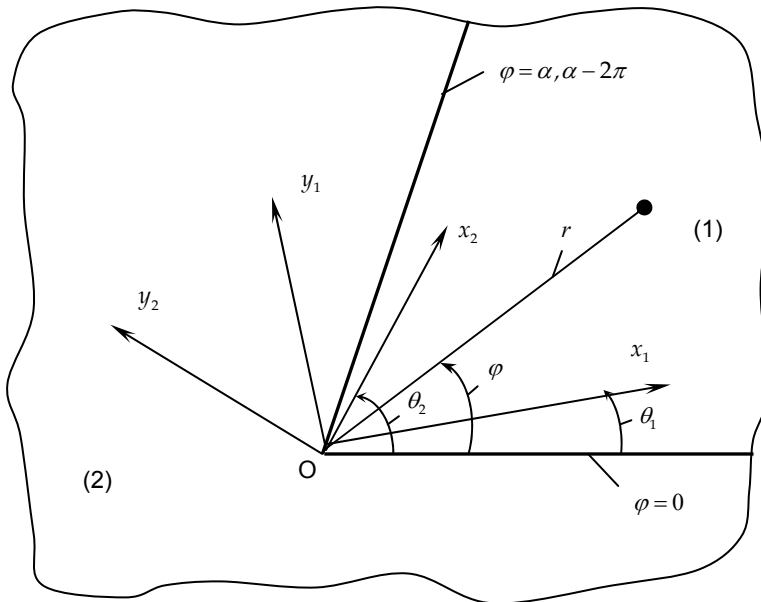


Fig. 1.

Hooke's law for these coordinate systems have the same form. In this case, the values of the strain coefficients are the same in both systems:  $a'_{11} = a_{11}$ ,  $a'_{12} = a_{12}$ ,  $a'_{13} = a_{13}$ , ...,  $a'_{66} = a_{66}$ .

Using the formulas of transformation of strain coefficients under the rotation of the coordinate system about the axis  $z = z'$  (Lekhnitskii, 1981), we obtain their new values expressed in terms of the old values (before the rotation of the coordinate system  $(x, y, z)$ ).

Comparing the strain coefficients in the same coordinate system  $(x', y', z')$ , we obtain,  $a_{11} = a_{12}$ ,  $a_{44} = a_{55}$ ,  $a_{13} = a_{23}$ ,  $a_{16} = a_{45} = a_{26} = a_{36} = 0$ .

Successively rotating the coordinate system  $(x, y, z)$  about the axes  $x$  and  $y$  by the angle  $90^\circ$  and repeating the same procedure, we finally obtain (1).

The transformation formulas for the strain coefficients under the rotation of the coordinate system about the  $x$ - and  $y$ -axes can also be obtained from the transformation formulas for the strain coefficients under the rotation of the coordinate system about the  $z$ -axis in the case of anisotropy of general form.

For example, to obtain the transformation formulas under the rotation of the coordinate system about the  $x$ -axis, it is necessary to rename the principal directions of elasticity as follows: the  $x$ -axis becomes the  $z$ -axis, the  $y$ -axis becomes the  $x$ -axis, and the  $z$ -axis becomes the  $y$ -axis. In this case, in the equations of generalized Hooke's law referred to the coordinate system  $(x, y, z)$ ,  $a_{22}$  plays the role of  $a_{11}$ ,  $a_{23}$  plays the role of  $a_{12}$ , and  $a_{24}$  plays the role of  $a_{16}$ . In a similar way, in the equations of generalized Hooke's law referred to the coordinate system  $(x', y', z')$ ,  $a'_{22}$  plays the role of  $a'_{11}$ ,  $a'_{23}$  plays the role of  $a'_{12}$ , and  $a'_{24}$  plays the role of  $a'_{16}$ . This implies that, in the case of an orthotropic body,  $a_{24} = 0$  under rotation of the coordinate system about the  $x$ -axis, but, in contrast to the case of rotation of the coordinate system about the  $z$ -axis,  $a'_{24}$  is generally nonzero.

In the case  $\theta_1 \neq \theta_2$ , the equations of generalized Hooke's law in the cylindrical coordinate system  $(r, \varphi, z)$  have the form

$$\begin{aligned}
 \varepsilon_r^{(i)} &= a_{11} \sigma_r^{(i)} + a_{12} (\sigma_\varphi^{(i)} + \sigma_z^{(i)}) - a [ (\sigma_r^{(i)} - \sigma_\varphi^{(i)}) \sin^2 2\alpha_i + \tau_{r\varphi}^{(i)} \sin 4\alpha_i ], \\
 \varepsilon_\varphi^{(i)} &= a_{11} \sigma_\varphi^{(i)} + a_{12} (\sigma_z^{(i)} + \sigma_r^{(i)}) + a [ (\sigma_r^{(i)} - \sigma_\varphi^{(i)}) \sin^2 2\alpha_i + \tau_{r\varphi}^{(i)} \sin 4\alpha_i ], \\
 \varepsilon_z^{(i)} &= a_{11} \sigma_z^{(i)} + a_{12} (\sigma_r^{(i)} + \sigma_\varphi^{(i)}), \\
 \gamma_{\varphi z}^{(i)} &= a_{44} \tau_{\varphi z}^{(i)} = 2(a_{11} - a_{12}) \tau_{\varphi z}^{(i)} - 4a \tau_{\varphi z}^{(i)}, \\
 \gamma_{zr}^{(i)} &= a_{44} \tau_{zr}^{(i)} = 2(a_{11} - a_{12}) \tau_{zr}^{(i)} - 4a \tau_{zr}^{(i)}, \\
 \gamma_{r\varphi}^{(i)} &= 2(a_{11} - a_{12}) \tau_{r\varphi}^{(i)} - a [ (\sigma_r^{(i)} - \sigma_\varphi^{(i)}) \sin 4\alpha_i + 4\tau_{r\varphi}^{(i)} \cos^2 2\alpha_i ], \\
 4a &= 2(a_{11} - a_{12}) - a_{44}, \quad \alpha_i = \varphi - \theta_i,
 \end{aligned} \tag{2}$$

where the above form of anisotropy is used. From now on, the first crystal is denoted by the index  $i=1$ , and the second, by  $i=2$ .

In the case of cubic symmetry of the material, we have the following dependencies between the moduli of elasticity  $A_{11}$ ,  $A_{12}$ ,  $A_{44}$  and the strain coefficients  $a_{11}$ ,  $a_{12}$ ,  $a_{44}$ :

$$A_{11} = \frac{a_{11} + a_{12}}{(a_{11} - a_{12})(a_{11} + 2a_{12})}, \quad A_{12} = -\frac{a_{12}}{(a_{11} - a_{12})(a_{11} + 2a_{12})}, \quad A_{44} = \frac{1}{a_{44}}$$

In the isotropic medium, we have  $a_{44} = 2(a_{11} - a_{12})$  and  $2A_{44} = A_{11} - A_{12}$ . For cubic crystals, the ratio  $\eta = 2A_{44}/(A_{11} - A_{12})$  is called a parameter of elastic anisotropy in (Vainstein et al., 1981). In contrast to  $\eta$ , we call  $a$  the coefficient of elastic anisotropy. For  $a=0$ , we have an anisotropic medium in Eqs. (2).

We also note that for  $\varphi = \theta_i = 0$ , Eqs. (2) correspond to generalized Hooke's law written for monocrystals and referred to the principal axes of elasticity.

### 3. Out-of-plane strain

In the case of longitudinal shear along the direction of the axis  $z$ , we have the following components of the displacement vector:  $u_r^{(i)} \equiv 0$ ,  $u_\varphi^{(i)} \equiv 0$ ,  $u_z^{(i)} = u_z^{(i)}(r, \varphi)$ .

For small strains, the strain components  $\gamma_{\varphi z}^{(i)}$  and  $\gamma_{rz}^{(i)}$ , not identically zero, are related to  $u_z^{(i)}$  by the Cauchy equations:  $\gamma_{\varphi z}^{(i)} = \partial u_z^{(i)}/r \partial \varphi$ ,  $\gamma_{rz}^{(i)} = \partial u_z^{(i)}/\partial r$ . According to Hooke's law (2), this implies that

$$\begin{aligned} \sigma_r = \sigma_\varphi = \sigma_z = \tau_{r\varphi} &\equiv 0, \\ \tau_{\varphi z}^{(i)} = \frac{1}{a_{44}^{(i)}} \frac{1}{r} \frac{\partial u_z^{(i)}}{\partial \varphi}, \quad \tau_{rz}^{(i)} &= \frac{1}{a_{44}^{(i)}} \frac{\partial u_z^{(i)}}{\partial r}. \end{aligned} \quad (3)$$

Substituting (3) into the differential equations of equilibrium, we obtain  $\Delta u_z^{(i)} = 0$ , where  $\Delta$  is the Laplace operator.

Since the crystals are rigidly joined, on the interface between the two crystals the displacements are continuous,

$$u_z^{(1)}(r, 0) = u_z^{(2)}(r, 0), \quad u_z^{(1)}(r, \alpha) = u_z^{(2)}(r, \alpha - 2\pi),$$

and the contact stresses are continuous,

$$\frac{\partial u_z^{(1)}(r, 0)}{\partial \varphi} = \frac{a_{44}^{(1)}}{a_{44}^{(2)}} \frac{\partial u_z^{(2)}(r, 0)}{\partial \varphi}, \quad \frac{\partial u_z^{(1)}(r, \alpha)}{\partial \varphi} = \frac{a_{44}^{(1)}}{a_{44}^{(2)}} \frac{\partial u_z^{(2)}(r, \alpha - 2\pi)}{\partial \varphi}.$$

Since  $a_{44}^{(1)} = a_{44}^{(2)} = a_{44}$ , this implies that, in the case of out-of-plane strain, the two-crystal composed of monocrystals of the same material behaves as a monocrystal corresponding to the case  $\theta_1 = \theta_2$ .

Thus, in the case of longitudinal shear in the direction of the  $z$ -axis, there is no stress concentration at the corner point of the interface between the two joined crystals regardless of the orientation of the principal directions  $x_1$  and  $x_2$ .

#### 4. Plane strain

In this case, we have

$$u_r^{(i)} = u_r^{(i)}(r, \varphi), \quad u_\varphi^{(i)} = u_\varphi^{(i)}(r, \varphi), \quad u_z^{(i)} \equiv 0.$$

Hence the following strain components are nonzero:

$$\varepsilon_r^{(i)} = \frac{\partial u_r^{(i)}}{\partial r}, \quad \varepsilon_\varphi^{(i)} = \frac{1}{r} \frac{\partial u_\varphi^{(i)}}{\partial \varphi} + \frac{u_r^{(i)}}{r}, \quad r\gamma_{r\varphi}^{(i)} = \frac{\partial u_r^{(i)}}{\partial \varphi} + r \frac{\partial u_\varphi^{(i)}}{\partial r} - u_\varphi^{(i)}. \quad (4)$$

Hooke's law (2) has the form

$$\begin{aligned} \varepsilon_r^{(i)} &= b_1 \sigma_r^{(i)} + b_2 \sigma_\varphi^{(i)} - a \left[ (\sigma_r^{(i)} - \sigma_\varphi^{(i)}) \sin^2 2\alpha_i + \tau_{r\varphi}^{(i)} \sin 4\alpha_i \right], \\ \varepsilon_\varphi^{(i)} &= b_2 \sigma_r^{(i)} + b_1 \sigma_\varphi^{(i)} + a \left[ (\sigma_r^{(i)} - \sigma_\varphi^{(i)}) \sin^2 2\alpha_i + \tau_{r\varphi}^{(i)} \sin 4\alpha_i \right], \\ \gamma_{r\varphi}^{(i)} &= 2(a_{11} - a_{12}) \tau_{r\varphi}^{(i)} - a \left[ (\sigma_r^{(i)} - \sigma_\varphi^{(i)}) \sin 4\alpha_i + 4\tau_{r\varphi}^{(i)} \cos^2 2\alpha_i \right], \\ b_1 &= a_{11} - \frac{a_{12}^2}{a_{11}}, \quad b_2 = a_{12} - \frac{a_{12}^2}{a_{11}}. \end{aligned} \quad (5)$$

In the absence of mass forces, we satisfy the differential equations of equilibrium by expressing  $\sigma_r^{(i)}$ ,  $\sigma_\varphi^{(i)}$  and  $\tau_{r\varphi}^{(i)}$  via the Airy stress function  $\Phi_i$ :

$$\sigma_r^{(i)} = \frac{1}{r^2} \frac{\partial^2 \Phi_i}{\partial \varphi^2} + \frac{1}{r} \frac{\partial \Phi_i}{\partial r}, \quad \sigma_\varphi^{(i)} = \frac{\partial^2 \Phi_i}{\partial r^2}, \quad \tau_{r\varphi}^{(i)} = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Phi_i}{\partial \varphi} \right). \quad (6)$$

By substituting (5) into the strain consistency condition

$$\frac{\partial^2 \gamma_{r\varphi}^{(i)}}{\partial r \partial \varphi} - r \frac{\partial^2 \varepsilon_\varphi^{(i)}}{\partial r^2} - \frac{1}{r} \frac{\partial^2 \varepsilon_r^{(i)}}{\partial \varphi^2} + \frac{1}{r} \frac{\partial \gamma_{r\varphi}^{(i)}}{\partial \varphi} - 2 \frac{\partial \varepsilon_\varphi^{(i)}}{\partial r} + \frac{\partial \varepsilon_r^{(i)}}{\partial r} = 0$$

after several simplifying transformations, according to (6), we obtain the basic equation of the problem:

$$\begin{aligned}
& \Delta^2 \Phi_i - \frac{a}{b_1} \left[ \left( \frac{\partial^4 \Phi_i}{\partial r^4} + \frac{1}{r^4} \frac{\partial^4 \Phi_i}{\partial \varphi^4} \right) \sin^2 2\alpha_i - 2 \left( \frac{1}{r^2} \frac{\partial^4 \Phi_i}{\partial r^2 \partial \varphi^2} + \frac{1}{r} \frac{\partial^3 \Phi_i}{\partial r^3} \right) \right. \\
& \times \left( 3 \sin^2 2\alpha_i - 2 \right) + \left( \frac{2}{r^3} \frac{\partial^3 \Phi_i}{\partial r \partial \varphi^2} + \frac{1}{r^2} \frac{\partial^2 \Phi_i}{\partial r^2} - \frac{1}{r^3} \frac{\partial \Phi_i}{\partial r} \right) \left( 15 \sin^2 2\alpha_i - 8 \right) \\
& - \frac{4}{r^4} \frac{\partial^2 \Phi_i}{\partial \varphi^2} \left( 11 \sin^2 2\alpha_i - 6 \right) + 2 \left( \frac{1}{r} \frac{\partial^4 \Phi_i}{\partial r^3 \partial \varphi} - \frac{1}{r^3} \frac{\partial^4 \Phi_i}{\partial r \partial \varphi^3} \right. \\
& \left. - \frac{6}{r^2} \frac{\partial^3 \Phi_i}{\partial r^2 \partial \varphi} + \frac{3}{r^4} \frac{\partial^3 \Phi_i}{\partial \varphi^3} + \frac{14}{r^3} \frac{\partial^2 \Phi_i}{\partial r \partial \varphi} - \frac{12}{r^4} \frac{\partial \Phi_i}{\partial \varphi} \right) \sin 4\alpha_i \Big] = 0, \\
& \Delta^2 = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right)^2.
\end{aligned} \tag{7}$$

The rigid connection of the crystals along their contact surface implies the continuity conditions for the displacements on this surface.

$$\begin{aligned}
\frac{\partial u_r^{(1)}(r, 0)}{\partial r} &= \frac{\partial u_r^{(2)}(r, 0)}{\partial r}, & \frac{\partial^2 u_\varphi^{(1)}(r, 0)}{\partial r^2} &= \frac{\partial^2 u_\varphi^{(2)}(r, 0)}{\partial r^2}, \\
\frac{\partial u_r^{(1)}(r, \alpha)}{\partial r} &= \frac{\partial u_r^{(2)}(r, \alpha - 2\pi)}{\partial r}, & \frac{\partial^2 u_\varphi^{(1)}(r, \alpha)}{\partial r^2} &= \frac{\partial^2 u_\varphi^{(2)}(r, \alpha - 2\pi)}{\partial r^2}.
\end{aligned} \tag{8}$$

and the continuity conditions for the contact stresses,

$$\begin{aligned}
\Phi_1(r, 0) &= \Phi_2(r, 0), & \frac{\partial \Phi_1(r, 0)}{\partial \varphi} &= \frac{\partial \Phi_2(r, 0)}{\partial \varphi}, \\
\Phi_1(r, \alpha) &= \Phi_2(r, \alpha - 2\pi), & \frac{\partial \Phi_1(r, \alpha)}{\partial \varphi} &= \frac{\partial \Phi_2(r, \alpha - 2\pi)}{\partial \varphi}.
\end{aligned} \tag{9}$$

If we set  $a=0$  in problem (7)-(9), then we obtain a plane problem for the homogeneous isotropic body.

According to (4), (5), and (6), we have

$$\begin{aligned}
\frac{\partial u_r^{(i)}}{\partial r} &= b_1 \left( \frac{1}{r^2} \frac{\partial^2 \Phi_i}{\partial \varphi^2} + \frac{1}{r} \frac{\partial \Phi_i}{\partial r} \right) + b_2 \frac{\partial^2 \Phi_i}{\partial r^2} \\
& - a \left[ \left( \frac{1}{r^2} \frac{\partial^2 \Phi_i}{\partial \varphi^2} + \frac{1}{r} \frac{\partial \Phi_i}{\partial r} - \frac{\partial^2 \Phi_i}{\partial r^2} \right) \sin^2 2\alpha_i + \left( \frac{1}{r^2} \frac{\partial \Phi_i}{\partial \varphi} - \frac{1}{r} \frac{\partial^2 \Phi_i}{\partial \varphi \partial r} \right) \sin 4\alpha_i \right],
\end{aligned} \tag{10}$$

$$\frac{\partial u_r^{(i)}}{\partial \varphi} + r \frac{\partial u_\varphi^{(i)}}{\partial r} - u_\varphi^{(i)} = 2(a_{11} - a_{12}) \left( \frac{1}{r} \frac{\partial \Phi_i}{\partial \varphi} - \frac{\partial^2 \Phi_i}{\partial \varphi \partial r} \right) + a \left[ \left( r \frac{\partial^2 \Phi_i}{\partial r^2} - \frac{1}{r} \frac{\partial^2 \Phi_i}{\partial \varphi^2} - \frac{\partial \Phi_i}{\partial r} \right) \sin 4\alpha_i - 4 \cos^2 2\alpha_i \left( \frac{1}{r} \frac{\partial \Phi_i}{\partial \varphi} - \frac{\partial^2 \Phi_i}{\partial \varphi \partial r} \right) \right]. \quad (11)$$

Differentiating (10) with respect to  $\varphi$  and (11) with respect to  $r$  and eliminating the derivative  $\partial^2 u_r^{(i)} / \partial r \partial \varphi$ , we obtain

$$\begin{aligned} \frac{\partial^2 u_\varphi^{(i)}}{\partial r^2} = a & \left[ \frac{\partial^3 \Phi_i}{\partial r^3} \sin 4\alpha_i + \frac{1}{r^3} \frac{\partial^3 \Phi_i}{\partial \varphi^3} \sin^2 2\alpha_i + \frac{1}{r} \frac{\partial^3 \Phi_i}{\partial \varphi \partial r^2} (4 - 5 \sin^2 2\alpha_i) \right. \\ & - \frac{2}{r^2} \frac{\partial^3 \Phi_i}{\partial \varphi^2 \partial r} \sin 4\alpha_i - \frac{2}{r} \frac{\partial^2 \Phi_i}{\partial r^2} \sin 4\alpha_i \\ & + \frac{4}{r^3} \frac{\partial^2 \Phi_i}{\partial \varphi^2} \sin 4\alpha_i + \frac{1}{r^2} \frac{\partial^2 \Phi_i}{\partial \varphi \partial r} (13 \sin^2 2\alpha_i - 8) \\ & \left. + \frac{2}{r^2} \frac{\partial \Phi_i}{\partial r} \sin 4\alpha_i + \frac{1}{r^3} \frac{\partial \Phi_i}{\partial \varphi} (8 - 12 \sin^2 2\alpha_i) \right] \\ & - b_1 \frac{1}{r^3} \frac{\partial^3 \Phi_i}{\partial \varphi^3} - \frac{1}{r} \frac{\partial^3 \Phi_i}{\partial \varphi \partial r^2} (a_{11} - a_{12} + b_1) \\ & + \frac{1}{r^2} \frac{\partial^2 \Phi_i}{\partial \varphi \partial r} (a_{11} - a_{12} - b_2) + \frac{2}{r^3} \frac{\partial \Phi_i}{\partial \varphi} (a_{12} - a_{11}). \end{aligned} \quad (12)$$

We use the expressions (10) and (12) to represent the continuity conditions (8) via the stress function  $\Phi$ .

## 5. Solution method

For  $a = 0$ , from (7) we derive the biharmonic equation and, solving it by separation of variables, obtain the following solution (Chobanyan, 1987; Chobanyan & Gevorkyan, 1971):

$$\Phi_i(r, \varphi) = r^{\lambda+1} F_i(\lambda; \varphi), \quad (13)$$

$$\begin{aligned} F_i(\lambda; \varphi) = & A_i \sin(\lambda+1)\varphi + B_i \cos(\lambda+1)\varphi \\ & + C_i \sin(\lambda-1)\varphi + D_i \cos(\lambda-1)\varphi. \end{aligned} \quad (14)$$

where  $\lambda$  is a parameter and  $A_i, B_i, C_i$  and  $D_i$  -are integration constants.

For  $a$  sufficiently small in absolute value, we replace the solution of Eq. (7) by the solution of the biharmonic equation (13). By substituting (13) into (7), we obtain a fourth-order ordinary differential equation for  $F_i(\lambda; \varphi)$ :



$$\begin{aligned}
& F_i^{IV} + 2(\lambda^2 + 1)F_i'' + (\lambda^2 - 1)^2 F_i - \frac{a}{b_1} \{ [F_i^{IV} + 2(\lambda^2 + 1)F_i'' + (\lambda^2 - 1)^2 F_i] \sin^2 2\alpha_i \\
& - 2(\lambda - 2)\sin 4\alpha_i F_i''' + 4(\lambda - 1)(\lambda - 2)\cos 4\alpha_i F_i'' \\
& + 2(\lambda - 2) \left[ (\lambda - 2)^2 - 5 \right] \sin 4\alpha_i F_i' + 4(\lambda^2 - 1)(\lambda - 2)\cos 4\alpha_i F_i \} = 0
\end{aligned} \quad (15)$$

whose general integral has the form (14) for  $a = 0$ .

After the substitution of (13) into (10) and (12), we can write

$$\begin{aligned}
\frac{\partial u_r^{(i)}}{\partial r} &= r^{\lambda-1} \left[ b_1 F_i''(\varphi) + (b_1 + \lambda b_2)(\lambda + 1) F_i(\varphi) \right] \\
& - r^{\lambda-1} a \left[ \frac{1}{2} F_i''(\varphi)(1 - \cos 4\alpha_i) - F_i'(\varphi) \lambda \sin 4\alpha_i \right. \\
& \left. - \frac{1}{2} (\lambda^2 - 1) F_i(\varphi)(1 - \cos 4\alpha_i) \right].
\end{aligned} \quad (16)$$

$$\begin{aligned}
\frac{\partial^2 u_\varphi^{(i)}}{\partial r^2} &= r^{\lambda-2} a \left\{ F_i(\varphi)(\lambda^2 - 1)(\lambda - 2)\sin 4\alpha_i + \frac{1}{2} F_i'(\varphi) \left[ 3\lambda^2 + 1 + (5\lambda^2 - 8\lambda - 1) \right. \right. \\
& \left. \left. \times \cos 4\alpha_i \right] - 2F_i''(\varphi)(\lambda - 1)\sin 4\alpha_i + \frac{1}{2} F_i'''(\varphi)(1 - \cos 4\alpha_i) \right\} - r^{\lambda-2} \left\{ b_1 F_i'''(\varphi) \right. \\
& \left. + F_i'(\varphi) \left[ \lambda(\lambda + 1)(a_{11} - a_{12} + b_1) - (\lambda + 1)(a_{11} - a_{12} - b_2) - 2(a_{12} - a_{11}) \right] \right\}.
\end{aligned} \quad (17)$$

According to (13), (16), and (17), the continuity conditions (8) and (9) acquire the form

$$\begin{aligned}
X_{1j} &= X_{2j} \quad (j = 1, 2, \dots, 8), \quad X_{i1} = F_i(0), \quad X_{i2} = F_i'(0), \\
X_{i3} &= 2b_1 F_i''(0) + 2(b_1 + \lambda b_2)(\lambda + 1) F_i(0) - a \left[ F_i''(0)(1 - \cos 4\theta_i) \right. \\
& \left. + 2F_i'(0)\lambda \sin 4\theta_i - (\lambda^2 - 1) F_i(0)(1 - \cos 4\theta_i) \right], \\
X_{i4} &= a \left\{ F_i'''(0)(1 - \cos 4\theta_i) + 4F_i''(0)(\lambda - 1)\sin 4\theta_i + F_i'(0) \left[ 3\lambda^2 + 1 \right. \right. \\
& \left. \left. + (5\lambda^2 - 8\lambda - 1)\cos 4\theta_i \right] - 2F_i(0)(\lambda^2 - 1)(\lambda - 2)\sin 4\theta_i \right\} - 2b_1 F_i'''(0) \\
& + 2F_i'(0) \left[ (\lambda + 1)(a_{11} - a_{12} - b_2) - \lambda(\lambda + 1)(a_{11} - a_{12} + b_1) + 2(a_{12} - a_{11}) \right], \\
X_{i5} &= F_i(\beta_i), \quad X_{i6} = F_i'(\beta_i), \quad \beta_1 = \alpha, \quad \beta_2 = \alpha - 2\pi, \\
X_{i7} &= 2b_1 F_i''(\beta_i) + 2(b_1 + \lambda b_2)(\lambda + 1) F_i(\beta_i) - a \left\{ [1 - \cos 4(\alpha - \theta_i)] F_i'''(\beta_i) \right. \\
& \left. - 2\lambda \sin 4(\alpha - \theta_i) F_i'(\beta_i) - (\lambda^2 - 1) [1 - \cos 4(\alpha - \theta_i)] F_i(\beta_i) \right\},
\end{aligned} \quad (18)$$

$$\begin{aligned}
X_{i8} = & a \left\{ F_i'''(\beta_i) [1 - \cos 4(\alpha - \theta_i)] - 4F_i''(\beta_i) (\lambda - 1) \sin 4(\alpha - \theta_i) \right. \\
& + F_i'(\beta_i) [3\lambda^2 + 1 + (5\lambda^2 - 8\lambda - 1) \cos 4(\alpha - \theta_i)] \\
& + 2F_i(\beta_i) (\lambda^2 - 1) (\lambda - 2) \sin 4(\alpha - \theta_i) \left. \right\} - 2b_1 F_i'''(\beta_i) \\
& - 2F_i''(\beta_i) [\lambda(\lambda + 1)(a_{11} - a_{12} + b_1) \\
& - (\lambda + 1)(a_{11} - a_{12} - b_2) - 2(a_{12} - a_{11})].
\end{aligned}$$

By substituting (14) into (18), we obtain a homogeneous system of linear algebraic equations for the constants  $A_i, B_i, C_i$  and  $D_i$ .

After some cumbersome calculations, from the existence condition for the nonzero solution of this system, we obtain the following characteristic equation for  $\lambda$ , which determines the stress concentration degree (6) see in (Galptshyan, 2008):

$$f(\lambda; a_{11}, a_{12}, a, \theta_1, \theta_2, \alpha) = 0 \quad (19)$$

Equation (19) contains six independent parameters  $a_{11}, a_{12}, a, \theta_1, \theta_2$  and  $\alpha$ .

|                  | $a/b_1$    |     | $a/b_1$   |
|------------------|------------|-----|-----------|
| Nb               | -0.6423463 | MgO | 0.2276457 |
| CaF <sub>2</sub> | -0.4838456 | Si  | 0.2498694 |
| FeS <sub>2</sub> | -0.4066341 | Ge  | 0.275492  |
| KCl              | -0.2682469 | Ta  | 0.2874998 |
| NaCl             | -0.2154233 | LiF | 0.3094264 |
| V                | -0.2139906 | Fe  | 0.4637442 |
| Mo               | -0.1877868 | Ni  | 0.4804368 |
| TiC              | -0.0664576 | Ag  | 0.5856406 |
| W                | 0          | Cu  | 0.593247  |
| Au               | 0.0556095  | Pb  | 0.7026827 |
| C                | 0.0965294  | Na  | 0.8089901 |
| Al               | 0.1403437  |     |           |

Table 1.

For certain specific values of these parameters, it follows from (6) and (13) that the stress components at the pole  $r = 0$  have an integrable singularity if  $0 < \text{Re } \lambda < 1$ . In this case, the order of the singularity is equal to  $|\text{Re } \lambda - 1|$ .

Thus, studying the singularity of the stress state near the corner point of the interface between two crystals in the case of plane strain is reduced to finding the root of the transcendental equation (19) with the least positive real part.

A structural analysis of Eq. (19) shows that its left-hand side is a polynomial of degree 18 in  $a/b_1$ . The absolute value of  $a/b_1$  is sufficiently small. Therefore, preserving only terms up to the first or the second degree in (19), instead of a polynomial of degree 18, we obtain a

polynomial of the first or the second degree, i.e., various approximations to Eq. (19). We also note that for  $a = 0$ , from the above system of algebraic equations, just as from Eq. (19), we obtain the equation  $\sin(\lambda+1)\pi = 0$  determining the eigenvalues  $\lambda = \lambda_k = k$  ( $k \in \mathbf{N}$ ) which correspond to the plane strain of a homogeneous isotropic body.

Preserving only terms up to the first or second degree in  $a/b_1$  in Eq. (19), we finally obtain

$$\begin{aligned}
& 2^8 (\lambda^2 - 1) \left\{ [\cos \lambda \alpha \cos \alpha - (\lambda + 1) \cos \lambda (\alpha - \pi) \cos(\pi \lambda + \alpha) + \sin[(\alpha - \pi)(\lambda - 1)] \right. \\
& \times \sin \lambda \pi] \sin(\pi \lambda + \alpha) \sin \lambda (\alpha - \pi) + 2 \sin^2 \lambda \pi \cos[(\lambda - 1)(\alpha - \pi)] \cos[\alpha + \lambda(\pi - \alpha)] \left. \right\} \\
& \times \cos^2[(\lambda + 1)(\alpha - \pi)] \sin^4 \lambda \pi + \frac{a}{b_1} \left\{ 2^4 (\lambda - 1) [\cos \lambda \alpha \cos \alpha - (\lambda + 1) \cos \lambda (\alpha - \pi) \right. \\
& \times \cos(\pi \lambda + \alpha) + \sin[(\alpha - \pi)(\lambda - 1)] \sin \lambda \pi] \rho_{22}(\lambda) \cos^2[(\lambda + 1)(\alpha - \pi)] \sin^4 \lambda \pi \\
& - \rho_{21}(\lambda) \sin[(\lambda - 1)(\alpha - \pi)] \sin \lambda \pi - \rho_{23}(\lambda) \sin[(\alpha - \pi)(\lambda - 1)] \sin \lambda \pi + 8(\lambda - 1) \quad (20) \\
& \times [\cos \lambda \alpha \cos \alpha - (\lambda + 1) \cos \lambda (\alpha - \pi) \cos(\pi \lambda + \alpha) + \sin[(\alpha - \pi)(\lambda - 1)] \sin \lambda \pi] \\
& \times [\eta_3 (5 - 3\lambda) \chi_1(\lambda) + (\lambda + 1) \rho_9(\lambda) \chi_2(\lambda) - 4 \chi_{22}(\lambda) (\lambda + 1) (1 - \cos 4\theta_1)] \chi_{11}(\lambda) \\
& \times \cos[(\lambda + 1)(\alpha - \pi)] \sin^3 \lambda \pi - \left\{ 2^4 [\eta_3 (\lambda - 1) (3\lambda - 5) \chi_2(\lambda) - \chi_3(\lambda) \rho_9(\lambda) \right. \\
& - 4 (1 - \cos 4\theta_1) \chi_{31}(\lambda)] (\lambda + 1) \chi_{11}(\lambda) \sin^3 \lambda \pi \cos[(\lambda + 1)(\alpha - \pi)] + (\lambda - 1) \left. \right\} 2^5 (\lambda + 1) \\
& \times \rho_1(\lambda) \sin^4 \lambda \pi \cos[(\lambda + 1)(\alpha - \pi)] \cos[(\lambda - 1)(\alpha - \pi)] + \chi_{11}(\lambda) \left\{ 8 \rho_3(\lambda) \sin \lambda \pi \right. \\
& - \rho_{18}(\lambda) + 4(\lambda - 1) [\cos \alpha \cos \lambda \alpha - (\lambda + 1) \cos \lambda (\alpha - \pi) \cos(\pi \lambda + \alpha) \\
& + \sin[(\alpha - \pi)(\lambda - 1)] \sin \lambda \pi] \rho_{17}(\lambda) \cos[(\lambda + 1)(\alpha - \pi)] \sin \lambda \pi \left. \right\} + 2^4 (\lambda + 1) \left\{ \zeta_3(\lambda) \right. \\
& \times [2 - 2 \cos 4(\alpha - \theta_1) + \rho_9(\lambda)] + (\lambda - 1) [8 \sin 4(\alpha - \theta_1) + \eta_3 (3\lambda - 5)] \zeta_2(\lambda) - 2 \rho_4(\lambda) \\
& - 4 \chi_{44}(\lambda) (1 - \cos 4\theta_1) \left. \right\} \chi_{11}(\lambda) \sin^3 \lambda \pi \cos[(\lambda + 1)(\alpha - \pi)] + \left\{ 2^5 (\lambda + 1) \rho_1(\lambda) \right. \\
& \times \sin^4 \lambda \pi \cos[(\lambda + 1)(\alpha - \pi)] \cos[(\lambda - 1)(\alpha - \pi)] + [8(\lambda + 1) \rho_{11}(\lambda) \sin \lambda \pi - \rho_{18}(\lambda)] \\
& \times \chi_{11}(\lambda) \left. \right\} \cos[(\lambda - 1)(\alpha - \pi)] \sin \lambda \pi \left. \right\} = 0,
\end{aligned}$$

$$\eta_1 = \sin 4(\alpha - \theta_2) - \sin 4(\alpha - \theta_1),$$

$$\eta_2 = \cos 4(\alpha - \theta_2) - \cos 4(\alpha - \theta_1), \quad \eta_3 = \sin 4\theta_2 - \sin 4\theta_1, \quad \eta_4 = \cos 4\theta_2 - \cos 4\theta_1,$$

$$\zeta_1(\lambda) = 2(\lambda \cos \alpha \sin \lambda \alpha - \sin \alpha \cos \lambda \alpha), \quad \zeta_2(\lambda) = 2 \cos \alpha \cos \lambda \alpha,$$

$$\zeta_3(\lambda) = 2(\lambda \sin \lambda \alpha \cos \alpha + \sin \alpha \cos \lambda \alpha), \quad \chi_{11}(\lambda) = 2 \sin \lambda \pi \cos[(\lambda + 1)(\alpha - \pi)],$$

$$\chi_1(\lambda) = (\lambda + 1) \sin(\lambda - 1) \alpha - (\lambda - 1) \sin(\lambda + 1) \alpha, \quad \chi_2(\lambda) = 2 \sin \alpha \sin \lambda \alpha,$$

$$\chi_{22}(\lambda) = 2 \sin(\pi \lambda + \alpha) \sin(\alpha - \pi) \lambda, \quad \chi_3(\lambda) = 2(\lambda \sin \alpha \cos \lambda \alpha + \sin \lambda \alpha \cos \alpha),$$

$$\begin{aligned}
\chi_{31}(\lambda) &= (\lambda - 1) \sin [(\lambda - 1)(\alpha - 2\pi)] - (\lambda + 1) \sin(\lambda + 1) \alpha, \\
\chi_{44}(\lambda) &= (\lambda - 1) \sin [(\lambda - 1)(\alpha - 2\pi)] + (\lambda + 1) \sin(\lambda + 1) \alpha, \\
\chi_{33}(\lambda) &= 2 \cos \lambda (\alpha - \pi) \cos(\pi \lambda + \alpha), \quad \chi_{12}(\lambda) = -2 \sin \lambda \pi \sin [(\lambda + 1)(\alpha - \pi)], \\
\chi_{13}(\lambda) &= (\lambda - 1) \sin(\lambda + 1) \alpha + (\lambda + 1) \sin [(\lambda - 1)(\alpha - 2\pi)], \\
\chi_{21}(\lambda) &= (\lambda + 1) \sin [(\lambda - 1)(\alpha - 2\pi)] - (\lambda - 1) \sin(\lambda + 1) \alpha, \\
\rho_9(\lambda) &= \eta_4 (\lambda - 1) + 4(1 - \cos 4\theta_1), \\
\rho_1(\lambda) &= 3\eta_4 \chi_1(\lambda) + \eta_3 \chi_2(\lambda) (\lambda + 1) - 4\chi_{11}(\lambda) (1 - \cos 4\theta_1), \\
\rho_2(\lambda) &= [3\eta_2 \lambda + 4 \cos 4(\alpha - \theta_1) - 3 \cos 4(\alpha - \theta_2) - 1] (\lambda + 1) \sin [(\lambda + 1)(\alpha - 2\pi)] \\
&\quad - [3\eta_1 \lambda + \sin 4(\alpha - \theta_1) + 3 \sin 4(\alpha - \theta_2)] (\lambda - 1) \cos [(\lambda + 1)(\alpha - 2\pi)] \\
&\quad + [1 - \cos 4(\alpha - \theta_1)] (\lambda + 1) \sin(\lambda + 1) \alpha + 4(\lambda - 1) \sin 4(\alpha - \theta_1) \cos(\lambda + 1) \alpha, \\
\rho_{13}(\lambda) &= (3\lambda - 5) \eta_4 + 4(1 - \cos 4\theta_1), \\
\rho_{14}(\lambda) &= [\lambda \cos 4(\alpha - \theta_1) - (\lambda - 1) \cos 4(\alpha - \theta_2) - 1] (\lambda + 1) \sin [(\lambda - 1)(\alpha - 2\pi)] \\
&\quad - (1 - \cos 4(\alpha - \theta_1)) (\lambda - 1) \sin(\lambda + 1) \alpha + \eta_1 (\lambda^2 - 1) \cos [(\lambda - 1)(\alpha - 2\pi)], \\
\rho_{16}(\lambda) &= (1 + \lambda \cos 4(\alpha - \theta_1) - (\lambda + 1) \cos 4(\alpha - \theta_2)) \cos [(\lambda + 1)(\alpha - 2\pi)] \\
&\quad - (1 - \cos 4(\alpha - \theta_1)) \cos(\lambda + 1) \alpha - \eta_1 (\lambda + 1) \sin [(\lambda + 1)(\alpha - 2\pi)], \\
\rho_6(\lambda) &= (1 + \lambda \cos 4(\alpha - \theta_1) - (\lambda + 1) \cos 4(\alpha - \theta_2)) \sin [(\lambda + 1)(\alpha - 2\pi)] \\
&\quad - (1 - \cos 4(\alpha - \theta_1)) \sin(\lambda + 1) \alpha + \eta_1 (\lambda + 1) \cos [(\lambda + 1)(\alpha - 2\pi)], \\
\rho_7(\lambda) &= 3\eta_4 \zeta_1(\lambda) + \eta_3 (\lambda + 1) \zeta_2(\lambda) + 2\rho_6(\lambda) + 4\chi_{11}(\lambda) (1 - \cos 4\theta_1), \\
\rho_8(\lambda) &= [\lambda \cos 4(\alpha - \theta_1) - (\lambda - 1) \cos 4(\alpha - \theta_2) - 1] \cos [(\lambda - 1)(\alpha - 2\pi)] \\
&\quad - [1 - \cos 4(\alpha - \theta_1)] \cos(\lambda + 1) \alpha - \eta_1 (\lambda - 1) \sin [(\lambda - 1)(\alpha - 2\pi)], \\
\rho_{19}(\lambda) &= (\lambda^2 - 1) [(3\eta_2 \lambda + 4 \cos 4(\alpha - \theta_1) - 5 \cos 4(\alpha - \theta_2) + 1) \cos [(\lambda - 1)(\alpha - 2\pi)] \\
&\quad + (3\eta_1 \lambda + \sin 4(\alpha - \theta_1) - 5 \sin 4(\alpha - \theta_2)) \sin [(\lambda - 1)(\alpha - 2\pi)] \\
&\quad + (1 - \cos 4(\alpha - \theta_1)) \cos(\lambda + 1) \alpha] - 4(\lambda - 1)^2 \sin 4(\alpha - \theta_1) \sin(\lambda + 1) \alpha,
\end{aligned}$$

$$\begin{aligned} \rho(\lambda) = & (3\eta_2\lambda + 4\cos 4(\alpha - \theta_1) - 3\cos 4(\alpha - \theta_2) - 1)(\lambda + 1)\cos[(\lambda + 1)(\alpha - 2\pi)] \\ & + (3\eta_1\lambda + \sin 4(\alpha - \theta_1) + 3\sin 4(\alpha - \theta_2))(\lambda - 1)\sin[(\lambda + 1)(\alpha - 2\pi)] \\ & + (1 - \cos 4(\alpha - \theta_1))(\lambda + 1)\cos(\lambda + 1)\alpha - 4(\lambda - 1)\sin 4(\alpha - \theta_1)\sin(\lambda + 1)\alpha, \end{aligned}$$

$$\rho_0(\lambda) = \eta_3(\lambda + 1)\zeta_3(\lambda) - 3\eta_4(\lambda^2 - 1)\zeta_2(\lambda) + 2\rho(\lambda) - 4(\lambda + 1)(1 - \cos 4\theta_1)\chi_{12}(\lambda),$$

$$\rho_5(\lambda) = 3\eta_4\chi_2(\lambda)(\lambda - 1) + \eta_3\chi_3(\lambda) - 4(1 - \cos 4\theta_1)\chi_{12}(\lambda),$$

$$\begin{aligned} \rho_3(\lambda) = & \chi_{11}(\lambda)[3\eta_3\zeta_2(\lambda)(\lambda^2 - 1) + \eta_4(\lambda + 1)\zeta_3(\lambda) - 2\rho_2(\lambda) \\ & + 4(1 - \cos 4\theta_1)(\lambda + 1)\chi_{11}(\lambda) - (\lambda + 1)\rho_1(\lambda)] \\ & + \chi_{12}(\lambda)[3\eta_3(\lambda + 1)\chi_1(\lambda) - \eta_4(\lambda + 1)^2\chi_2(\lambda) \\ & + 4(\lambda + 1)(1 - \cos 4\theta_1)\chi_{12}(\lambda) - \rho_0(\lambda)], \end{aligned}$$

$$\begin{aligned} \rho_{11}(\lambda) = & \eta_4[\chi_{11}(\lambda)\chi_3(\lambda) - \chi_{12}(\lambda)\chi_2(\lambda)(\lambda + 1)] - 3\eta_3[\chi_{11}(\lambda)\chi_2(\lambda)(\lambda - 1) \\ & - \chi_1(\lambda)\chi_{12}(\lambda)] + 2^4(1 - \cos 4\theta_1)\sin^2\lambda\pi - \rho_1(\lambda)\chi_{11}(\lambda) - \rho_5(\lambda)\chi_{12}(\lambda), \end{aligned}$$

$$\rho_{12}(\lambda) = \rho_{11}(\lambda)\chi_{11}(\lambda) - 4\rho_1(\lambda)\sin^2\lambda\pi,$$

$$\rho_{15}(\lambda) = 4\rho_7(\lambda)\sin^2\lambda\pi + \rho_{11}(\lambda)\chi_{11}(\lambda),$$

$$\rho_{17}(\lambda) = 2(\lambda + 1)(\eta_4\chi_{11}(\lambda) - \eta_3\chi_{12}(\lambda))\cos(\lambda - 1)\alpha - 6(\lambda + 1)$$

$$\times (\eta_3\chi_{11}(\lambda) + \eta_4\chi_{12}(\lambda))\sin(\lambda - 1)\alpha + 2(\chi_{11}(\lambda)\rho_{16}(\lambda) - \chi_{12}(\lambda)\rho_6(\lambda)),$$

$$\begin{aligned} \rho_{18}(\lambda) = & -4\chi_{11}(\lambda)\{[\zeta_1(\lambda)(2 - 2\cos 4(\alpha - \theta_1) + \rho_{13}(\lambda)) + \eta_3\zeta_2(\lambda)(\lambda^2 - 1) \\ & + 2\rho_{14}(\lambda) - 4\chi_{13}(\lambda)(1 - \cos 4\theta_1)]\sin^2(\lambda + 1)\pi + \rho_{13}(\lambda)\chi_1(\lambda) + \eta_3\chi_2(\lambda)(\lambda^2 - 1) \\ & - 4\chi_{21}(\lambda)(1 - \cos 4\theta_1)\} + 2(\lambda + 1)\rho_{12}(\lambda)\sin\lambda\pi\cos[(\lambda - 1)(\alpha - \pi)] \end{aligned}$$

$$+ 2(\lambda + 1)[\rho_{15}(\lambda) + \rho_{17}(\lambda)\chi_{12}(\lambda)]\sin\lambda\pi\cos[(\alpha - \pi)(\lambda - 1)]$$

$$+ 2\rho_{17}(\lambda)\chi_{11}(\lambda)(\lambda - 1)\sin\lambda\pi\sin[(\alpha - \pi)(\lambda - 1)],$$

$$\begin{aligned} \rho_4(\lambda) = & (\lambda - 1)[(3\eta_2\lambda + 4\cos 4(\alpha - \theta_1) - 5\cos 4(\alpha - \theta_2) + 1)\sin[(\lambda - 1)(\alpha - 2\pi)] \\ & - (3\eta_1\lambda + \sin 4(\alpha - \theta_1) - 5\sin 4(\alpha - \theta_2))\cos[(\lambda - 1)(\alpha - 2\pi)] + 4\sin 4(\alpha - \theta_1) \\ & \times \cos(\lambda + 1)\alpha] + (\lambda + 1)[1 - \cos 4(\alpha - \theta_1)]\sin(\lambda + 1)\alpha \end{aligned}$$

$$\begin{aligned}
\rho_{20}(\lambda) = & -4\chi_{11}(\lambda) \left[ \eta_3 \zeta_3(\lambda) (\lambda^2 - 1) - (2 - 2\cos 4(\alpha - \theta_1) + \rho_{13}(\lambda)) (\lambda^2 - 1) \zeta_2(\lambda) \right. \\
& + 8(\lambda - 1) \zeta_1(\lambda) \sin 4(\alpha - \theta_1) + 2(\lambda + 1) \rho_{19}(\lambda) \\
& \left. + 2\chi_{33}(\lambda) (1 - \cos 4\theta_1) (\lambda^2 - 1) \right] \sin^2 \lambda \pi + \rho_{12}(\lambda) (\lambda^2 - 1) \left[ \zeta_2(\lambda) - \chi_{33}(\lambda) (\lambda + 1) \right] \\
& + 4(\lambda + 1) \chi_{12}(\lambda) \left[ \rho_{13}(\lambda) \chi_1(\lambda) + \eta_3 \chi_2(\lambda) (\lambda^2 - 1) - 4\chi_{21}(\lambda) (1 - \cos 4\theta_1) \right] \sin^2 \lambda \pi \\
& - 2 \left[ \rho_{11}(\lambda) \chi_{12}(\lambda) (\lambda + 1) - 4\rho_0(\lambda) \sin^2 \lambda \pi \right] (\lambda + 1) \sin \lambda \pi \cos [(\alpha - \pi) (\lambda - 1)] \\
& + 4\chi_{11}(\lambda) (\lambda^2 - 1) \left[ \rho_{13}(\lambda) \chi_2(\lambda) + \eta_3 \chi_3(\lambda) \right. \\
& - 4\chi_{22}(\lambda) (1 - \cos 4\theta_1) \left. \right] \sin^2 \lambda \pi + 2(\lambda - 1) \left[ \rho_3(\lambda) \chi_{11}(\lambda) \right. \\
& - 4\rho_1(\lambda) (\lambda + 1) \sin^2 \lambda \pi \left. \right] \sin \lambda \pi \sin [(\alpha - \pi) (\lambda - 1)] \\
& - 4\chi_{12}(\lambda) (\lambda + 1) \left[ \rho_{13}(\lambda) \chi_1(\lambda) + \eta_3 \chi_2(\lambda) (\lambda^2 - 1) \right. \\
& - 4\chi_{21}(\lambda) (1 - \cos 4\theta_1) \left. \right] \sin^2 \lambda \pi \\
& + 2(\lambda + 1) \left[ \rho_3(\lambda) \chi_{12}(\lambda) - \rho_5(\lambda) (\lambda + 1) \right] \sin \lambda \pi \cos [(\alpha - \pi) (\lambda - 1)], \\
\rho_{22}(\lambda) = & \eta_3 \zeta_1(\lambda) (5 - 3\lambda) + \zeta_2(\lambda) (\lambda + 1) \left[ 2 - 2\cos 4(\alpha - \theta_1) + \rho_9(\lambda) \right] \\
& + 2(\lambda + 1) \left[ \rho_8(\lambda) - 2\chi_{33}(\lambda) (1 - \cos 4\theta_1) \right],
\end{aligned}$$

$$\begin{aligned}
\rho_{23}(\lambda) = & 4(1 - \lambda^2) \left[ \cos \alpha \cos \lambda \alpha - (\lambda + 1) \cos \lambda (\alpha - \pi) \cos(\pi \lambda + \alpha) \right. \\
& + \sin [(\alpha - \pi) (\lambda - 1)] \sin \lambda \pi \left. \right] (\rho_{17}(\lambda) \chi_{12}(\lambda) + \rho_{15}(\lambda)) \cos [(\lambda + 1) (\alpha - \pi)] \sin \lambda \pi \\
& - \rho_{20}(\lambda) \chi_{11}(\lambda) - 8(\lambda + 1) \left[ \rho_{11}(\lambda) \chi_{12}(\lambda) (\lambda + 1) - 4\rho_0(\lambda) \sin^2 \lambda \pi \right. \\
& \left. - \rho_3(\lambda) \chi_{12}(\lambda) + \rho_5(\lambda) (\lambda + 1) \right] \sin^2 \lambda \pi \cos [(\lambda + 1) (\alpha - \pi)] \cos [(\lambda - 1) (\alpha - \pi)] \\
\rho_{21}(\lambda) = & 4(1 - \lambda^2) \left[ \cos \alpha \cos \lambda \alpha - (\lambda + 1) \cos \lambda (\alpha - \pi) \cos(\pi \lambda + \alpha) \right. \\
& + \sin [(\alpha - \pi) (\lambda - 1)] \sin \lambda \pi \left. \right] \left[ \chi_{11}(\lambda) \rho_{11}(\lambda) - 4\rho_1(\lambda) \sin^2 \lambda \pi \right] \\
& \times \cos [(\lambda + 1) (\alpha - \pi)] \sin \lambda \pi - \chi_{11}(\lambda) \rho_{20}(\lambda).
\end{aligned}$$

## 6. Study of the roots of the characteristic equation

Table 1 shows the values of the dimensionless ratio  $a/b_1$  for some cubic crystals at room temperature. Moreover for all the considered materials  $b_1 > 0$  and with the exception of cubic pyrites ( $FeS_2$ ), for which  $b_2 = 0,00365798 \cdot 10^{-11} Pa^{-1}$ ,  $b_2 < 0$ .

The least value of the ratio is attained for the niobium crystal (Nb) and the largest, for the sodium crystal (Na). In absolute value,  $|a/b_1| < 1$ .

To study the roots of Eq. (19) in the interval  $10 < \text{Re} \lambda < 1$ , in Table 1 we choose six real materials and two imaginary materials for which  $|a/b_1| = 10^{-5}$ . To investigate whether

there is a singularity in the stress concentration at the corner point of the interface between the two joined crystals, for each of the materials, we choose seven versions of variations in the parameters  $\alpha$ ,  $\theta_1$  and  $\theta_2$ , which are given in Tables 2 and 3. For example, the first

|     | $\frac{a}{b_1}$ | $\sigma_*, MPa$     | $\alpha = \pi/2$<br>$\theta_1 = \pi/4$<br>$\theta_2 = 0$ | $\alpha = \pi/4$<br>$\theta_1 = \pi/4$<br>$\theta_2 = 0$                   |
|-----|-----------------|---------------------|--|--|
| Mo  | -0.1877868      | 800 – 1200          | 0.647029   | 0.0174393<br>$\mp i \cdot 0.058343$<br>0.688156                            |
| TiC | -0.0664576      | 560                 | 0.0153193<br>0.6899690                                   | 0.01012946<br>$\mp i \cdot 0.047990$<br>0.72254                            |
|     | $-10^{-5}$      |                     | $7.3815 \cdot 10^{-6}$<br>0.987524                       | 0.996981   |
| W   | 0               | 1100<br>1800 – 4150 | -  | -  |
|     | $10^{-5}$       |                     | 0.994154<br>$\mp i \cdot 0.0107712$                      | 0.000786231<br>0.9276061<br>$\mp i \cdot 0.1747073$                        |
| Au  | 0.0556095       | 140                 | 0.0497266<br>0.422350                                    | 0.0809312<br>0.2043483<br>0.4714287<br>0.9546085<br>$\mp i \cdot 0.216914$ |
| C   | 0.0965294       |                     | 0.032592<br>0.5279915                                    | 0.5889015  |
| Al  | 0.1403437       | 50<br>115           | 0.0284796<br>0.560425                                    | 0.0522492<br>$\mp i \cdot 0.073474$<br>0.617351                            |

Table 2.

version, where  $\alpha = \pi / 2$ ,  $\theta_1 = \pi / 4$ ,  $\theta_2 = 0$ , concerns the case in which the interface between two crystals is formed by the plane of elastic symmetry of the second crystal but not of the first crystal. In the fourth version ( $\alpha = \pi/4$ ,  $\theta_1 = 0$ ,  $\theta_2 = \pi/4$ ), the part  $\varphi_1 = \theta_1 = 0$  of the interface is the plane of elastic symmetry of the first crystal, and the other part ( $\varphi = \theta_2 = \pi/4$ ) is the plane of elastic symmetry of the second crystal.

For all materials given in Tables 2 and 3 and for all versions, we found, in general, all real and complex roots of Eq. (20) with  $0 < \text{Re } \lambda < 1$ , including all (without any exception) roots with minimum positive real part.

It follows from Tables 2 and 3 that, for all two-crystals except tungsten and for all the versions, there are stress concentrations near the corner point of the interface between the crystals. If we compare the two crystals of molybdenum (*Mo*) and titanium carbide (*TiC*) for which  $a/b_1 < 0$ , then it follows from the results obtained for seven versions that, in general, the stress concentration degree (the order of singularity) of molybdenum is less than that of titanium carbide. It is of interest to note that the ultimate strength of polycrystalline molybdenum  $\sigma_*$  is larger than the ultimate strength of polycrystalline titanium carbide, which is an integral characteristic of strength. In Table 2, we present the ultimate strengths under tension at temperature 20°C for molybdenum and titanium carbide.

For the two-crystal of tungsten (*W*), we have  $a/b_1 = 0$  and hence, according to (20), there is no singularity of stress concentration near the corner point of the interface between two crystals. This may be one of the causes of the fact that the polycrystalline tungsten materials have very high ultimate strength.

In Table 2, we present the ultimate strengths under tension of the polycrystalline tungsten annealed wire (1100 MPa) and unannealed wire (from 1800 MPa to 4150 MPa, depending on the diameter). We draw the reader's attention to the fact that the ultimate strength of the diamond monocrystal at temperature 20°C is equal to 1800 MPa.

Note that for the polycrystalline metals listed in Table 2 there is a correspondence between the ultimate strength  $\sigma_*$  and the modulus of elasticity  $E$  (here the quantity  $E$  is treated as an integral characteristic of elasticity of a metal). The moduli of elasticity of the polycrystalline metals *Mo*, *W*, *Au* and *Al* listed in Table 2 are, respectively, equal to (285-300) GPa, (350-380) GPa, 79 GPa, and 70 GPa. The ultimate strength is larger for a metal with larger modulus of elasticity.

All numerical values of strength limit brought in the table (2) as well as elastic modulus for the discussed materials considered to be a published data taken from various sources. For example, these data for tungsten (*W*) are taken from the book (Knunians and etc. 1961).

Strength limit of unannealed tungsten wire is depended from the diameter and could be explained by the existence defects of crystal lattice.

Here we also note that there is no such correspondence if molybdenum and titanium carbide are compared. Although the ultimate strength of molybdenum is larger than the ultimate strength of titanium carbide, the modulus of elasticity of molybdenum is less than



the modulus of elasticity of titanium carbide, which is equal to 460 GPa. We note that the titanium carbide is a compound matter.

|     | $\alpha = \pi/2$<br>$\theta_1 = 0$<br>$\theta_2 = \pi/4$     | $\alpha = \pi/4$<br>$\theta_1 = 0$<br>$\theta_2 = \pi/4$  | $\alpha = 3\pi/4$<br>$\theta_1 = \pi/4$<br>$\theta_2 = 0$                   | $\alpha = 3\pi/4$<br>$\theta_1 = \pi/6$<br>$\theta_2 = \pi/3$                          | $\alpha = \pi/2$<br>$\theta_1 = \pi/6$<br>$\theta_2 = \pi/3$ |
|-----|--|---|---|--|--|
| Mo  | 0.710072   | 0.0775947   | 0.0411225   | 0.957018   | 0.0094110<br>$\mp i \cdot 0.0739984$<br>0.656204             |
| TiC | 0.774834   | 0.0542824   | 0.0365724   | 0.95741  | 0.0071750<br>$\mp i \cdot 0.064893$<br>0.702898              |
|     | 0.993177   | $7.87311 \cdot 10^{-4}$<br>0.927667<br>$\mp i \cdot 0.1746807$<br>0.999978<br>$\mp i \cdot 0.0029137$ | $8.82199 \cdot 10^{-4}$<br>0.9905998<br>$\mp i \cdot 0.1403189$<br>0.995236 | 0.9906207<br>$\mp i \cdot 0.1403553$<br>0.995563                                       | 0.9960624<br>$\mp i \cdot 0.0057440$                         |
| W   | -  | -   | -   | -  | -  |
|     | $1.0567 \cdot 10^{-5}$<br>0.9970875<br>$\mp i \cdot 0.00590$ | 0.9276334<br>$\mp i \cdot 0.174718$<br>0.997109   | 0.9912171<br>$\mp i \cdot 0.140329$<br>0.99971<br>$\mp i \cdot 0.005039$    | 0.0011696<br>0.991197<br>$\mp i \cdot 0.1402922$<br>0.999734<br>$\mp i \cdot 0.004690$ | 0.0013220<br>0.993404  |
| Au  | 0.0644867<br>0.8890054<br>$\mp i \cdot 0.4002118$            | 0.8730987<br>$\mp i \cdot 0.236169$   | 0.0779069<br>0.2491514<br>0.7677329   | 0.1780295<br>0.7444930   | 0.4007251  |
| C   | 0.1243433  | 0.8400085<br>$\mp i \cdot 0.272280$   | 0.0557915<br>0.4864447<br>0.7224011   | 0.332059<br>0.6977614  | 0.5246499  |
| Al  | 0.215732   | 0.0206982<br>$\mp i \cdot 0.113575$   | 0.0512975<br>0.612502<br>0.65982  | 0.451447<br>0.655004   | 0.0154580<br>$\mp i \cdot 0.094053$<br>0.56112               |

Table 3.

Discussing the results obtained for two-crystals of gold (*Au*) and aluminum (*Al*) (Tables 2 and 3), for which  $a/b_1 > 0$ , we conclude that, according to the root of Eq. (20) obtained for seven versions, the stress concentration degree (the order of singularity) near the corner point of the interface between two crystals is larger for the two-crystal of aluminum. Here we also note that the ultimate strength and the modulus of elasticity of polycrystalline gold are larger than those of polycrystalline aluminum. In Table 2, we present the ultimate strengths under tension for polycrystalline aluminum annealed wire (50 MPa) and cold-rolled wire (115 MPa).

For a two-crystal of diamond (*C*), the stress concentrations near the corner point of the interface between two crystals are rather large (see Tables 2 and 3).

Depending on the choice of the coordinate axes, the modulus of elasticity of the diamond monocrystal varies from 1049.67 GPa to 1206.63 GPa, and, as was already noted, the ultimate strength is approximately equal to 1800 MPa. But for diamond polycrystalline formations (edge, aggregate), we did not find the corresponding integral characteristics of elasticity and strength in the literature. We assume that these characteristics, numerically, must be less than the modulus of elasticity and the ultimate strength of the diamond monocrystal, because there is no stress concentration in the interior of a polycrystalline body.

As follows from Tables 2 and 3, for the imaginary materials with the ratios  $|a/b_1| = 10^{-5}$ , there are very strong stress concentrations for some of the versions.

In Figs. 2-5, we present graphs of variation of the function  $r_*^{\text{Re}\lambda-1}$  as  $r_*$  approaches the pole  $r = 0$

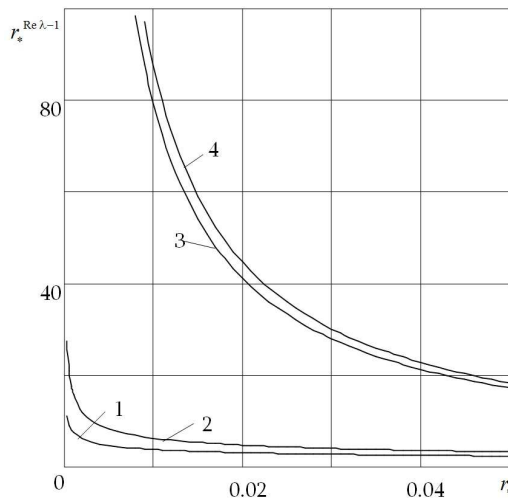


Fig. 2.

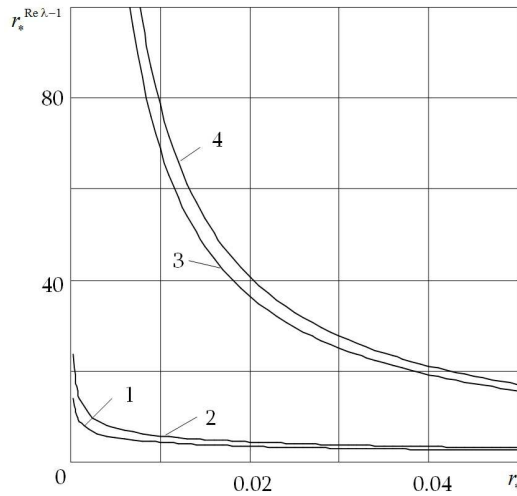


Fig. 3.

( $r_*$  is the ratio of the coordinate  $r$  to the characteristic dimension of the two-crystal). Curves 3 and 4 correspond to the two-crystal of gold ( $Au$ ) and the two-crystal of aluminum ( $Al$ ), respectively. Curves 1 and 2 correspond to a two-component piecewise homogeneous isotropic body with shear moduli ratio  $\mu = G_1 / G_2 = a_{44}^{(2)} / a_{44}^{(1)} = 20$  and Poisson ratios  $\nu_1 = 0.2, \nu_2 = 0.4$  and to the two-component piecewise homogeneous isotropic body with shear moduli ratio  $\mu = G_1 / G_2 = a_{44}^{(2)} / a_{44}^{(1)} = 0.05$  and the Poisson ratios  $\nu_1 = 0.2, \nu_2 = 0.3$ , respectively. Moreover,  $\nu_1 = -E_1 a_{12}^{(1)}, \nu_2 = -E_2 a_{12}^{(2)}$ ,

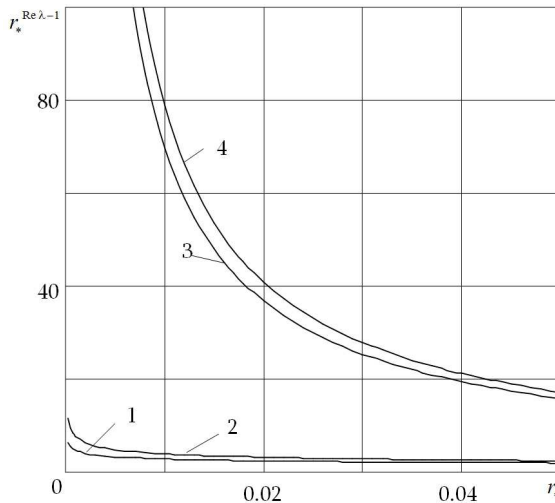


Fig. 4.

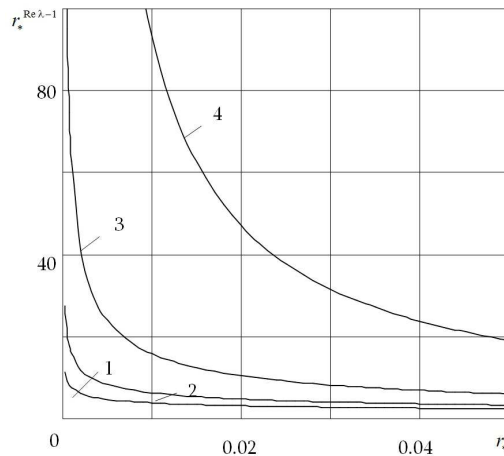


Fig. 5.

where  $a_{12}^{(1)}$  and  $a_{12}^{(2)}$  are the strain coefficients of homogeneous isotropic parts and  $E_1$  and  $E_2$  are the Young moduli of the same homogeneous isotropic parts. Figures 2-5 correspond to the first, second, fifth, and seventh versions given in Tables 2 and 3, respectively; curves 1 and 2 in the same figures correspond to the four values of the linear angle  $\alpha$  formed by the contact surfaces of homogeneous isotropic parts of the compound body. The values of the angle  $\alpha$  in Figs. 2-5 are respectively equal to:  $\alpha = \pi/2, \pi/4, 3\pi/4$  and  $\pi/2$ . The values of the ratio  $\mu$  and the Poisson ratios  $\nu_1$  and  $\nu_2$ , and the corresponding values of the orders of singularities, are taken from Table 1 presented in (Chobanyan, 1987; Chobanyan & Gevorkyan, 1971).

The graphs show that the order of singularity of the stresses at the corner point of the contact surface of aluminum crystals is larger than the order of singularity of stresses at the corner point of the contact surface of gold crystals. The graphs also show that, for the piecewise homogeneous isotropic bodies under study, the order of singularity of the stresses is much lower than that for two-crystals of aluminum and gold.

## 7. Conclusion

From the analysis performed in Section 6, we draw the following conclusions.

Although we considered specific cases of stress state, namely, the out-of-plane strain and the plane strain of two-crystals whose separate crystals consist of one and the same material with cubic symmetry and with different orientations of the principal directions of elasticity, we can state that, in the general case of loading of a polycrystalline body, there are stress concentrations at the corner points of the interface between the joined crystals.

It is well known that the structure of the crystal lattice of a given matter plays a definite role in the process of formation of its mechanical properties and characteristics, in particular, the strength of monocrystals. But in polycrystalline materials, along with this factor, the

strength of the joint of crystals and the fact that there are stress singularities at the corner points of the interface between the crystals totally play the decisive role in the process of formation of these characteristics. This can be observed in the process of mechanical fragmentation of polycrystalline materials. They split and form small crystals of certain shape. Of course, the separate crystals are also deformed in this process. The modulus of elasticity and the ultimate strength of a monocrystal with cubic symmetry for simple matters is larger than the corresponding characteristics of the polycrystalline material of the same matter.

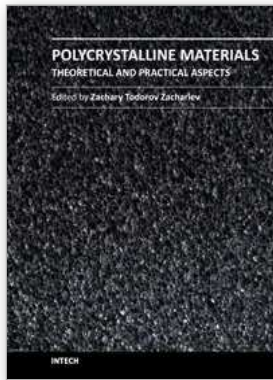
In the problem of plane strain, the existence of stress concentration (singularity) at the corner point of the interface between the two joined crystals with cubic symmetry made of the same material, just as the degree of stress concentration (the order of singularity), depends on the parameters  $a/b_1$ ,  $\alpha$ ,  $\theta_1$ , and  $\theta_2$ , which are determined in Sections 1-4.

In the case of out-of-plane strain of the two-crystal under study, there is no stress concentration at the corner point of the interface between the two joined crystals.

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