

Photon Localization Revisited

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1. Introduction

Whilst quantum electrodynamics (QED) underwent an impressive development and reached its maturity in the middle of the last century, one of its basic concepts—the photon wave function in free space—was deprived of such fortune. Although the photon wave function in coordinate representation was introduced already in 1930 by Landau and Peierls, the concept was found to suffer from inherent difficulties that were not overcome during the century—see review (Bialynicki-Birula, 1996). The common explanation presented in textbooks, e.g., (Akhiezer & Berestetskii, 1965; Mandel & Wolf, 1995), may be summed up as follows: (i) no position operator exists for the photon, (ii) while the position wave function may be localized near a space-time point, the measurable quantities like the electromagnetic field vectors, energy, and the photodetection probability remain spread out due to their non-local relation with the position wave function.

However, just before the turn of the century both of these widely-espoused notions were disproved (Bialynicki-Birula, 1998; Hawton, 1999) and in the new century a fresh interest in the photon localization problem seems to have been awakened, see, e.g., (Bialynicki-Birula & Bialynicka-Birula, 2009; Chan et al., 2002; Hawton, 2007; Keller, 2000; 2005), meeting the needs of developments in near-field optics, cavity QED, and quantum computing. Recently, into the study of quantum phenomena in general and photon localization in particular, the so-called localized waves were involved (Belgiorno et al., 2010; Besieris et al., 1994; Ciattoni & Conti, 2007; Jáuregui & Hacyan, 2005; Saari et al., 2005). These belong to the propagation-invariant non-diffracting localized solutions to the linear wave equation—a research subject emerged in the 1980-ies, see the 1st collective monograph on the field (Hernández-Figueroa et al., 2008). Experimental feasibility of some of the localized waves has been demonstrated already (Alexeev et al., 2002; Bowlan et al., 2009; Grunwald et al., 2003; Reivelt & Saari, 2002; Saari et al., 2010; Saari & Reivelt, 1997; Sönajalg et al., 1997).

(Bialynicki-Birula, 1998) writes that the statement “even when the position wave function is strongly concentrated near the origin, the energy wave function is spread out over space asymptotically like $r^{-7/2}$ ” —citation from (Mandel & Wolf, 1995), p. 638—is incorrect and that both wave functions may be strongly concentrated near the origin. He demonstrates, on one hand, that photons can be essentially better localized in space—with an exponential falloff of the photon energy density and the photodetection rates. On the other hand, he establishes—and it is even somewhat startling that nobody has done it earlier—that certain localization restrictions arise out of a mathematical property of the positive frequency solutions which therefore are of a universal character and apply not only to photon states but hold for all particles. More specifically, it has been proven in the Letter (Bialynicki-Birula,

1998) for the case of spherically imploding-exploding one-photon wavepacket that the Paley-Wiener theorem allows even at instants of maximal localization only such asymptotic decrease of the modulus of the wave function with the radial distance r that is *slower* than the linear exponential one, i.e., anything slower than $\sim \exp(-Ar)$, where A is a constant. The latter is what the Paley-Wiener theorem says about a function whose Fourier spectrum contains no negative frequencies. In (Bialynicki-Birula & Bialynicka-Birula, 2009) the same results have been obtained using a set of space-dependent photon creation and annihilation operators without any reference to the mode decomposition. The latter paper shows also that one may sharply localize either electric or magnetic but not both footprints of photons.

The purpose of the chapter is to give an overview and an analysis of the most striking contradictions in the notions of photon localization presented in textbooks and in abovementioned new studies.

The first Section reproduces derivation of main textbook formulas concerning the photon localization problem. In the next Section we demonstrate, following (Bialynicki-Birula, 1998), a superficiality of the common textbook notion. Here we also publish for the first time our generalizations to the exponential localization models found in (Bialynicki-Birula, 1998).

In Section 4 we take a closer look at the localization restrictions caused by the circumstance that the photon wave function is mathematically an analytic signal with respect to time variable, which obeys the Paley-Wiener theorem. Here we present some supporting graphical illustrations as well.

Section 5 reproduces results of our paper (Saari et al., 2005) on two-dimensional localization of photon packets constructed from certain localized wave solutions to the wave equation. A discussion follows why such packets seemingly violate the localization restrictions set by the Paley-Wiener theorem.

2. Common treatment of the problem

In this Section we outline the standard approach to the photon localizability problem following the monograph (Mandel & Wolf, 1995). Only these formulas will be presented that are requisite for introduction, comparison and contrasting with what follows in the next Sections.

A field state containing one photon with wavevector \mathbf{k} and polarization s is given through acting of the photon creation operator $a^+(\mathbf{k}, s)$ on the vacuum state $|vac\rangle$. Any one-photon state which is at least partially localized needs a linear superposition of such plane-wave states of the form

$$|1ph\rangle = \sum_s \int d^3k \phi(\mathbf{k}, s) a^+(\mathbf{k}, s) |vac\rangle, \quad (1)$$

in which $\phi(\mathbf{k}, s)$ is any properly normalized weight function. The vector function

$$\Phi(\mathbf{r}, t) = \sum_s \int d^3k \phi(\mathbf{k}, s) \boldsymbol{\varepsilon}_{\mathbf{k}, s} e^{i(\mathbf{k}\mathbf{r} - \omega t)}, \quad (2)$$

where $\boldsymbol{\varepsilon}_{\mathbf{k}, s}$ are (generally complex) unit polarization vectors and the frequency $\omega = c|\mathbf{k}|$, then represents the corresponding position space wave function (known as the Landau-Peierls wave function) of the photon in state $|1ph\rangle$. Its modulus squared, integrated over a volume V , gives the probability of locating a particle within the volume V .

However, the vector function defined in Eq. (2) does not transform locally under Lorentz transformations. Moreover, there is no procedure to measure $\Phi(\mathbf{r}, t)$. Measurable may be the quantities like energy or the photoelectric detection probability of the photon. It can be shown that neither of them are localized in the volume into which $\Phi(\mathbf{r}, t)$ has been confined.

The average photon energy is

$$\langle 1ph | \hat{H} | 1ph \rangle = \sum_s \int d^3k \hbar\omega |\phi(\mathbf{k}, s)|^2 .$$

By introducing the function

$$\Phi_E(\mathbf{r}, t) = \sum_s \int d^3k \sqrt{\hbar\omega} \phi(\mathbf{k}, s) \epsilon_{\mathbf{k},s} e^{i(\mathbf{k}\mathbf{r} - \omega t)} , \quad (3)$$

cf. Eq. (2), which might be called the energy wave function to distinguish it from $\Phi(\mathbf{r}, t)$, and differs from $\Phi(\mathbf{r}, t)$ only in having the \mathbf{k} -dependent factor $\sqrt{\hbar\omega}$ in the expansion, we readily find that

$$\langle 1ph | \hat{H} | 1ph \rangle = \int d^3r |\Phi_E(\mathbf{r}, t)|^2 . \quad (4)$$

$|\Phi_E(\mathbf{r}, t)|^2$ therefore plays the role of energy density. But this energy density is not locally connected with the photon density $|\Phi(\mathbf{r}, t)|^2$. Indeed, from the Fourier expansions of $\Phi(\mathbf{r}, t)$ and $\Phi_E(\mathbf{r}, t)$ we find with the help of the convolution theorem that they are connected through the spatial convolution

$$\Phi_E(\mathbf{r}, t) = \int d^3r' G(\mathbf{r} - \mathbf{r}') \Phi(\mathbf{r}', t) , \quad (5)$$

where the spatial function $G(\dots)$ is the three-dimensional Fourier transform of $\sqrt{\hbar\omega}$, or

$$G(\mathbf{r}) = \frac{(\hbar c)^{1/2}}{(2\pi)^3} \int d^3k \sqrt{k} e^{i\mathbf{k}\mathbf{r}} . \quad (6)$$

Because of the spread associated with $G(\mathbf{r})$, $\Phi_E(\mathbf{r}, t)$ can be non-zero at positions where $\Phi(\mathbf{r}, t)$ is zero.

Following (Mandel & Wolf, 1995) and by introducing an exponential factor with vanishing parameter ε in order to regularize the integral, one gets

$$G(\mathbf{r}, \varepsilon) = \frac{(\hbar c)^{1/2}}{(2\pi)^3} \int d^3k \sqrt{k} e^{i\mathbf{k}\mathbf{r}} e^{-\varepsilon k} .$$

This integral can be taken in spherical coordinates

$$\begin{aligned} G(\mathbf{r}, \varepsilon) &= \frac{(\hbar c)^{1/2}}{(2\pi)^3} \int_0^\infty dk k^2 \sqrt{k} e^{-\varepsilon k} \int_0^\pi d\theta e^{ikr \cos \theta} \sin \theta \int_0^{2\pi} d\phi \\ &= \frac{(\hbar c)^{1/2}}{2\pi^2} \frac{1}{r} \int_0^\infty dk k^{3/2} e^{-\varepsilon k} \sin kr \\ &= \frac{3(\hbar c)^{1/2}}{8\pi^{3/2}} \frac{1}{r(r^2 + \varepsilon^2)^{5/4}} \sin \left(\frac{5}{2} \arctan \frac{r}{\varepsilon} \right) . \end{aligned} \quad (7)$$

When $\varepsilon \rightarrow 0$, or more generally whenever $\varepsilon \ll r$, this reduces to

$$G(\mathbf{r}) = \pm \frac{3(\hbar c)^{1/2}}{8\sqrt{2}\pi^{3/2}} r^{-7/2}. \quad (8)$$

For the derivations in the next sections it is useful to notice that when in the spherically symmetric case a 3D Fourier integral is transformed into corresponding 1D one, the factor $(kr)^{-1} \sin kr$ appears in the integrand. This factor represents spherically symmetric standing wave and the integral is nothing but a superposition over such waves of different frequencies.

One can conclude from Eqs. (5) and (8) that—citation from (Mandel & Wolf, 1995) follows—"even when the position wave function $\Phi(\mathbf{r}, t)$ is strongly concentrated near the origin, the energy wave function is spread out over space asymptotically like $r^{-7/2}$. Alternatively, we may say that even when the probability distribution of the photon is strongly localized near the origin, the energy distribution extends over large distances and falls off as r^{-7} ." As the EM field operators contain the same \mathbf{k} -dependent factor $\sqrt{\hbar\omega}$ and are proportional to $\Phi_E(\mathbf{r}, t)$, the corresponding measurable quantities, incl. the probability of photodetection, bear the same non-local relation to the photon probability distribution $|\Phi(\mathbf{r}, t)|^2$. Again, one can conclude that—citation from (Mandel & Wolf, 1995) continues—"for a photon which is strongly localized close to the origin, there is a non-vanishing probability falling off as r^{-7} that it will be detected by a photoelectric detector at a distance r ."

3. Exponential localization

In this section we see that the statements cited in the previous paragraph are incorrect.

Due to the orthogonality of the polarization vectors $\mathbf{e}_{\mathbf{k},s}^* \cdot \mathbf{e}_{\mathbf{k},s'} = \delta_{ss'}$ ($s, s' = 1, 2$) the total energy of a one-photon state Eq. (4) breaks into two non-interfering contributions from both polarization states. Therefore one may consider only one polarization at a time (or assume $\phi(\mathbf{k}, s = 2) \equiv 0$). Thus, the problem of the best localization of a photon reduces to the question: what is the fastest possible falloff with the distance \mathbf{r} of the modulus of vector function $\Phi_E(\mathbf{r}, t)$?

The most consistent treatment of the photon (energy) wave function is based on the Riemann-Silberstein (RS) complex vector which is given (in the SI system) by the following linear combination of the EM field vectors (Bialynicki-Birula, 1996; 1998)

$$\mathbf{F}(\mathbf{r}, t) = \frac{\mathbf{D}(\mathbf{r}, t)}{\sqrt{2\varepsilon_0}} + i \frac{\mathbf{B}(\mathbf{r}, t)}{\sqrt{2\mu_0}}.$$

Upon quantization of the electromagnetic field, the RS vector becomes the field operator $\hat{\mathbf{F}}(\mathbf{r}, t)$. It can be most conveniently decomposed into circularly polarized plane-wave modes with polarization vectors $\mathbf{e}_{\pm}(\mathbf{k})$ for left-handed and right-handed photons, which are related as $\mathbf{e}_-(\mathbf{k}) = (\mathbf{e}_+(\mathbf{k}))^*$. Using the RS operator with such decomposition for expressing the energy density of a one-photon state, the energy breaks into two independent contributions from both polarizations already before integration over space, i. e., the energy density turns out to be given as

$$|\Phi_E(\mathbf{r}, t)|^2 = |\mathbf{F}_+(\mathbf{r}, t)|^2 + |\mathbf{F}_-(\mathbf{r}, t)|^2. \quad (9)$$

where

$$\mathbf{F}_{\pm}(\mathbf{r}, t) = \int d^3k d(k) \mathbf{e}_{\pm}(\mathbf{k}) f_{\pm}(\mathbf{k}) e^{i(\mathbf{k}\mathbf{r} - \omega t)}. \quad (10)$$

Here $d(k)$ is the frequency dependent normalization factor $d(k) = (2\pi)^{-3/2}\sqrt{\hbar\omega} = \sqrt{\hbar c / (2\pi)^3 k^{1/2}}$ and the weight functions $\phi(\mathbf{k}, s = 1, 2)$ have been expressed for the circularly polarized one-photon states as $(2\pi)^{-3/2}f_{\pm}(\mathbf{k})$, cf. Eq. (3).

Again, since both polarization states in Eq. (9) contribute independently (incoherently) to the total energy density, one may consider only one polarization at a time. Thus, the problem of the best localization of a photon reduces to the question: what is the fastest possible falloff with the distance $r = |\mathbf{r}|$ of the modulus of vector functions $\mathbf{F}_{\pm}(\mathbf{r}, t)$?

Since we are interested in a uniform localization in all directions (in 3D space, later in a 2D plane), the expansion in Eq. (10) has to contain plane waves with many very different directions of vectors \mathbf{k} . As the constituents of the expansion are transversal plane waves obeying the condition $\mathbf{k} \times \mathbf{e}_+(\mathbf{k}) = -ik\mathbf{e}_+(\mathbf{k})$, the polarization vectors $\mathbf{e}_{\pm}(\mathbf{k})$ correspondingly possess various directions as well—that highly complicates study of falloff properties of $|\mathbf{F}_{\pm}(\mathbf{r}, t)|$.

Fortunately, such study can be conveniently carried out if we express, following (Bialynicki-Birula, 1996; 1998), the RS vector in terms of a “superpotential” $\mathbf{Z}(\mathbf{r}, t)$,

$$\mathbf{F}(\mathbf{r}, t) = \nabla \times \left[i \frac{\partial}{\partial ct} \mathbf{Z}(\mathbf{r}, t) + \nabla \times \mathbf{Z}(\mathbf{r}, t) \right]. \tag{11}$$

The complex vector field $\mathbf{Z}(\mathbf{r}, t)$ is a complexified version of the Hertz potential, which must obey the homogeneous wave equation and therefore has the following decomposition into plane waves

$$\mathbf{Z}(\mathbf{r}, t) = \int d^3k \left[\mathbf{h}_+(\mathbf{k})e^{i(\mathbf{k}\mathbf{r}-\omega t)} + \mathbf{h}_-(\mathbf{k})e^{-i(\mathbf{k}\mathbf{r}-\omega t)} \right], \tag{12}$$

where $\mathbf{h}_{\pm}(\mathbf{k})$ now are *arbitrary* vector functions of \mathbf{k} , whose directions are not governed by any transversality condition. Like in the case of well-known problem of dipole radiation we may take unidirectional set of vectors $\mathbf{h}_{\pm}(\mathbf{k})$ which depend only on the modulus of \mathbf{k} . In such case we stand a good chance of finding model closed-form expressions for the integral in Eq. (12) with the help of tables of integrals and/or transforms, as we see below. Thereafter one can derive $\mathbf{F}_{\pm}(\mathbf{r}, t)$ (or $\mathbf{F}_{-}(\mathbf{r}, t)$) from Eq. (11) and study falloff behavior of photon energy density $|\mathbf{F}_{\pm}(\mathbf{r}, t)|^2$ (or $|\mathbf{F}_{-}(\mathbf{r}, t)|^2$), which is also proportional to the photon detection probability.

As we are interested in one polarization state we will deal with the first (positive frequency) term in Eq. (12) only and let us choose $\mathbf{h}_+(\mathbf{k})$ in the form (Bialynicki-Birula, 1998)

$$\mathbf{h}_+(\mathbf{k}) \equiv \mathbf{h}(\mathbf{k}) = \mathbf{m}l^3 \frac{h(\sigma)}{\sigma}, \tag{13}$$

where \mathbf{m} is a constant vector that includes the normalization factor and $\sigma = kl$ is wavenumber measured in units of a characteristic length l that will play the role of photon wave function falloff parameter, i. e., l controls the volume of spherically symmetric localization. Since the wave vector dependence of $\mathbf{h}(\mathbf{k})$ in Eq. (13) is isotropic, the 3D integral in Eq. (12) – like in derivation Eq. (7)—reduces into 1D integral in spherical coordinates yielding

$$\mathbf{Z}(\mathbf{r}, t) = 4\pi\mathbf{m}l^2 \int_0^{\infty} dk h(kl) \frac{\sin kr}{r} e^{-ikct} \tag{14}$$

$$= 2\pi i \mathbf{m} \frac{l}{r} \left[g\left(\frac{ct+r}{l}\right) - g\left(\frac{ct-r}{l}\right) \right], \tag{15}$$

where the function g is given by the Fourier transform of $h(\sigma)$ over positive frequencies only

$$g(\tau) = \int_0^{\infty} d\sigma h(\sigma) e^{-i\sigma\tau} . \quad (16)$$

If we split the function under the transform into two factors $h(\sigma) = \bar{h}(\sigma) \exp(-\sigma)$, we notice that the Fourier integral can be expressed as the Laplace transform at a point in the complex plane, whose real coordinate has value 1

$$g(\tau) = \mathcal{L} \{ \bar{h}(\sigma); \sigma, 1 + i\tau \} . \quad (17)$$

Eq. (17) opens possibility to carry out search of such functions $\bar{h}(\sigma)$ in rich tables of the Laplace transform, which correspond to strong falloff of $g(\tau)$. Indeed, for all spectra of the form

$$\bar{h}_{H_n}(\sigma) = 2^{-1} \sigma^{-\frac{n+1}{2}} \exp\left(-\frac{1}{\sigma}\right) H_n\left(\sqrt{\frac{1}{\sigma}}\right) , \quad (18)$$

which contains only half-integer negative powers of the wavenumber irrespective of the order n of the Hermite polynom H_n and due to the exponential factor approaches zero very rapidly as the wavenumber approaches zero, there is a closed-form Laplace transform in (Bateman & Erdelyi, 1954) Sect. 4.11, Eq. (18) or Sect. 5.6, Eq. (8), which yields

$$g_{H_n}(\tau) = 2^{n-1} \sqrt{\pi} (1 + i\tau)^{\frac{n-1}{2}} \exp\left(-2\sqrt{1 + i\tau}\right) , \quad (19)$$

or, consequently, with the help of Eq. (15)

$$\mathbf{Z}(\mathbf{r}, t) = 2^n \pi^{3/2} i \mathbf{m} \frac{l}{r} \left[\begin{array}{l} (1 + i\frac{ct+r}{l})^{\frac{n-1}{2}} \exp\left(-2\sqrt{1 + i\frac{ct+r}{l}}\right) - \\ - (1 + i\frac{ct-r}{l})^{\frac{n-1}{2}} \exp\left(-2\sqrt{1 + i\frac{ct-r}{l}}\right) \end{array} \right] \quad (20)$$

A particular case with $n = 1$, when

$$\bar{h}_1(\sigma) = \sigma^{-\frac{3}{2}} \exp\left(-\frac{1}{\sigma}\right) , \quad (21)$$

$$\mathbf{Z}(\mathbf{r}, t) = 2\pi^{3/2} i \mathbf{m} \frac{l}{r} \left[\exp\left(-2\sqrt{1 + i\frac{ct+r}{l}}\right) - \exp\left(-2\sqrt{1 + i\frac{ct-r}{l}}\right) \right] \quad (22)$$

was found in (Bialynicki-Birula, 1998) and from Eq. (22) $\mathbf{Z}(\mathbf{r}, t = 0)$ and $i\partial_{ct}\mathbf{Z}(\mathbf{r}, t = 0)$, which turn out to be real quantities, were calculated there as well.

The Hertz vector given by Eqs. (20), (22) describes a broadband single photon state having the form of a spherical shell converging for negative and diverging for positive values of time t , attaining the maximal localization at instant $t = 0$. The function $\mathbf{Z}(\mathbf{r}, t)$ and its time derivative $\partial_{ct}\mathbf{Z}(\mathbf{r}, t)$ fall off exponentially at large r as $\exp(-\sqrt{2r/l})$ (multiplied by some power of r), and this property will be shared by their space derivatives involved in Eq. (11). Hence, the photon energy density as defined by Eq. (9) will also exhibit an exponential falloff.

Browsing tables of integrals reveals that for obtaining the exponential falloff the spectra need not contain negative powers of the wavenumber in combinations prescribed by special

polynomials like in Eq. (18). For example, the spectra containing only a single half-integer negative power

$$\bar{h}_n(\sigma) = \sigma^{-n-\frac{1}{2}} \exp\left(-\frac{q}{\sigma}\right), \tag{23}$$

(were q is an optional dimensionless parameter) with the help of (Gradshteyn & Ryzhik, 2000) Eq. (3.472-5) gives

$$g_n(\tau) = (-1)^n \sqrt{\frac{\pi}{1+i\tau}} \frac{\partial^n}{\partial q^n} \exp\left[-2\sqrt{q(1+i\tau)}\right].$$

We see that by putting $q = 1$ we reach exactly the same exponential factor as in Eq. (19).

As shown in the previous Section, the spectrum of the Landau-Peierls (LP) wave function Eq. (2) contains factor $k^{-1/2}$ as compared to the spectrum of photon wave function Eq. (3) or Eq. (10) due to the extra factor $\sqrt{\hbar\omega}$ or $d(k)$ in the expansion of the energy wave function. Hence, in order to obtain the photon position wave function from the Hertz potential with the model spectra Eqs. (19), (21), (23) by the same procedure, one has to use the spectra with an additional factor $(\hbar c)^{-1/2} \sigma^{-1/2}$, i.e., the spectra with integer powers of the wavenumber. Specifically, instead of Eq. (23) we must start (if omitting constants and taking $q = 1$) with

$$\bar{h}_n^{LP}(\sigma) = \sigma^{-n-1} \exp\left(-\frac{1}{\sigma}\right).$$

Fortunately, for such spectrum a closed-form Laplace transform exists (Bateman & Erdelyi, 1954) Eq. (5.16-40) yielding

$$g_n^{LP}(\tau) = \mathcal{L}\left(\sigma^{-\eta-1} \exp\left(-\frac{1}{\sigma}\right); \sigma, 1+i\tau\right) = 2(1+i\tau)^{\eta/2} K_\eta(2\sqrt{1+i\tau}), \eta = n, \tag{24}$$

where K_η is the modified Bessel function (or the Macdonald function) of order η . This result can be used also for obtaining $g_{H_n}^{LP}(\tau)$ corresponding to energy wave function spectrum Eq. (18) if to write out explicitly the Hermite polynomial. Parenthetically, as the last transform formula is valid for any, even complex value of η , then for the half-integer value $\eta = 1/2$ we recover Eq. (22), since $K_{\pm\frac{1}{2}}(z) = \sqrt{\pi/2z}e^{-z}$. Moreover, for any, incl. integer order, the last equality holds *asymptotically*

$$\lim_{|1+i\tau| \rightarrow \infty} \mathcal{L}\left(\sigma^{-\eta-1} \exp\left(-\frac{1}{\sigma}\right); \sigma, 1+i\tau\right) = \sqrt{\pi}(1+i\tau)^{\eta/2-1/4} e^{-2\sqrt{1+i\tau}}.$$

For the particular case $\eta = 1$ this result coincides with (Bialynicki-Birula, 1998) Eq. (34).

Hence, we have reached a general result: if the wavenumber spectrum of decomposition of the Hertz vector $\mathbf{Z}(\mathbf{r}, t)$ into plane waves has the form

$$h(k) \propto \left(\frac{1}{kl}\right)^\alpha \exp\left(-kl - \frac{1}{kl}\right)$$

then, irrespective of the value of power α , at large distances the falloff of $\mathbf{Z}(\mathbf{r}, t)$ as well as of wave functions $\mathbf{F}(\mathbf{r}, t)$, $\Phi(\mathbf{r}, t)$ and corresponding photon energy and probability densities is

dominantly governed by exponential factor, the exponent being proportional to square root of the distance.

Follows inescapable conclusion—for the first time made by (Bialynicki-Birula, 1998) on the basis of one particular spectrum—that the textbook statements, cited in the end of previous Section, are incorrect. Both functions—the position wave function and the energy wave function—may be strongly (= exponentially) concentrated near the origin. The incorrect statements seemingly stem from an idea that asymptotic behavior of convolution of two functions is governed by the one with slower falloff. Apparently such understanding is superficial and need not to be correct if none of the functions has finite support.

4. Limits of uniform localization in all directions

Having shown possibility of asymptotic falloff as $\sim \exp(-A\sqrt{r})$, where A is a constant, naturally the question arises whether a stronger localization is allowed, e. g., according to the exponent with a higher power of the radius r^γ , $1/2 < \gamma \geq 1$. The answer comes from inspection of Eq. (16) with the help of the Paley-Wiener theorem—or criterion, see (Bialynicki-Birula, 1998). According to this theorem, the Fourier transform $g(\tau)$ of a square-integrable function $h(\sigma)$ that vanishes for all negative values of σ (i. e., for negative frequencies in our context) must obey the following integrability condition:

$$\int_{-\infty}^{\infty} d\tau \frac{|\log |g(\tau)||}{1 + \tau^2} < \infty. \quad (25)$$

This condition does not allow for the exponential falloff with $\gamma \geq 1$ but anything arbitrarily weaker than that is allowed. For example, $g(\tau) \sim \exp(-A\tau^{n/(n+1)})$ and even $g(\tau) \sim \exp(-A\tau/\log \tau)$, etc, i. e., almost linear exponential functions are allowed (Bialynicki-Birula, 1998).

Let us take a closer look at how the Paley-Wiener criterion restricts localization of a function whose spectrum contains only positive frequencies, i. e., constitutes an analytic signal.

An 1D right-moving wave function with white spectrum is ultimately (i. e., delta-) localized forever: $\Psi(x, t) \sim \delta(x - ct)$. If we cut off the negative frequency half of the spectrum, we get

$$\Psi_+(x, t) \sim \delta_-(x - ct),$$

$$\delta_-(y) \equiv \frac{1}{2\pi} \int_0^\infty dk e^{iky} = \frac{1}{2} \left[\delta(x - ct) + \frac{i}{\pi} P \frac{A}{x - ct} \right],$$

where the principal value (P) term corrupts the delta-localization. Since $|\delta_-(y)| = 1/2\pi |y|$, the falloff is slow: reciprocally proportional to the distance from the wave peak. Parenthetically, two counterpropagating δ_- -pulses colliding at the origin do not constitute an analytical wave (but still an analytic signal in respect of time) because the wavenumber takes now values of both signs. The imaginary part of such standing-wave-type wave function vanishes at the instant $t = 0$, i. e., it is delta-localized at that instant.

This simple example of ultimate localization in 1D helps to study the case of uniform localization in 3D space. Let us take the spectrum $h(lk)$ in the integral Eq. (14), which is nothing but a superposition of standing spherical waves, in the form $h(lk) =$

$4\pi^{-1} \sin^2(lk/2) / k$. Then for the instant $t = 0$ with the help of Eq. (3.828-3) of Ref. (Gradshteyn & Ryzhik, 2000) we get

$$\int_0^\infty dk 4\pi^{-1} k^{-1} \sin^2(lk/2) \sin kr = 1, \quad r < l, \tag{26}$$

$$= 1/2, \quad r = l,$$

$$= 0, \quad r > l.$$

In other words, in the case of such spectrum $\mathbf{Z}(\mathbf{r}, t = 0)$ is confined into spherical cell of radius l . The support of $\mathbf{Z}(\mathbf{r}, t = 0)$ is finite and in this sense the localization is the strongest. There is no restrictions by the Paley-Wiener theorem here, because strictly at the instant $t = 0$ of maximal localization the integral is nothing but the sine transform for which the theorem does not apply. Indeed, the sine transform tables give examples of the resultant functions with arbitrarily abrupt falloff. However, it does not mean as if the photon localization restriction was lifted at the instant $t = 0$. The explanation is that according to Eq. (11) the energy wave function involves also the time derivative of $\mathbf{Z}(\mathbf{r}, t)$ at $t = 0$, but the sine transforms of two functions $h(lk)$ and $h(lk)k$ (the additional frequency factor k enters the integrand as the Fourier image of $\partial/\partial ct$) cannot simultaneously possess arbitrarily abrupt falloffs.

As soon as $t \neq 0$ Eq. (26) is replaced by the Fourier transform over positive frequencies, which can be evaluated through the Laplace transform \mathcal{L} as follows

$$\mathbf{Z}(\mathbf{r}, t) = 4\pi \mathbf{m} l r^{-1} I(r, t),$$

$$I(r, t) = 4\pi^{-1} \int_0^\infty dk k^{-1} \sin^2(lk/2) \sin kr e^{-ikct} \tag{27}$$

$$= 4\pi^{-1} \lim_{\varepsilon \rightarrow 0} \mathcal{L} \left\{ k^{-1} \sin^2(lk/2) \sin kr; k, \varepsilon + ict \right\}$$

$$= \pi^{-1} \lim_{\varepsilon \rightarrow 0} \left[\arctan\left(\frac{l-r}{\varepsilon + ict}\right) - \arctan\left(\frac{l+r}{\varepsilon + ict}\right) + 2 \arctan\left(\frac{r}{\varepsilon + ict}\right) \right], \tag{28}$$

where ε is the real part of the Laplace transform variable. As one can see in Fig. 1, falloff of $I(r, t \neq 0)$ obeys the Paley-Wiener restriction indeed—asymptotically it is slower than exponential decay with linear exponent. Naturally, $I(r, t = 0)$ returns the behavior of Eq. (26), i. e., the strict confinement into spherical cell of radius l . Sharp peaks and long tails of the modulus of $I(r, t \neq 0)$ originate from the imaginary part of $I(r, t)$ which is—as it is known for an analytic signal—related to the real part through the Hilbert transform. The latter resembles operation of taking derivative but is non-local, thus explaining appearance of the peaks and tails.

Let us take a closer look at the real part of $I(r, t)$, because it deserves interest not only as the main contribution to $|I(r, t)|$. Common classical Hertz potential is a real quantity and contains negative frequencies $\omega = -kc$ as well. Since k as wavenumber in spherical coordinates is by definition positive everywhere in the integrand of Eq. (27) except in the exponent, where it stands for the frequency $\omega = \pm kc$, integration over negative frequencies as well is equivalent to adding to the integral its complex conjugate $I^*(r, t)$. Hence, expressions for classical fields would be governed by the real part of Eq. (28), which is free from the Paley-Wiener restrictions. As one can see in Fig. 2, it constitutes a spherical bipolar pulse of rectangular profile, which collapses—with negative half-cycle ahead—to the origin at negative times and expands – with negative half-cycle behind—from the origin at positive times. At all times it preserves

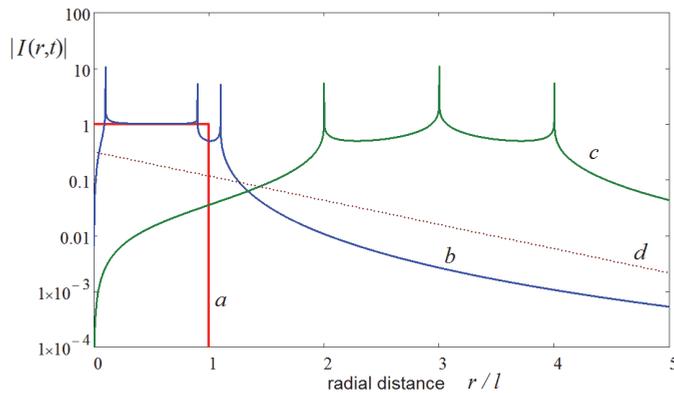


Fig. 1. (color online). Radial dependence of Eq.(28) at various time instants in units l/c : a (red line), $t = 0$; b (blue), $t = 0.1$; c (green), $t = 3$. For comparison falloff of $\pi^{-1} \exp(-r/l)$ is shown (dotted line d).

strict confinement. Between the stages of collapse and expansion the profile undergoes a transformation which is characteristic for all few- and sub-cycle focusing pulses and is caused by the Gouy phase shift (Saari, 2001). Having in mind that sections along horizontal lines in Fig. 2 give radial dependences at fixed instants like in Fig. 1, one can verify that the sharp peaks in the imaginary part correspond indeed to the abrupt steps in the real part.

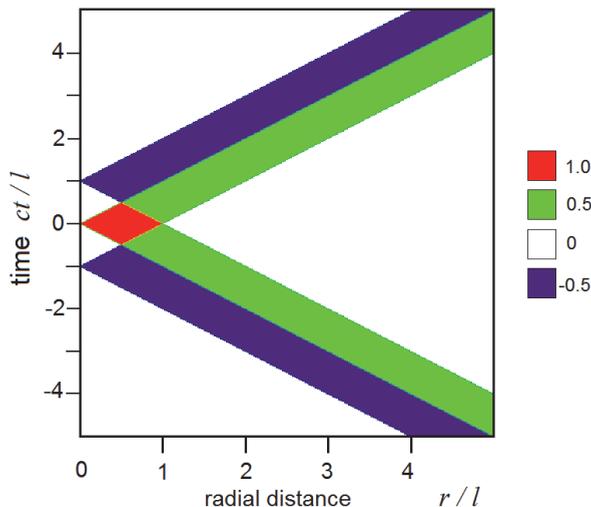


Fig. 2. (color online). Four value levels of the real part of $I(r,t)$ depicted in 2D plane of its arguments. In white areas the function $\text{Re}I(r,t)$ equals strictly to zero, i. e., its support has finite volume.

5. Localization in two dimensions

What if we allow nonuniform localization? Is a two-dimensional localization of the one-photon state also restricted with the Paley-Wiener criterion? As shown in (Saari et al., 2005) the answer to the latter question is NO, if we construct the wave functions from certain so-called localized waves (Besieris et al., 1998; Bialynicki-Birula & Bialynicka-Birula, 2006; Hernández-Figueroa et al., 2008; Recami et al., 2003; Saari & Reivelt, 2004; Salo et al., 2000), which are recently discovered solutions to the linear wave equation. This Section reproduces examples from (Saari et al., 2005).

As the first example leading to a stronger localization that one might expect from the Paley-Wiener theorem, we will consider the photon field which is a superposition of cylindrical solutions of the wave equation. Let us again use the Hertz potential approach, this time putting $\mathbf{Z}(\mathbf{r}, \tau) = \mathbf{m}\Psi(\mathbf{r}, \tau)$, where $\tau \equiv ct$, \mathbf{m} is again a constant vector that includes the proper normalization factor and is directed along the axis z (any other orientation gives similar results), and $\Psi(\mathbf{r}, \tau)$ is a wavepacket of Bessel functions J_0 with the exponential spectrum and a specific dispersion law for the axial wavenumber $k_z(\omega) = \text{const} = k_0$

$$\Psi(\rho, z, \tau) = \Delta \int_{|k_0|}^{\infty} dk J_0(k\rho) e^{-k\Delta} e^{-i(k\tau - k_0z)}, \tag{29}$$

where the radial coordinate ρ has been introduced and $k_\rho = (k^2 - k_0^2)^{1/2}$ is the lateral component of the wave vector of the monochromatic plane-wave constituents represented with the weight function $e^{-k\Delta}$ whose width is Δ^{-1} (spectral width of the packet). The integral can be taken with the help of a Laplace transform table and we obtain

$$\mathbf{Z}(\rho, z, \tau) = \mathbf{m}\Delta \frac{\exp\left(-|k_0| \sqrt{\rho^2 + (\Delta + i\tau)^2}\right)}{\sqrt{\rho^2 + (\Delta + i\tau)^2}} e^{ik_0z}. \tag{30}$$

Eq. (30) describes a simple cylindrical pulse modulated harmonically in the axial direction and radially converging (when $\tau < 0$) to the axis and thereafter (when $\tau > 0$) expanding from it, the intensity distribution resembling an infinitely long tube coaxial with the z axis and with a time-dependent diameter (see Fig. 6. in (Saari & Reivelt, 2004)). It follows from Eqs. (30) and (11) that

$$|\mathbf{Z}(\rho \rightarrow \infty, z, \tau = 0)| \sim \rho^{-1} \exp(-\rho/l), \tag{31}$$

$$|\mathbf{F}(\rho \rightarrow \infty, z, \tau = 0)|^2 \sim \left[\rho^{-2} + O(\rho^{-3})\right] \exp(-2\rho/l), \tag{32}$$

where $l \equiv |k_0|^{-1}$ is the characteristic length (or length unit). Thus, while the photon is completely delocalized in the axial direction, its energy density falloff in the lateral directions is exponential with the linear exponent at all times the conditions $\tau \ll \rho \gg \Delta$ are fulfilled, see Fig. 3. The time derivative as well as the spatial derivatives contain the same exponential factor, ensuring the exponential falloff of the the Riemann-Silberstein vector in Eq. (32).

Hence, a one-photon field given by Eq. (30) serves as the first and simplest example where the localization in two transversal dimensions is governed by different rules than uniform localization in three dimensions.

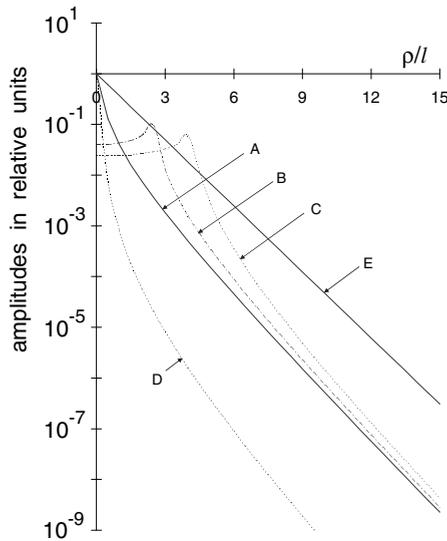


Fig. 3. Curves of the radial dependence in a decimal logarithmic scale. Curve A is for $|Z(\rho, 0, \tau = 0)|$; B, $|Z(\rho, 0, \tau = 2.5l)|$; C is the same as B but with Ψ taken from Eq. (33); D, $|\frac{\partial}{\partial \tau} Z(\rho, 0, \tau = 0)|$; E is a reference curve $\exp(-\rho/l)$. The curves A, B, and C have been normalized so that $|Z(0, 0, 0)| = 1$. The values of the remaining free parameters are $\Delta = 0.1l$ and $\beta = 0.8$.

The next example is readily available via the Lorentz transformation of the wave function given by Eq. (29) along the axis z , which gives another possible solution of the scalar wave equation. The result is a new independent solution but it can also be considered as the wave given by Eqs. (29) and (30), which is observed in another inertial reference frame (Saari & Reivelt, 2004):

$$\Psi(\rho, z, \tau) = \Delta \frac{\exp\left(-|k_0| \sqrt{\rho^2 + (\Delta - i\gamma(\beta z - \tau))^2}\right)}{\sqrt{\rho^2 + (\Delta - i\gamma(\beta z - \tau))^2}} \times \exp(i\gamma k_0(z - \beta\tau)), \tag{33}$$

where the relativistic factors $\gamma \equiv (1 - \beta^2)^{-1/2}$ and $\beta \equiv v/c < 1$ have been introduced, v being a free parameter—the relative speed between the frames. In the waist region (see Fig. 4) this wave function has the same radial falloff as was given by Eq. (31), see curve "C" in Fig. 1, while the axial localization follows a power law. The strongly localized waist and the whole amplitude distribution move rigidly and without any spread along the axis z with a superluminal speed c/β . Such wave with intriguing properties, named the focused X wave (FXW) (Besieris et al., 1998; Saari & Reivelt, 2004), belongs to the so-called superluminal propagation-invariant localized waves. Although the FXW is not experimentally generated yet, a set-up based on a cylindrical diffraction grating has been proposed and its properties analyzed (Valtna et al., 2007). It should be noted here that there is nothing unphysical in the

superluminality of the localized waves since a superluminal group velocity does not mean as if energy or information could be transmitted faster than c . This is an experimentally verified fact for the so-called Bessel-X pulse which is another representative of the family of superluminal waves (Alexeev et al., 2002; Bowlan et al., 2009; Saari et al., 2010; Saari & Reivelt, 1997). For a detailed analysis of the startling superluminality see (Saari, 2004; Saari et al., 2010) and references therein.

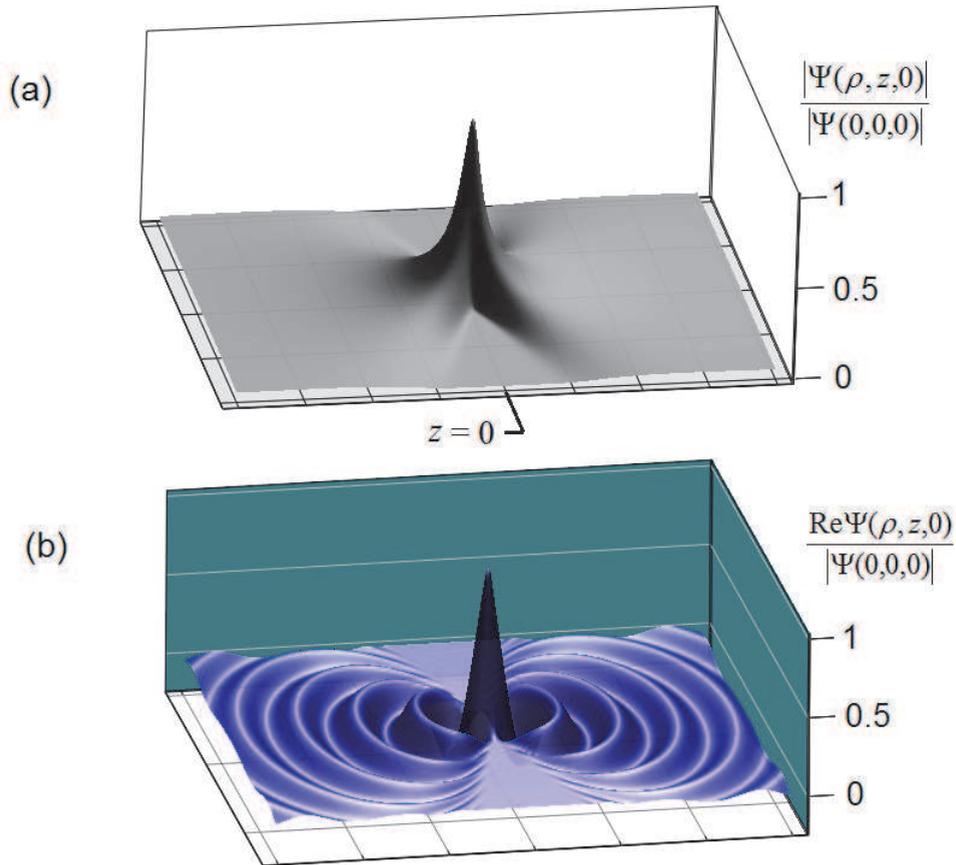


Fig. 4. (color online). The superluminal FXW given by Eq. (33). Shown are the dependences (a) of the modulus and (b) of the real part of the wavefunction on the longitudinal (z , increasing to the right) and a transverse (say, x) coordinates. The distance between the grid lines on the basal plane (x, z) is 22λ , where $\lambda = 2\pi|k_0|^{-1}$, k_0 being negative. The values of the remaining free parameters are $\Delta = 30\lambda$ and $\beta = 0.995$ or $\gamma = 10$.

Hence, in its waist (cross-sectional) plane a one-photon field given by the FXW possesses the same strong localization at any time as the previously considered cylindrical field does in any transversal plane at the instant $t = 0$.

By making use of the historically first representative of localized waves – the so-called focus wave mode (FWM)—see (Brittingham, 1983; Sezginer, 1985) and also reviews (Besieris et al., 1998; Saari & Reivelt, 2004) and references therein – one readily obtains an example of the field that exhibits even much stronger than simple exponential localization. The FWM is given by the scalar function

$$\Psi(\rho, z, \tau) = a \frac{\exp\left[-\frac{\rho^2}{2l(a-i(z-\tau))}\right]}{a-i(z-\tau)} \exp\left[-\frac{i(z+\tau)}{2l}\right], \quad (34)$$

where again l is a wavelength-type characteristic length and the constant a controls the axial localization length. This wave function is depicted in Fig. 5. Since the FXW in the limit $\beta \rightarrow 1$ becomes a FWM (Besieris et al., 1998; Saari & Reivelt, 2004), Fig. 5 qualitatively resembles Fig. 4(b) (a 3D animated color plot of FWM is available in open-access on-line paper (Sheppard & Saari, 2008)).

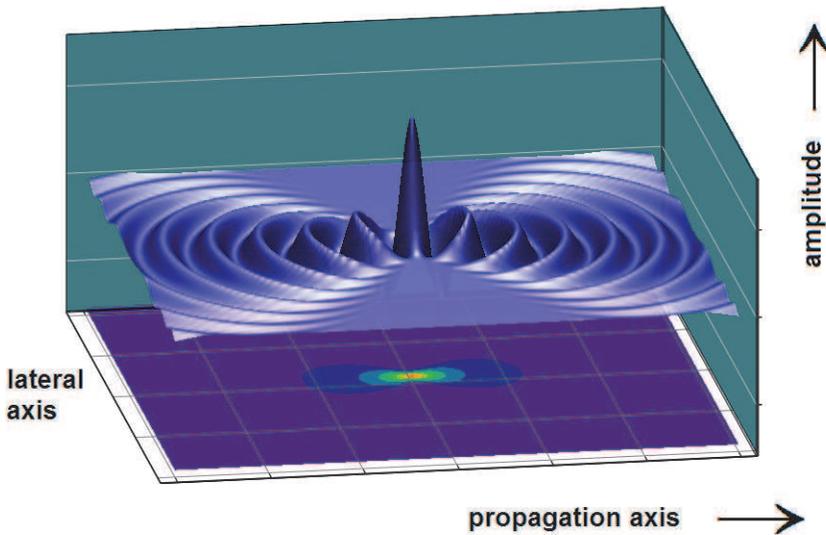


Fig. 5. (color online). The luminal FWM given by Eq. (34). Shown are a surface plot of the real part of the wave function and (in the basement plane) a contour plot of its modulus. For details and animation of the time dependence see (Sheppard & Saari, 2008), <http://www.opticsinfobase.org/oe/viewmedia.cfm?uri=oe-16-1-150&seq=3>.

Multiplying Eq. (34) by \mathbf{m} to build the vector $\mathbf{Z}(\rho, z, \tau)$ and inserting the latter into Eq. (11) we obtain that in this example the photon localization in the waist plane is quadratically exponential (Gaussian falloff):

$$|\mathbf{Z}(\rho \rightarrow \infty, z = \tau)| \sim \exp(-\rho^2/2la), \quad (35)$$

$$|\mathbf{F}(\rho \rightarrow \infty, z = \tau)|^2 \sim \rho^6 \exp(-\rho^2/la). \quad (36)$$

In Eq. (36) only the highest-power term with respect to ρ is shown.

6. Discussion

To start discussing our results that seem to be at variance with the Paley-Wiener restriction, let us ask first whether the wave functions considered are something extraordinary. The answer is: yes, they are indeed, since browsing various integral transform tables reveals rather few examples where both the real and imaginary part of a wave function and of its time derivative have simultaneously an exponential or stronger localization in conjunction with other requisite properties. Fortunately, the list of proper wave functions with an extraordinary strong localization is growing—in addition to an optically feasible version (Reivelt & Saari, 2002; 2004) of the FWM various new interesting solutions with Gaussian falloff can be derived (Kiselev, 2007). Yet, it could be argued that the well-known Gaussian beam pulse has the same quadratically exponential radial profile in the waist region. However, resorting to the family of the Gaussian beams (the Gauss-Laguerre and Gauss-Hermite beams, etc.) is irrelevant here. The reason is that all these beams are solutions of the wave equation only in the paraxial approximation not valid in the case of any significant localization of wide-band (pulsed) superpositions of the beams and at large values of the radial distance. As a matter of fact, e.g., an exact solution corresponding to a lowest-order (axisymmetric) Gaussian beam has a weak power-law radial falloff in the waist region (Saari, 2001; Sheppard & Saghafi, 1999).

The next possible objection to the physical significance of the results obtained might arise from the infinite total energy (Besieris et al., 1998; Hernández-Figueroa et al., 2008) of the waves given by Eqs. (30), (33), and (34). However, at any spatial location the wave function is square integrable with respect to time, thus the condition of the Paley-Wiener theorem has been satisfied. Moreover, physically feasible finite-energy (i. e., finite-aperture) versions of localized waves generally exhibit even better localization properties, although not persistently. A finite-energy version of the FXW, called the modified focused X wave (MFXW) (Besieris et al., 1998; Valtna et al., 2006), has the same exponential factor as in Eq. (33), which is multiplied by a fraction that allows to force the axial localization to follow an arbitrarily strong power-law. The latter circumstance indicates that the strong lateral localization of the fields considered does not appear somehow at the expense of their axial localization.

As a matter of fact, energy-normalization of a wave function depends on how many photons it describes. It is easy to see that derivations and results presented here hold for any number state with $N \geq 1$ and also for incoherent mixtures of such states (which is important for experimental studies). Here it is not of interest to consider coherent states since generally for states of electromagnetic field that have classical counterparts one can escape—already in the case of uniform spherical localization—the constraints imposed by the Paley-Wiener theorem (Bialynicki-Birula, 1998).

The final crucial question is: are our results in contradiction with those of (Bialynicki-Birula, 1998) reproduced in Section 3? The answer is *no*, since in the case of the cylindrical waves the radial distance and temporal frequency are not directly Fourier-conjugated variables. In order to clarify this point, let us first take a closer look at Eqs. (14) and (15). The sine in Eq. (14) results from the imploding and exploding spherical wave constituents of the standing wave, like an odd one-dimensional standing wave arises from counterpropagating waves. We saw in Section 4 that although strictly at the instant $\tau = 0$ the function $\mathbf{Z}(\mathbf{r}, t)$ can possess arbitrarily abrupt falloff, simultaneously its time derivative and hence the wave function $\mathbf{F}(\mathbf{r}, t)$ cannot. In contrast, the time derivative of the wave function given by Eq. (30) or Eq. (29) has the same

strong exponential falloff as the function itself, which persists for some (not too long) time, see Fig. 3. By comparing Eqs. (29) and (14) we notice that while in Eq. (14) the argument of the sine function is the product of the distance with the Fourier variable, in Eq. (29) the argument of the Bessel function is the product of the radial distance ρ with the radial wavenumber k_ρ , the latter depending on the Fourier variable through the square-root expression with the constant parameter k_0 —the lower limit of the integration. As it follows also from Eqs. (31) and (32) the condition $k_0 \neq 0$ is crucial for obtaining the exponential falloff. Hence, in the case of the cylindrical waves considered by us, the apparent violation of the rules set by the Paley-Wiener theorem results from the specific complicated relation between the radial distance and the Fourier variable.

7. Conclusion

The problem of photon localization is of rather fundamental nature in quantum electrodynamics. Despite of almost 80-year history of the problem – and the related problem of the photon wave function—the interest in the revision of it has quickened in the recent years. One of the stimulus for that might be developments in modern optics, particularly in femtosecond and quantum optics, thanks to which the somewhat academic problem is transforming into a practical one. Indeed, e. g., availability and applications of single- and sub-cycle photon pulses will force a revision of traditional notions in optics based on the narrow-band approximation. In particular, phrases like "localization cannot be better than wavelength" are losing sense in the case of such pulses.

Ultrawideband by definition are the so-called localized waves—an emerging new field in wave acoustics and physical optics. We have shown that an interdisciplinary "technology transfer"—application of methods and solutions found in the field of localized waves—is productive for the study of photon localization.

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The book embraces a wide spectrum of problems falling under the concepts of "Quantum optics" and "Laser experiments". These actively developing branches of physics are of great significance both for theoretical understanding of the quantum nature of optical phenomena and for practical applications. The book includes theoretical contributions devoted to such problems as providing a general approach to describe electromagnetic field states with correlation functions of different nature, nonclassical properties of some superpositions of field states in time-varying media, photon localization, mathematical apparatus that is necessary for field state reconstruction on the basis of restricted set of observables, and quantum electrodynamics processes in strong fields provided by pulsed laser beams. Experimental contributions are presented in chapters about some quantum optics processes in photonic crystals - media with spatially modulated dielectric properties - and chapters dealing with the formation of cloud of cold atoms in magneto optical trap. All chapters provide the necessary basic knowledge of the phenomena under discussion and well-explained mathematical calculations.

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