

Identifiability of Piecewise Constant Conductivity

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1. Introduction

Consider the heat conduction in a nonhomogeneous insulated rod of a unit length, with the ends kept at zero temperature at all times. Our main interest is in the identification and identifiability of the discontinuous conductivity (thermal diffusivity) coefficient $a(x)$, $0 \leq x \leq 1$. The **identification** problem consists of finding a conductivity $a(x)$ in an admissible set K for which the temperature $u(x, t)$ fits given observations in a prescribed sense.

Under a wide range of conditions one can establish the continuity of the objective function $J(a)$ representing the best fit to the observations. Then the existence of the best fit to data conductivity follows if the admissible set K is compact in the appropriate topology. However, such an approach usually does not guarantee the uniqueness of the found conductivity $a(x)$. Establishing such a uniqueness is referred to as the **identifiability** problem. For an extensive survey of heat conduction, including inverse heat conduction problems see (Beck et al., 1985; Cannon, 1984; Ramm, 2005)

From physical considerations the conductivity coefficients $a(x)$ are assumed to be in

$$A_{\text{ad}} = \{a \in L^\infty(0, 1) : 0 < \nu \leq a(x) \leq \mu\}. \quad (1)$$

The temperature $u(a) = u(x, t; a)$ inside the rod satisfies

$$\begin{aligned} u_t - (a(x)u_x)_x &= f(x, t), & Q &= (0, 1) \times (0, T), \\ u(0, t) &= q_1(t), \quad u(1, t) = q_2(t), & t &\in (0, T), \\ u(x, 0) &= g(x), & x &\in (0, 1), \end{aligned} \quad (2)$$

where $g \in H = L^2(0, 1)$, $q_1, q_2 \in C^1[0, \infty)$. Suppose that one is given an observation $z(t) = u(p, t; a)$ of the heat conduction process (2) for $t_1 < t < t_2$ at some observation point $0 < p < 1$. From the series solution for (2) and the uniqueness of the Dirichlet series expansion (see Section 5), one can, in principle, recover all the eigenvalues of the associated Sturm-Liouville problem. If one also knows the eigenvalues for the heat conduction process with the same coefficient a and different boundary conditions, then classical results of Gelfand and Levitan (Gelfand & Levitan, 1955) show that the conductivity $a(x)$ can be uniquely identified from the knowledge of the two spectral sequences.

Alternatively, the conductivity is identifiable if the entire spectral function is known (i.e. the eigenvalues and the values of the derivatives of the normalized eigenfunctions at $x = 0$). However, such results have little practical value, since the observation data $z(t)$ always

contain some noise, and therefore one cannot hope to adequately identify more than just a few first eigenvalues of the problem.

A different approach is taken in (Duchateau, 1995; Kitamura & Nakagiri, 1977; Nakagiri, 1993; Orlov & Bentsman, 2000; Pierce, 1979). These works show that one can identify a constant conductivity a in (2) from the measurement $z(t)$ taken at one point $p \in (0, 1)$. These works also discuss problems more general than (2), including problems with a broad range of boundary conditions, non-zero forcing functions, as well as elliptic and hyperbolic problems. In (Elayyan & Isakov, 1997; Kohn & Vogelius, 1985) and references therein identifiability results are obtained for elliptic and parabolic equations with discontinuous parameters in a multidimensional setting. A typical assumption there is that one knows the normal derivative of the solution at the boundary of the region for every Dirichlet boundary input. For more recent work see (Benabdallah et al., 2007; Demir & Hasanov, 2008; Isakov, 2006).

In our work we examine piecewise constant conductivities $a(x)$, $x \in [0, 1]$. Suppose that the conductivity a is known to have sufficiently separated points of discontinuity. More precisely, let $a \in PC(\sigma)$ defined in Section 2. Let $u(x, t; a)$ be the solution of (2). The eigenfunctions and the eigenvalues for (2) are defined from the associated Sturm-Liouville problem (5).

In our approach the identifiability is achieved in two steps:

First, given finitely many equidistant observation points $\{p_m\}_{m=1}^{M-1}$ on interval $(0, 1)$ (as specified in Theorem 5.5), we extract the first eigenvalue $\lambda_1(a)$ and a constant nonzero multiple of the first eigenfunction $G_m(a) = C(a)\psi_1(p_m; a)$ from the observations $z_m(t; a) = u(p_m, t; a)$. This defines the M -tuple

$$\mathcal{G}(a) = (\lambda_1(a), G_1(a), \dots, G_{M-1}(a)) \in \mathbb{R}^M. \quad (3)$$

Second, the Marching Algorithm (see Theorem 5.5) identifies the conductivity a from $\mathcal{G}(a)$.

We start by recalling some basic properties of the eigenvalues and the eigenfunctions for (2) in Section 2. Our main identifiability result is Theorem 5.5. It is discussed in Section 5. The continuity properties of the solution map $a \rightarrow \mathcal{G}(a)$ are established in Section 4, and the continuity of the identification map $\mathcal{G}^{-1}(a)$ is proved in Section 8. Computational algorithms for the identification of $a(x)$ from noisy data are presented in Section 10.

This exposition outlines main results obtained in (Gutman & Ha, 2007; 2009). In (Gutman & Ha, 2007) the case of distributed measurements is considered as well.

2. Properties of the eigenvalues and the eigenfunctions

The admissible set A_{ad} is too wide to obtain the desired identifiability results, so we restrict it as follows.

Definition 2.1. (i) $a \in \mathcal{PS}_N$ if function a is piecewise smooth, that is there exists a finite sequence of points $0 = x_0 < x_1 < \dots < x_{N-1} < x_N = 1$ such that both $a(x)$ and $a'(x)$ are continuous on every open subinterval (x_{i-1}, x_i) , $i = 1, \dots, N$ and both can be continuously extended to the closed intervals $[x_{i-1}, x_i]$, $i = 1, \dots, N$. For definiteness, we assume that a and a' are continuous from the right, i.e. $a(x) = a(x+)$ and $a'(x) = a'(x+)$ for all $x \in [0, 1)$. Also let $a(1) = a(1-)$.

(ii) Define $\mathcal{PS} = \bigcup_{N=1}^{\infty} \mathcal{PS}_N$.

(iii) Define $\mathcal{PC} \subset \mathcal{PS}$ as the class of piecewise constant conductivities, and $\mathcal{PC}_N = \mathcal{PC} \cap \mathcal{PS}_N$. Any $a \in \mathcal{PC}_N$ has the form $a(x) = a_i$ for $x \in [x_{i-1}, x_i)$, $i = 1, 2, \dots, N$.

(iv) Let $\sigma > 0$. Define

$$\mathcal{PC}(\sigma) = \{a \in \mathcal{PC} : x_i - x_{i-1} \geq \sigma, \quad i = 1, 2, \dots, N\},$$

where x_1, x_2, \dots, x_{N-1} are the discontinuity points of a , and $x_0 = 0, x_N = 1$.

Note that $a \in \mathcal{PC}(\sigma)$ attains at most $N = \lceil [1/\sigma] \rceil$ distinct values $a_i, 0 < v \leq a_i \leq \mu$. For $a \in \mathcal{PS}_N$ the governing system (2) is given by

$$\begin{cases} u_t - (a(x)u_x)_x = f(x, t), & x \neq x_i, \quad t \in (0, T), \\ u(0, t) = q_1(t), \quad u(1, t) = q_2(t), & t \in (0, T), \\ u(x_i+, t) = u(x_i-, t), & t \in (0, T), \\ a(x_i+)u_x(x_i+, t) = a(x_i-)u_x(x_i-, t), & t \in (0, T), \\ u(x, 0) = g(x), & x \in (0, 1). \end{cases} \tag{4}$$

The associated Sturm-Liouville problem for (4) is

$$\begin{cases} (a(x)\psi(x)')' = -\lambda\psi(x), \quad x \neq x_i, \\ \psi(0) = \psi(1) = 0, \\ \psi(x_i+) = \psi(x_i-), \\ a(x_i+)\psi_x(x_i+) = a(x_i-)\psi_x(x_i-). \end{cases} \tag{5}$$

For convenience we collect basic properties of the eigenvalues and the eigenfunctions of (5). Additional details can be found in (Birkhoff & Rota, 1978; Evans, 2010; Gutman & Ha, 2007).

Theorem 2.2. *Let $a \in \mathcal{PS}$. Then*

- (i) *The associated Sturm-Liouville problem (5) has infinitely many eigenvalues*

$$0 < \lambda_1 < \lambda_2 < \dots \rightarrow \infty.$$

The eigenvalues $\{\lambda_k\}_{k=1}^\infty$ and the corresponding orthonormal set of eigenfunctions $\{\psi_k\}_{k=1}^\infty$ satisfy

$$\lambda_k = \int_0^1 a(x)[\psi'_k(x)]^2 dx, \tag{6}$$

$$\lambda_k = \inf \left\{ \frac{\int_0^1 a(x)[\psi'(x)]^2 dx}{\int_0^1 [\psi(x)]^2 dx} : \psi \perp \text{span}\{\psi_1, \dots, \psi_{k-1}\} \subset H_0^1(0, 1) \right\}. \tag{7}$$

The normalized eigenfunctions $\{\psi_k\}_{k=1}^\infty$ form a basis in $L^2(0, 1)$. Eigenfunctions $\{\psi_k/\sqrt{\lambda_k}\}_{k=1}^\infty$ form an orthonormal basis in

$$V_a = \{\psi \in H_0^1(0, 1) : \int_0^1 a(x)[\psi'(x)]^2 dx < \infty\}.$$

- (ii) *Each eigenvalue is simple. For each eigenvalue λ_k there exists a unique continuous, piecewise smooth normalized eigenfunction $\psi_k(x)$ such that $\psi'_k(0+) > 0$, and the function $a(x)\psi'_k(x)$ is continuous on $[0, 1]$.*
- (iii) *Eigenvalues $\{\lambda_k\}_{k=1}^\infty$ satisfy Courant min-max principle*

$$\lambda_k = \min_{V_k} \max \left\{ \frac{\int_0^1 a(x)[\psi'(x)]^2 dx}{\int_0^1 [\psi(x)]^2 dx} : \psi \in V_k \right\},$$

where V_k varies over all subspaces of $H_0^1(0, 1)$ of finite dimension k .

(iv) Eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ satisfy the inequality

$$v\pi^2k^2 \leq \lambda_k \leq \mu\pi^2k^2.$$

(v) First eigenfunction ψ_1 satisfies $\psi_1(x) > 0$ for any $x \in (0, 1)$.

(vi) First eigenfunction ψ_1 has a unique point of maximum $q \in (0, 1)$: $\psi_1(x) < \psi_1(q)$ for any $x \neq q$.

Proof. (i) See (Evans, 2010).

(ii) On any subinterval (x_i, x_{i+1}) the coefficient $a(x)$ has a bounded continuous derivative. Therefore, on any such interval the initial value problem $(a(x)v'(x))' + \lambda v = 0$, $v(x_i) = A$, $v'(x_i) = B$ has a unique solution. Suppose that two eigenfunctions $w_1(x)$ and $w_2(x)$ correspond to the same eigenvalue λ_k . Then they both satisfy the condition $w_1(0) = w_2(0) = 0$. Therefore their Wronskian is equal to zero at $x = 0$. Consequently, the Wronskian is zero throughout the interval (x_0, x_1) , and the solutions are linearly dependent there. Thus $w_2(x) = Cw_1(x)$ on (x_0, x_1) , $w_2(x_1-) = Cw_1(x_1-)$ and $w_2'(x_1-) = Cw_1'(x_1-)$. The linear matching conditions imply that $w_2(x_1+) = Cw_1(x_1+)$ and $w_2'(x_1+) = Cw_1'(x_1+)$. The uniqueness of solutions implies that $w_2(x) = Cw_1(x)$ on (x_1, x_2) , etc. Thus $w_2(x) = Cw_1(x)$ on $(0, 1)$ and each eigenvalue λ_k is simple. In particular λ_1 is a simple eigenvalue. The uniqueness and the matching conditions also imply that any solution of $(a(x)v'(x))' + \lambda v = 0$, $v(0) = 0$, $v'(0) = 0$ must be identically equal to zero on the entire interval $(0, 1)$. Thus no eigenfunction $\psi_k(x)$ satisfies $\psi_k'(0) = 0$. Assuming that the eigenfunction ψ_k is normalized in $L^2(0, 1)$ it leaves us with the choice of its sign for $\psi_k'(0)$. Letting $\psi_k'(0) > 0$ makes the eigenfunction unique.

(iii) See (Evans, 2010).

(iv) Suppose $a(x) \leq b(x)$ for $x \in [0, 1]$. The min-max principle implies $\lambda_k(a) \leq \lambda_k(b)$. Since the eigenvalues of (7) with $a(x) = 1$ are π^2k^2 the required inequality follows.

(v) Recall that $\psi_1(x)$ is a continuous function on $[0, 1]$. Suppose that there exists $p \in (0, 1)$ such that $\psi_1(p) = 0$. Let $w_l(x) = \psi_1(x)$ for $0 \leq x < p$, and $w_l(x) = 0$ for $p \leq x \leq 1$. Let $w_r(x) = \psi_1(x) - w_l(x)$, $x \in [0, 1]$. Then w_l, w_r are continuous, and, moreover, $w_l, w_r \in H_0^1(0, 1)$. Also

$$\int_0^1 w_l(x)w_r(x)dx = 0, \quad \text{and} \quad \int_0^1 a(x)w_l'(x)w_r'(x)dx = 0.$$

Suppose that w_l is not an eigenfunction for λ_1 . Then

$$\int_0^1 a(x)[w_l'(x)]^2dx > \lambda_1 \int_0^1 [w_l(x)]^2dx.$$

Since

$$\int_0^1 a(x)[w_r'(x)]^2dx \geq \lambda_1 \int_0^1 [w_r(x)]^2dx$$

we have

$$\lambda_1 = \frac{\int_0^1 a(x)[\psi_1'(x)]^2dx}{\int_0^1 [\psi_1(x)]^2dx} = \frac{\int_0^1 a(x)([w_l'(x)]^2 + [w_r'(x)]^2)dx}{\int_0^1 ([w_l(x)]^2 + [w_r(x)]^2)dx} >$$

$$\frac{\int_0^1 (\lambda_1 [w_l(x)]^2 + \lambda_1 [w_r(x)]^2) dx}{\int_0^1 ([w_l(x)]^2 + [w_r(x)]^2) dx} = \lambda_1.$$

This contradiction implies that w_l (and w_r) must be an eigenfunction for λ_1 . However, $w_l(x) = 0$ for $p \leq x \leq 1$, and as in (ii) it implies that $w_l(x) = 0$ for all $x \in [0, 1]$ which is impossible. Since $\psi'_1(0) > 0$ the conclusion is that $\psi_1(x) > 0$ for $x \in (0, 1)$.

(vi) From part (ii), any eigenfunction ψ_k is continuous and satisfies

$$(a(x)\psi'_k(x))' = -\lambda_k\psi_k(x)$$

for $x \neq x_i$. Also function $a(x)\psi'_k(x)$ is continuous on $[0, 1]$ because of the matching conditions at the points of discontinuity $x_i, i = 1, 2, \dots, N - 1$ of a . The integration gives

$$a(x)\psi'_k(x) = a(p)\psi'_k(p) - \lambda_k \int_p^x \psi_k(s) ds,$$

for any $x, p \in (0, 1)$.

Let $p \in (0, 1)$ be a point of maximum of ψ_k . If $p \neq x_i$ then $\psi'_k(p) = 0$. If $p = x_i$, then $\psi'_k(x_i-) \geq 0$ and $\psi'_k(x_i+) \leq 0$. Therefore $\lim_{x \rightarrow p} a(x)\psi'_k(x) = 0$, and $\psi'_k(p+) = \psi'_k(p-) = 0$ since $a(x) \geq \nu > 0$. In any case for such point p we have

$$a(x)\psi'_k(x) = -\lambda_k \int_p^x \psi_k(s) ds, \quad x \in (0, 1). \tag{8}$$

Since $\psi_1(x) > 0, a(x) > 0$ on $(0, 1)$ equation (8) implies that $\psi'_1(x) > 0$ for any $0 \leq x < p$ and $\psi'_1(x) < 0$ for any $p < x \leq 1$. Since the derivative of ψ_1 is zero at any point of maximum, we have to conclude that such a maximum p is unique. □

3. Representation of solutions

First, we derive the solution of (4) with $f = q_1 = q_2 = 0$. Then we consider the general case.

Theorem 3.1. (i) Let $g \in H = L^2(0, 1)$. For any fixed $t > 0$ the solution $u(x, t)$ of

$$\begin{aligned} u_t - (a(x)u_x)_x &= 0, & Q &= (0, 1) \times (0, T), \\ u(0, t) &= 0, \quad u(1, t) &= 0, & t \in (0, T), \\ u(x, 0) &= g(x), & x &\in (0, 1) \end{aligned} \tag{9}$$

is given by

$$u(x, t; a) = \sum_{k=1}^{\infty} \langle g, \psi_k \rangle e^{-\lambda_k t} \psi_k(x),$$

and the series converges uniformly and absolutely on $[0, 1]$.

(ii) For any $p \in (0, 1)$ function

$$z(t) = u(p, t; a), \quad t > 0$$

is real analytic on $(0, \infty)$.

Proof. (i) Note that the eigenvalues and the eigenfunctions satisfy

$$\nu \|\psi'_k\|^2 \leq \int_0^1 a(x) [\psi'_k(x)]^2 dx = \lambda_k \|\psi_k\|^2 = \lambda_k.$$

Thus

$$\|\psi'_k\| \leq \frac{\sqrt{\lambda_k}}{\sqrt{v}},$$

and

$$|\psi_k(x)| \leq \int_0^x |\psi'_k(s)| ds \leq \|\psi'_k\| \leq \frac{\sqrt{\lambda_k}}{\sqrt{v}}.$$

Bessel's inequality implies that the sequence of Fourier coefficients $\langle g, \psi_k \rangle$ is bounded. Therefore, denoting by C various constants and using the fact that the function $s \rightarrow \sqrt{s}e^{-\sigma s}$ is bounded on $[0, \infty)$ for any $\sigma > 0$ one gets

$$|\langle g, \psi_k \rangle e^{-\lambda_k t} \psi_k(x)| \leq C \frac{\sqrt{\lambda_k}}{\sqrt{v}} e^{-\frac{\lambda_k t}{2}} e^{-\frac{\lambda_k t}{2}} \leq C e^{-\frac{\lambda_k t}{2}}.$$

From (iv) of Theorem 2.2 $\lambda_k \geq v\pi^2 k^2$. Thus

$$\sum_{k=1}^{\infty} |\langle g, \psi_k \rangle e^{-\lambda_k t} \psi_k(x)| \leq C \sum_{k=1}^{\infty} e^{-\frac{v\pi^2 k^2 t}{2}} \leq C \sum_{k=1}^{\infty} \left(e^{-\frac{v\pi^2 t}{2}} \right)^k < \infty.$$

By Weierstrass M -test the series converges absolutely and uniformly on $[0, 1]$.

- (ii) Let $t_0 > 0$ and $p \in (0, 1)$. From (i), the series $\sum_{k=1}^{\infty} \langle g, \psi_k \rangle e^{-\lambda_k t_0} \psi_k(p)$ converges absolutely. Therefore $\sum_{k=1}^{\infty} \langle g, \psi_k \rangle e^{-\lambda_k s} \psi_k(p)$ is analytic in the part of the complex plane $\{s \in \mathbb{C} : \operatorname{Re} s > t_0\}$, and the result follows. \square

Next we establish a representation formula for the solutions $u(x, t; a)$ of (4) under more general conditions. Suppose that $u(x, t; a)$ is a strong solution of (4), i.e. the equation and the initial condition in (4) are satisfied in $H = L^2(0, 1)$. Let

$$\Phi(x, t; a) = \frac{q_2(t) - q_1(t)}{\int_0^1 \frac{1}{a(s)} ds} \int_0^x \frac{1}{a(s)} ds + q_1(t). \quad (10)$$

Then $v(x, t; a) = u(x, t; a) - \Phi(x, t; a)$ is a strong solution of

$$\begin{cases} v_t - (av_x)_x = -\Phi_t + f, & 0 < x < 1, 0 < t < T, \\ v(0, t) = 0, & 0 < t < T, \\ v(1, t) = 0, & 0 < t < T, \\ v(x, 0) = g(x) - \Phi(x, 0), & 0 < x < 1. \end{cases} \quad (11)$$

Accordingly, the weak solution u of (4) is defined by $u(x, t; a) = v(x, t; a) + \Phi(x, t; a)$ where v is the weak solution of (11). For the existence and the uniqueness of the weak solutions for such evolution equations see (Evans, 2010; Lions, 1971).

Let $V = H_0^1(0, 1)$ and $X = C[0, 1]$.

Theorem 3.2. *Suppose that $T > 0$, $a \in \mathcal{PS}$, $g \in H$, $q_1, q_2 \in C^1[0, T]$ and $f(x, t) = h(x)r(t)$ where $h \in H$ and $r \in C[0, T]$. Then*

- (i) *There exists a unique weak solution $u \in C((0, T]; X)$ of (4).*

(ii) Let $\{\lambda_k, \psi_k\}_{k=1}^\infty$ be the eigenvalues and the eigenfunctions of (5). Let $g_k = \langle g, \psi_k \rangle$, $\phi_k(t) = \langle \Phi(\cdot, t), \psi_k \rangle$ and $f_k(t) = \langle f(\cdot, t), \psi_k \rangle$ for $k = 1, 2, \dots$. Then the solution $u(x, t; a)$, $t > 0$ of (4) is given by

$$u(x, t; a) = \Phi(x, t; a) + \sum_{k=1}^\infty B_k(t; a)\psi_k(x), \tag{12}$$

where

$$B_k(t; a) = e^{-\lambda_k t}(g_k - \phi_k(0; a)) + \int_0^t e^{-\lambda_k(t-\tau)}(f_k(\tau) - \phi'_k(\tau; a))d\tau \tag{13}$$

for $k = 1, 2, \dots$.

(iii) For each $t > 0$ and $a \in \mathcal{PS}$ the series in (12) converges in X . Moreover, this convergence is uniform with respect to t in $0 < t_0 \leq t \leq T$ and $a \in \mathcal{PS}$.

Proof. Under the conditions specified in the Theorem the existence and the uniqueness of the weak solution $v \in C([0, T]; H) \cap L^2([0, T]; V)$ of (11) is established in (Evans, 2010; Lions, 1971). By the definition $u = v + \Phi$. Thus the existence and the uniqueness of the weak solution u of (4) is established as well.

Let $\{\psi_k\}_{k=1}^\infty$ be the orthonormal basis of eigenfunctions in H corresponding to the conductivity $a \in \mathcal{PS}$. Let $B_k(t) = \langle v(\cdot, t), \psi_k \rangle$. To simplify the notation the dependency of B_k on a is suppressed. Then $v = \sum_{k=1}^\infty B_k(t)\psi_k$ in H for any $t \geq 0$, and

$$B'_k(t) + \lambda_k B_k(t) = -\phi'_k(t) + f_k(t), \quad B_k(0) = g_k - \phi_k(0).$$

Therefore $B_k(t)$ has the representation stated in (13).

Let $0 < t_0 < T$. Our goal is to show that v defined by $v = \sum_{k=1}^\infty B_k(t)\psi_k$ is in $C([t_0, T]; X)$. For this purpose we establish that this series converges in $X = C[0, 1]$ uniformly with respect to $t \in [t_0, T]$ and $a \in A_{ad}$.

Note that V is continuously embedded in X . Furthermore, since $0 < v \leq a(x) \leq \mu$ the original norm in V is equivalent to the norm $\|\cdot\|_{V_a}$ defined by $\|w\|_{V_a}^2 = \int_0^1 a|w'|^2 dx$. Thus it is enough to prove the uniform convergence of the series for v in V_a . The uniformity follows from the fact that the convergence estimates below do not depend on a particular $t \in [t_0, T]$ or $a \in A_{ad}$. By the definition of the eigenfunctions ψ_k one has $\langle a\psi'_k, \psi'_j \rangle = \lambda_k \langle \psi_k, \psi_j \rangle$ for all k and j .

Thus the eigenfunctions are orthogonal in V_a . In fact, $\{\psi_k / \sqrt{\lambda_k}\}_{k=1}^\infty$ is an orthonormal basis in V_a , see (Evans, 2010). Therefore the series $\sum_{k=1}^\infty B_k(t)\psi_k$ converges in V_a if and only if $\sum_{k=1}^\infty \lambda_k |B_k(t)|^2 = \|v(\cdot, t; a)\|_{V_a}^2 < \infty$ for any $t > 0$. This convergence follows from the fact that the function $s \rightarrow \sqrt{s}e^{-\sigma s}$ is bounded on $[0, \infty)$ for any $\sigma > 0$, see (Gutman & Ha, 2009). \square

4. Continuity of the solution map

In this section we establish the continuous dependence of the eigenvalues λ_k , eigenfunctions ψ_k and the solution u of (4) on the conductivities $a \in \mathcal{PS} \subset A_{ad}$, when A_{ad} is equipped with the $L^1(0, 1)$ topology. For smooth a see (Courant & Hilbert, 1989).

Theorem 4.1. *Let $a \in \mathcal{PS}$, $\mathcal{PS} \subset A_{ad}$ be equipped with the $L^1(0, 1)$ topology, and $\{\lambda_k(a)\}_{k=1}^\infty$ be the eigenvalues of the associated Sturm-Liouville system (5). Then the mapping $a \rightarrow \lambda_k(a)$ is continuous for every $k = 1, 2, \dots$.*

Proof. Let $a, \hat{a} \in \mathcal{PS}$, $\{\lambda_k, \psi_k\}_{k=1}^\infty$ be the eigenvalues and the eigenfunctions corresponding to a , and $\{\hat{\lambda}_k, \hat{\psi}_k\}_{k=1}^\infty$ be the eigenvalues and the eigenfunctions corresponding to \hat{a} . According

to Theorem 2.2 the eigenfunctions form a complete orthonormal set in H . Since $\int_0^1 a\psi'_i\psi'_j dx = \lambda_j \int_0^1 \psi_j \psi_i dx$ for any $\psi \in H_0^1(0,1)$ we have $\int_0^1 a\psi'_i\psi'_j dx = 0$ for $i \neq j$.

Let $W_k = \text{span}\{\psi_j\}_{j=1}^k$. Then W_k is a k -dimensional subspace of $H_0^1(0,1)$, and any $\psi \in W_k$ has the form $\psi(x) = \sum_{j=1}^k \alpha_j \psi_j(x)$, $\alpha_j \in \mathbb{R}$. From the min-max principle (Theorem 2.2(iii))

$$\hat{\lambda}_k \leq \max_{\psi \in W_k} \frac{\int_0^1 \hat{a}(x)[\psi'(x)]^2 dx}{\int_0^1 [\psi(x)]^2 dx}.$$

Note that

$$\max_{\psi \in W_k} \frac{\int_0^1 a(x)[\psi'(x)]^2 dx}{\int_0^1 [\psi(x)]^2 dx} = \max \left\{ \frac{\sum_{j=1}^k \alpha_j^2 \lambda_j}{\sum_{j=1}^k \alpha_j^2} : \alpha_j \in \mathbb{R}, j = 1, 2, \dots, k \right\} = \lambda_k.$$

Therefore

$$\begin{aligned} \hat{\lambda}_k &\leq \max_{\psi \in W_k} \frac{\int_0^1 a(x)[\psi'(x)]^2 dx}{\int_0^1 [\psi(x)]^2 dx} + \max_{\psi \in W_k} \frac{\int_0^1 (\hat{a}(x) - a(x))[\psi'(x)]^2 dx}{\int_0^1 [\psi(x)]^2 dx} \\ &\leq \lambda_k + \|a - \hat{a}\|_{L^1} \max_{\alpha_j} \frac{\|\sum_{j=1}^k \alpha_j \psi'_j\|_{\infty}^2}{\sum_{j=1}^k \alpha_j^2}, \end{aligned}$$

where $\|\cdot\|_{\infty}$ is the norm in $L^{\infty}(0,1)$. Estimates from Theorem 3.1 and the Cauchy-Schwarz inequality give

$$\frac{|\sum_{j=1}^k \alpha_j \psi'_j(x)|^2}{\sum_{j=1}^k \alpha_j^2} \leq \frac{\sum_{j=1}^k \alpha_j^2 \sum_{j=1}^k |\psi'_j(x)|^2}{\sum_{j=1}^k \alpha_j^2} \leq \frac{\lambda_k^2 k}{v^2} \leq \frac{(\mu\pi^2 k^2)^2 k}{v^2} = C(k).$$

Therefore

$$|\lambda_k - \hat{\lambda}_k| \leq C(k) \|a - \hat{a}\|_{L^1}$$

and the desired continuity is established. \square

The following theorem is established in (Gutman & Ha, 2007).

Theorem 4.2. Let $a \in \mathcal{PS}$, $\mathcal{PS} \subset A_{\text{ad}}$ be equipped with the $L^1(0,1)$ topology, and $\{\psi_k(x; a)\}_{k=1}^{\infty}$ be the unique normalized eigenfunctions of the associated Sturm-Liouville system (5) satisfying the condition $\psi'_k(0+; a) > 0$. Then the mapping $a \rightarrow \psi_k(a)$ from \mathcal{PS} into $X = C[0,1]$ is continuous for every $k = 1, 2, \dots$.

Theorem 4.3. Let $a \in \mathcal{PS} \subset A_{\text{ad}}$ equipped with the $L^1(0,1)$ topology, and $u(a)$ be the solution of the heat conduction process (4), under the conditions of Theorem 3.2. Then the mapping $a \rightarrow u(a)$ from \mathcal{PS} into $C([0, T]; X)$ is continuous.

Proof. According to Theorem 3.2 the solution $u(x, t; a)$ is given by $u(x, t; a) = v(x, t; a) + \Phi(x, t; a)$, where $v(x, t; a) = \sum_{k=1}^{\infty} B_k(t; a) \psi_k(x)$ with the coefficients $B_k(t; a)$ given by (13). Let

$$v^N(x, t; a) = \sum_{k=1}^N B_k(t; a) \psi_k(x).$$

By Theorems 4.1 and 4.2 the eigenvalues and the eigenfunctions are continuously dependent on the conductivity a . Therefore, according to (13), the coefficients $B_k(t, a)$ are continuous as functions of a from \mathcal{PS} into $C([0, T]; X)$. This implies that $a \rightarrow v^N(a)$ is continuous. By Theorem 3.2 the convergence $v^N \rightarrow v$ is uniform on A_{ad} as $N \rightarrow \infty$ and the result follows. \square

5. Identifiability of piecewise constant conductivities from finitely many observations

Series of the form $\sum_{k=1}^{\infty} C_k e^{-\lambda_k t}$ are known as Dirichlet series. The following lemma shows that a Dirichlet series representation of a function is unique. Additional results on Dirichlet series can be found in Chapter 9 of (Saks & Zygmund, 1965).

Lemma 5.1. *Let $\mu_k > 0, k = 1, 2, \dots$ be a strictly increasing sequence, and $0 \leq T_1 < T_2 \leq \infty$. Suppose that either*

- (i) $\sum_{k=1}^{\infty} |C_k| < \infty,$
or
 - (ii) $\gamma > 0, \mu_k \geq \gamma k^2, k = 1, 2, \dots,$ and $\sup_k |C_k| < \infty.$
- Then

$$\sum_{k=1}^{\infty} C_k e^{-\mu_k t} = 0 \quad \text{for all } t \in (T_1, T_2)$$

implies $C_k = 0$ for $k = 1, 2, \dots$

Proof. In both cases the series $\sum_{k=1}^{\infty} C_k e^{-\mu_k z}$ converges uniformly in $Re z > 0$ region of the complex plane, implying that it is an analytic function there. Thus

$$\sum_{k=1}^{\infty} C_k e^{-\mu_k t} = 0 \quad \text{for all } t > 0.$$

Suppose that some coefficients C_k are nonzero. Without loss of generality we can assume $C_1 \neq 0$. Then

$$0 = e^{\mu_1 t} \sum_{k=1}^{\infty} C_k e^{-\mu_k t} = C_1 + \sum_{k=2}^{\infty} C_k e^{(\mu_1 - \mu_k)t} \rightarrow C_1, \quad t \rightarrow \infty,$$

which is a contradiction. \square

Remark. According to Theorem 3.1 for each fixed $p \in (0, 1)$ the solution $z(t) = u(p, t; a)$ of (4) is given by a Dirichlet series. The series coefficients $C_k = \langle g, v_k \rangle v_k(p)$ are square summable, therefore they form a bounded sequence. The growth condition for the eigenvalues stated in (iv) of Theorem 2.2 shows that Lemma 5.1(ii) is applicable to the solution $z(t)$.

Functions $a \in \mathcal{PC}_N$ have the form $a(x) = a_i$ for $x \in [x_{i-1}, x_i], i = 1, 2, \dots, N$. Assuming $f = q_1 = q_2 = 0$, in this case the governing system (4) is

$$\begin{aligned} u_t - a_i u_{xx} &= 0, & x \in (x_{i-1}, x_i), \quad t \in (0, T), \\ u(0, t) = u(1, t) &= 0, & t \in (0, T), \\ u(x_i+, t) = u(x_i-, t), & & t \in (0, T), \\ a_{i+1} u_x(x_i+, t) = a_i u_x(x_i-, t), & & t \in (0, T), \\ u(x, 0) &= g(x), & x \in (0, 1), \end{aligned} \tag{14}$$

where $g \in L^2(0,1)$ and $i = 1, 2, \dots, N-1$. The associated Sturm-Liouville problem is

$$\begin{aligned} a_i \psi''(x) &= -\lambda \psi(x), & x \in (x_{i-1}, x_i), \\ \psi(0) &= \psi(1) = 0, \\ \psi(x_{i+}) &= \psi(x_{i-}), \\ a_{i+1} \psi'(x_{i+}) &= a_i \psi'(x_{i-}) \end{aligned} \quad (15)$$

for $i = 1, 2, \dots, N-1$.

The central part of the identification method is the Marching Algorithm contained in Theorem 5.5. Recall that it uses only the M -tuple $\mathcal{G}(a)$, see (3). That is we need only the first eigenvalue λ_1 and a nonzero multiple of the first eigenfunction ψ_1 of (15) for the identification of the conductivity $a(x)$.

Suppose that $p^* \in (x_{i-1}, x_i)$. Then ψ_1 can be expressed on (x_{i-1}, x_i) as

$$\psi_1(x) = A \cos \left(\sqrt{\frac{\lambda_1}{a_i}} (x - p^*) + \gamma \right), \quad -\frac{\pi}{2} < \gamma < \frac{\pi}{2}$$

with $A > 0$. The range for γ in the above representation follows from the fact that $\psi_1(p^*) = A \cos \gamma > 0$ by Theorem 2.2(5).

The identifiability of piecewise constant conductivities is based on the following three Lemmas, see (Gutman & Ha, 2007).

Lemma 5.2. *Suppose that $\delta > 0$. Assume $Q_1, Q_3 \geq 0$, $Q_2 > 0$ and $0 < Q_1 + Q_3 < 2Q_2$. Let*

$$\Gamma = \left\{ (A, \omega, \gamma) : A > 0, 0 < \omega < \frac{\pi}{2\delta}, -\frac{\pi}{2} < \gamma < \frac{\pi}{2} \right\}.$$

Then the system of equations

$$A \cos(\omega\delta - \gamma) = Q_1, \quad A \cos \gamma = Q_2, \quad A \cos(\omega\delta + \gamma) = Q_3$$

has a unique solution $(A, \omega, \gamma) \in \Gamma$ given by

$$\begin{aligned} \omega &= \frac{1}{\delta} \arccos \frac{Q_1 + Q_3}{2Q_2}, \quad \gamma = \arctan \left(\frac{Q_1 - Q_3}{2Q_2 \sin \omega\delta} \right), \\ A &= \frac{Q_2}{\cos \gamma}. \end{aligned}$$

Lemma 5.3. *Suppose that $\delta > 0$, $0 < p \leq x_1 < p + \delta < 1$, $0 < \omega_1, \omega_2 < \pi/2\delta$.*

Let $w(x)$, $v(x)$, $x \in [p, p + \delta]$ be such that

$$\begin{aligned} w(x) &= A_1 \cos \omega_1 x + B_1 \sin \omega_1 x, \\ v(x) &= A_2 \cos \omega_2 x + B_2 \sin \omega_2 x. \end{aligned}$$

Suppose that

$$\begin{aligned} v(x_1) &= w(x_1), \quad \omega_1^2 v'(x_1) = \omega_2^2 w'(x_1), \\ v'(x_1) &> 0, \quad v(x_1) > 0. \end{aligned}$$

Then

(i) *Conditions $v(p + \delta) = w(p + \delta)$, $v'(p + \delta) \geq 0$ and $\omega_1 \leq \omega_2$ imply $\omega_1 = \omega_2$.*

(ii) Conditions $v(p + \delta) = w(p + \delta)$, $w'(p + \delta) \geq 0$ and $\omega_1 \geq \omega_2$ imply $\omega_1 = \omega_2$.

Lemma 5.4. Let $\delta > 0$, $0 < \eta \leq 2\delta$, $\omega_1 \neq \omega_2$ with $0 < \omega_1\delta, \omega_2\delta < \pi/2$. Also let $A, B > 0$, $0 \leq p < p + \eta \leq 1$ and

$$\begin{aligned} w(x) &= A \cos[\omega_1(x - p) + \gamma_1], \\ v(x) &= B \cos[\omega_2(x - p - \eta) + \gamma_2] \end{aligned}$$

with $|\gamma_1|, |\gamma_2| < \pi/2$. Then system

$$w(q) = v(q), \tag{16}$$

$$\omega_2^2 w'(q) = \omega_1^2 v'(q), \tag{17}$$

$$w(q) > 0, \quad v(q) > 0 \tag{18}$$

admits at most one solution q on $[p, p + \eta]$. This unique solution q can be computed as follows:

If $\gamma_1 \geq 0$ then

$$q = p + \frac{1}{\omega_1} \left[\arctan \left(\omega_1 \sqrt{\left| \frac{B^2 - A^2}{A^2\omega_2^2 - B^2\omega_1^2} \right|} \right) - \gamma_1 \right]. \tag{19}$$

If $\gamma_2 \leq 0$ then

$$q = p + \eta + \frac{1}{\omega_2} \left[-\arctan \left(\omega_2 \sqrt{\left| \frac{B^2 - A^2}{A^2\omega_2^2 - B^2\omega_1^2} \right|} \right) - \gamma_2 \right]. \tag{20}$$

Otherwise compute q_1 and q_2 according to formulas (19) and (20) and discard the one that does not satisfy the conditions of the Lemma.

By the definition of $a \in \mathcal{PC}$ there exist $N \in \mathbb{N}$ and a finite sequence $0 = x_0 < x_1 < \dots < x_{N-1} < x_N = 1$ such that a is a constant on each subinterval (x_{n-1}, x_n) , $n = 1, \dots, N$. Let $\sigma > 0$. The following Theorem is our main result.

Theorem 5.5. Given $\sigma > 0$ let an integer M be such that

$$M \geq \frac{3}{\sigma} \quad \text{and} \quad M > 2\sqrt{\frac{\mu}{\nu}}.$$

Suppose that the initial data $g(x) > 0$, $0 < x < 1$ and the observations $z_m(t) = u(p_m, t; a)$, $p_m = m/M$ for $m = 1, 2, \dots, M - 1$ and $0 \leq T_1 < t < T_2$ of the heat conduction process (14) are given. Then the conductivity $a \in A_{\text{ad}}$ is identifiable in the class of piecewise constant functions $\mathcal{PC}(\sigma)$.

Proof. The identification proceeds in two steps. In step I the M -tuple $\mathcal{G}(a)$ is extracted from the observations $z_m(t)$. In step II the Marching Algorithm identifies $a(x)$.

Step I. Data extraction.

By Theorem 3.1 we get

$$z_m(t) = \sum_{k=1}^{\infty} g_k e^{-\lambda_k t} \psi_k(p_m), \quad m = 1, 2, \dots, M - 1, \tag{21}$$

where $g_k = \langle g, \psi_k \rangle$ for $k = 1, 2, \dots$. By Theorem 2.2(5) $\psi_1(x) > 0$ on interval $(0, 1)$. Since g is positive on $(0, 1)$ we conclude that $g_1\psi_1(p_m) > 0$. Since $z_m(t)$ is represented by a Dirichlet

series, Lemma 5.1 assures that all nonzero coefficients (and the first term, in particular) are defined uniquely.

An algorithm for determining the first eigenvalue λ_1 , and the coefficient $g_1\psi_1(p_m)$ from (21) is given in Section 10. Repeating this process for every m one gets the values of

$$G_m = g_1\psi_1(p_m) > 0, \quad p_m = m/M \tag{22}$$

for $m = 1, 2, \dots, M - 1$. This determines the M -tuple $\mathcal{G}(a)$, see (3). Because of the zero boundary conditions we let $G_0 = G_M = 0$.

Step II. *Marching Algorithm.*

The algorithm marches from the left end $x = 0$ to a certain observation point $p_{l-1} \in (0, 1)$ and identifies the values a_n and the discontinuity points x_n of the conductivity a on $[0, p_{l-1}]$. Then the algorithm marches from the right end point $x = 1$ to the left until it reaches the observation point $p_{l+1} \in (0, 1)$ identifying the values and the discontinuity points of a on $[p_{l+1}, 1]$. Finally, the values of a and its discontinuity are identified on the interval $[p_{l-1}, p_{l+1}]$.

The overall goal of the algorithm is to determine the number $N - 1$ of the discontinuities of a on $[0, 1]$, the discontinuity points $x_n, n = 1, 2, \dots, N - 1$ and the values a_n of a on $[x_{n-1}, x_n], n = 1, 2, \dots, N$ ($x_0 = 0, x_N = 1$). As a part of the process the algorithm determines certain functions $H_n(x)$ defined on intervals $[x_{n-1}, x_n], n = 1, 2, \dots, N$. The resulting function $H(x)$ defined on $[0, 1]$ is a multiple of the first eigenfunction v_1 over the entire interval $[0, 1]$. An illustration of the Marching Algorithm is given in Figure 1.

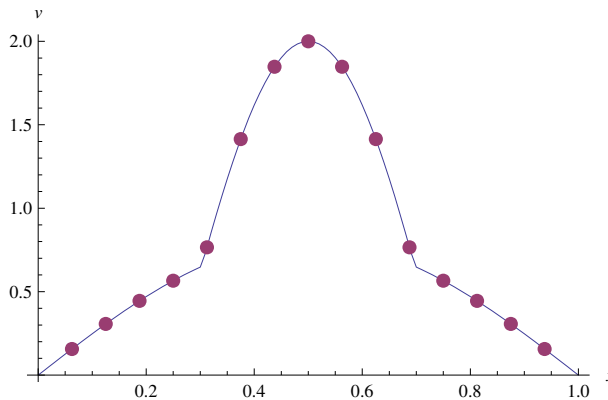


Fig. 1. Conductivity identification by the Marching Algorithm. The dots are a multiple of the first eigenfunction at the observation points p_m . The algorithm identifies the values of the conductivity a and its discontinuity points

- (i) Find $l, 0 < l < M$ such that $G_l = \max\{G_m : m = 1, 2, \dots, M - 1\}$ and $G_m < G_l$ for any $0 \leq m < l$.
- (ii) Let $i = 1, m = 0$.
- (iii) Use Lemma 5.2 to find A_i, ω_i and γ_i from the system

$$\begin{cases} A_i \cos(\omega_i\delta - \gamma_i) = G_m, \\ A_i \cos \gamma_i = G_{m+1}, \\ A_i \cos(\omega_i\delta + \gamma_i) = G_{m+2}. \end{cases} \tag{23}$$

Let

$$H_i(x) = A_i \cos(\omega_i(x - p_{m+1}) + \gamma_i).$$

(iv) If $m + 3 \geq l$ then go to step (vii). If $H_i(p_{m+3}) \neq G_{m+3}$, or $H_i(p_{m+3}) = G_{m+3}$ and $H'_i(p_{m+3}) \leq 0$ then a has a discontinuity x_i on interval $[p_{m+2}, p_{m+3}]$. Proceed to the next step (v).

If $H_i(p_{m+3}) = G_{m+3}$ and $H'_i(p_{m+3}) > 0$ then let $m := m + 1$ and repeat this step (iv).

(v) Use Lemma 5.2 to find A_{i+1} , ω_{i+1} and γ_{i+1} from the system

$$\begin{cases} A_{i+1} \cos(\omega_{i+1}\delta - \gamma_{i+1}) = G_{m+3}, \\ A_{i+1} \cos \gamma_{i+1} = G_{m+4}, \\ A_{i+1} \cos(\omega_{i+1}\delta + \gamma_{i+1}) = G_{m+5}. \end{cases} \tag{24}$$

Let

$$H_{i+1}(x) = A_{i+1} \cos(\omega_{i+1}(x - p_{m+4}) + \gamma_{i+1}).$$

(vi) Use formulas in Lemma 5.4 to find the unique discontinuity point $x_i \in [p_{m+2}, p_{m+3}]$. The parameters and functions used in Lemma 5.4 are defined as follows. Let $p = p_{m+2}$, $\eta = \delta$. To avoid a confusion we are going to use the notation $\Omega_1, \Omega_2, \Gamma_1, \Gamma_2$ for the corresponding parameters $\omega_1, \omega_2, \gamma_1, \gamma_2$ required in Lemma 5.4. Let $\Omega_1 = \omega_i$, $\Omega_2 = \omega_{i+1}$. For $w(x)$ use function $H_i(x)$ recentered at $p = p_{m+2}$, i.e. rewrite $H_i(x)$ in the form

$$w(x) = H_i(x) = A \cos(\Omega_1(x - p_{m+2}) + \Gamma_1), \quad |\Gamma_1| < \pi/2.$$

For $v(x)$ use function H_{i+1} recentered at $p + \eta = p_{m+3}$, i.e.

$$v(x) = H_{i+1}(x) = B \cos(\Omega_2(x - p_{m+3}) + \Gamma_2), \quad |\Gamma_2| < \pi/2.$$

Let $i := i + 1$, $m := m + 3$. If $m < l$ then return to step (iv). If $m \geq l$ then go to the next step (vii).

(vii) Do steps (ii)-(vi) in the reverse direction of x , advancing from $x = 1$ to $x = p_{l+1}$. Identify the values and the discontinuity points of a on $[p_{l+1}, 1]$, as well as determine the corresponding functions $H_i(x)$.

(viii) Using the notation introduced in (vi) let $H_j(x)$ be the previously determined function H on interval $[p_{l-2}, p_{l-1}]$. Recenter it at $p = p_{l-1}$, i.e. $w(x) = H_j(x) = A \cos(\Omega_1(x - p_{l-1}) + \Gamma_1)$. Let $H_{j+1}(x)$ be the previously determined function H on interval $[p_{l+1}, p_{l+2}]$. Recenter it at p_{l+1} : $v(x) = H_{j+1}(x) = B \cos(\Omega_2(x - p_{l+1}) + \Gamma_2)$. If $\Omega_1 = \Omega_2$ then stop, otherwise use Lemma 5.4 with $\eta = 2\delta$, and the above parameters to find the discontinuity $x_j \in [p_{l-1}, p_{l+1}]$. Stop.

The justification of the Marching Algorithm is given in (Gutman & Ha, 2007). □

6. Identifiability of piecewise constant conductivity with one discontinuity

The Marching Algorithm of Theorem 5.5 requires measurements of the system at possibly large number of observation points. Our next Theorem shows that if a piecewise constant conductivity a is known to have just one point of discontinuity x_1 , and its values a_1 and a_2 are known beforehand, then the discontinuity point x_1 can be determined from just one measurement of the heat conduction process.

Theorem 6.1. Let $p \in (0, 1)$ be an observation point, $g(x) > 0$ on $(0, 1)$, and the observation $z_p(t) = u(x_p, t; a)$, $t \in (T_1, T_2)$ of the heat conduction process (14) be given. Suppose that the conductivity $a \in A_{\text{ad}}$ is piecewise constant and has only one (unknown) point of discontinuity $x_1 \in (0, 1)$. Given positive values $a_1 \neq a_2$ such that $a(x) = a_1$ for $0 \leq x < x_1$ and $a(x) = a_2$ for $x_1 \leq x < 1$ the point of discontinuity x_1 is constructively identifiable.

Proof. Arguing as in the previous Theorem

$$z_p(t) = \sum_{k=1}^{\infty} g_k e^{-\lambda_k t} \psi_k(p), \quad 0 \leq T_1 < t < T_2,$$

where $g_k = \langle g, \psi_k \rangle$ for $k = 1, 2, \dots$. Since $g_1 \psi_1(p) > 0$ the uniqueness of the Dirichlet series representation implies that one can uniquely determine the first eigenvalue λ_1 and the value of $G_p = g_1 \psi_1(p)$.

Without loss of generality one can assume that $a_1 > a_2$. In this case we show that the first eigenvalue λ_1 is strictly increasing as a function of the discontinuity point $x_1 \in [0, 1]$. Indeed, suppose that

$$0 \leq x_1^a < x_1^b \leq 1,$$

that is

$$a(x) = \begin{cases} a_1, & 0 < x < x_1^a \\ a_2, & x_1^a < x < 1 \end{cases} \quad \text{and} \quad b(x) = \begin{cases} a_1, & 0 < x < x_1^b \\ a_2, & x_1^b < x < 1 \end{cases}.$$

By Theorem 2.2(i)

$$\lambda_1^b = \frac{\int_0^1 b(x) [\psi'_{1,b}(x)]^2 dx}{\int_0^1 [\psi_{1,b}(x)]^2 dx} > \frac{\int_0^1 a(x) [\psi'_{1,b}(x)]^2 dx}{\int_0^1 [\psi_{1,b}(x)]^2 dx} \geq \inf_{\psi \in H_0^1(0,1)} \frac{\int_0^1 a(x) [\psi'(x)]^2 dx}{\int_0^1 [\psi(x)]^2 dx} = \lambda_1^a$$

provided that the derivative $\psi'_{1,b}(x)$ of the first eigenfunction $\psi_{1,b}(x)$ is not identically zero on (x_1^a, x_1^b) . But, from $(b(x)\psi'_{1,b}(x))' = -\lambda_1^b \psi_{1,b}(x)$, the assumption $\psi'_{1,b}(x) = 0$ on (x_1^a, x_1^b) implies $\psi_{1,b}(x) = 0$ on (x_1^a, x_1^b) . However, this is impossible, since $\psi_{1,b}(x) > 0$ on $(0, 1)$. Thus there exists a unique conductivity of the type sought in the Theorem for which its first eigenvalue is equal to λ_1 , i.e. a is identifiable.

Now the unique discontinuity point x_1 of a can be determined as follows. Let

$$\omega_1 = \sqrt{\frac{\lambda_1}{a_1}}, \quad \omega_2 = \sqrt{\frac{\lambda_1}{a_2}}.$$

Then the first eigenfunction ψ_1 is given by

$$\psi_1(x) = \begin{cases} A \sin \omega_1 x, & 0 < x < x_1, \\ B \sin \omega_2 (1 - x), & x_1 < x < 1 \end{cases} \quad (25)$$

for some $A, B > 0$. The matching conditions at x_1 give

$$A \sin \omega_1 x_1 = B \sin \omega_2 (1 - x_1) \quad \text{and} \quad \frac{A}{\omega_1} \cos \omega_1 x_1 = \frac{B}{\omega_2} \cos \omega_2 (1 - x_1).$$

Since $\psi_1(x_1) > 0$ we have $0 < \omega_1 x_1 < \pi$ and $0 < \omega_2(1 - x_1) < \pi$. Therefore x_1 satisfies

$$\frac{1}{\omega_1} \cot \omega_1 x = \frac{1}{\omega_2} \cot \omega_2(1 - x).$$

The existence and the uniqueness of the solution x_1 of the above nonlinear equation follows from the monotonicity and the continuity of the cotangent functions. Practically, the value of x_1 can be found by a numerical method. \square

7. Identifiability with non-zero boundary conditions

Let $a \in \mathcal{PS}$, and $u(x, t; a)$ be the unique solution of the heat conduction process (4). Next Theorem describes some conditions under which the identifiability for (4) is possible.

Theorem 7.1. *Given $\sigma > 0$ let an integer M be such that*

$$M \geq \frac{3}{\sigma} \quad \text{and} \quad M > 2\sqrt{\frac{\mu}{\nu}}.$$

Suppose that the observations $z_m(t; a) = u(p_m, t; a)$ for $p_m = m/M$, $m = 1, 2, \dots, M - 1$ and $t > 0$ of the heat conduction process (4) are given. Then the conductivity $a \in A_{ad}$ is identifiable in the class of piecewise constant functions $\mathcal{PC}(\sigma)$ in each one of the following four cases.

- (i) $f = 0, q_1 = 0, q_2 = 0, g > 0, g \in L^2(0, 1)$.
- (ii) $g = 0, q_1 = 0, q_2 = 0, f(x, t) = h(x)r(t) \neq 0, h > 0, h \in L^2(0, 1), r \in C[0, \infty)$.
- (iii) $g = 0, f = 0, q_2 = 0, q_1 \neq 0, q_1(0) = 0, q_1 \in C^1[0, \infty)$.
- (iv) $g = 0, f = 0, q_1 = 0, q_2 \neq 0, q_2(0) = 0, q_2 \in C^1[0, \infty)$.

Proof. Case (i) is considered in Theorem 5.5. In case (ii) of the Theorem let

$$y_m(t) = \sum_{k=1}^{\infty} \langle h, \psi_k \rangle \psi_k(p_m) e^{-\lambda_k t}. \tag{26}$$

Then $y_m(t)$ is the solution of (4) with $g = h, f = 0$ and zero boundary conditions, observed at $p_m \in (0, 1)$. It is shown in Theorem 3.2 that such a solution is a continuous function for $t > 0$. Furthermore, using the estimate $|\psi_k(x)| \leq \sqrt{\lambda_k}/\sqrt{\nu}$ established in Theorem 3.1, and the Cauchy-Schwarz inequality we get

$$\int_0^{\infty} |y_m(t)| dt \leq \sum_{k=1}^{\infty} \frac{1}{\lambda_k} |h_k| |\psi_k(p_m)| \leq \frac{1}{\sqrt{\nu}} \sum_{k=1}^{\infty} \frac{|h_k|}{\sqrt{\lambda_k}} \leq C \|h\| < \infty. \tag{27}$$

Therefore $y_m(t) \in L^1[0, \infty)$.

Returning to the observation $z_m(t)$, Theorem 3.2 shows that it is given by

$$z_m(t) = u(p_m, t) = \int_0^t \left[\sum_{k=1}^{\infty} \langle h, \psi_k \rangle \psi_k(p_m) e^{-\lambda_k(t-\tau)} \right] r(\tau) d\tau.$$

That is

$$z_m(t) = \int_0^t y_m(t - \tau) r(\tau) d\tau.$$

Since $y_m(t) \in L^1[0, \infty)$ and $r(t)$ is continuous and bounded on $[0, \infty)$, Titchmarsh Theorem (Titchmarsh, 1962), Theorem 152, Chap. XI, p. 325, implies that this Volterra integral equation is uniquely solvable for $y_m(t)$.

Since $h > 0$ is assumed to be in $L^2(0, 1)$, one has $C(a) = \langle h, \psi_1(a) \rangle \neq 0$. The uniqueness of the Dirichlet series representation (26) and rest of the argument is the same as in the proof of case (i).

In case (iii) of the Theorem function $\Phi(x, t; a)$ has the form $\Phi(x, t; a) = q_1(t)\zeta(x; a)$, where

$$\zeta(x; a) = 1 - \frac{1}{\int_0^1 \frac{1}{a(s)} ds} \int_0^x \frac{1}{a(s)} ds.$$

Note that $\zeta(x; a)$ is bounded, continuous and strictly positive on $(0, 1)$. Thus $\zeta \in L^2(0, 1)$. Let $\xi_k = \langle \zeta(x; a), \psi_k(x; a) \rangle$ for $k = 1, 2, \dots$. Then $\phi_k(t; a) = q_1(t)\xi_k$, $\phi_k(0; a) = 0$ and $\phi'_k(t; a) = q'_1(t)\xi_k$.

Let

$$y_m(t) = - \sum_{k=1}^{\infty} \xi_k \psi_k(p_m) e^{-\lambda_k t}. \quad (28)$$

Arguing as in case (ii), we conclude that $y_m(t)$ is continuous on $[0, \infty)$ and $y_m(t) \in L^1[0, \infty)$. Also, by Theorem 3.2

$$z_m(t) = u(p_m, t) = - \int_0^t \left[\sum_{k=1}^{\infty} \xi_k \psi_k(p_m) e^{-\lambda_k(t-\tau)} \right] q'_1(\tau) d\tau.$$

That is

$$z_m(t) = \int_0^t y_m(t-\tau) q'_1(\tau) d\tau.$$

Since $y_m(t) \in L^1[0, \infty)$ and $q'_1(t)$ is continuous and bounded on $[0, \infty)$, Titchmarsh Theorem (Titchmarsh, 1962), Theorem 152, Chap. XI, p. 325, implies that this Volterra integral equation is uniquely solvable for $y_m(t)$.

Since $\xi_1 > 0$ and $\psi_1(p_m) > 0$, the uniqueness of the Dirichlet series representation (28) implies that the M -tuple $\mathcal{G}(a)$ is recoverable from the observations $z_m(t)$. In this case $C(a) = \langle \zeta(x; a), \psi_1(x; a) \rangle$. Finally, the Marching Algorithm identifies the unknown conductivity a .

Case (iv) of the Theorem is treated in the same way as case (iii). \square

8. Continuity of the identification map

The Marching Algorithm establishes the identifiability of the conductivity $a \in \mathcal{PC}(\sigma)$ from the data $\mathcal{G}(a)$. In other words, the inverse mapping \mathcal{G}^{-1} is well defined on $\mathcal{G}(\mathcal{PC}(\sigma))$. To prove our main result that the identifiability map \mathcal{G}^{-1} is continuous, first we show that the set $\mathcal{PC}(\sigma) \subset A_{\text{ad}}$ is compact in $L^1(0, 1)$. A proof of this result can be found in (Gutman & Ha, 2009).

Theorem 8.1. *Let A_{ad} be equipped with the $L^1(0, 1)$ topology. Let $N \in \mathbb{N}$ and $\sigma > 0$. Then*

- (i) *Set $\mathcal{PC}_N \subset A_{\text{ad}}$ is compact.*
- (ii) *Set $\mathcal{PC}(\sigma) \subset A_{\text{ad}}$ is compact.*

Theorem 8.2. *Let A_{ad} be equipped with the $L^1(0, 1)$ topology, and the data map $\mathcal{G} : \mathcal{PC}(\sigma) \rightarrow \mathbb{R}^M$ be defined as in (3). Then the identifiability map $\mathcal{G}^{-1} : \mathcal{G}(\mathcal{PC}(\sigma)) \rightarrow \mathcal{PC}(\sigma)$ is continuous.*

Proof. Theorem 7.1 shows that in every case specified there the data map $a \rightarrow \mathcal{G}(a)$ is defined everywhere on $\mathcal{PC}(\sigma)$ and that the conductivity a is identifiable from $\mathcal{G}(a)$, i.e. \mathcal{G} is invertible on $\mathcal{G}(\mathcal{PC}(\sigma))$. By Theorem 8.1 the set $\mathcal{PC}(\sigma)$ is compact in $L^1(0, 1)$. Thus the Theorem would be established if the injective map $a \rightarrow \mathcal{G}(a)$ were shown to be continuous.

Recall that $\mathcal{G}(a) = (\lambda_1(a), G_1(a), \dots, G_{M-1}(a)) \in \mathbb{R}^M$. The continuity of $a \rightarrow \lambda_1(a)$ was established in Theorem 4.1. In every case of Theorem 7.1 the data G_m has the form $G_m(a) = C(a)\psi_1(p_m; a)$, where p_m are the observation points. By Theorem 4.2 the mapping $a \rightarrow \psi_1(\cdot; a)$ is continuous from $\mathcal{PC}(\sigma) \subset L^1(0, 1)$ into $C[0, 1]$. Thus the evaluation maps $a \rightarrow \psi_1(p_m; a) \in \mathbb{R}$ are continuous for every $p_m \in [0, 1]$.

To see that $a \rightarrow C(a)$ is continuous we have to examine it separately for each case of Theorem 7.1. In case (i) $C(a) = \langle g, \psi_1(a) \rangle$, where $g \in L^2(0, 1)$ is a fixed initial condition. The continuity of the inner product and of $a \rightarrow \psi_1(\cdot; a)$ imply the continuity of $C(a)$. In case (ii) $C(a) = \langle h, \psi_1(a) \rangle$ for an $h \in L^2(0, 1)$ and the continuity of $C(a)$ follows. In cases (iii) and (iv) the continuity of $C(a)$ is established similarly. \square

9. Identifiability with a known heat flux

Let Π be the set of piecewise constant functions on $[0, 1]$ with finitely many discontinuity points,

$$\Pi = \{a(x) : 0 < \nu \leq a(x) \leq \mu, a(x) = a_j, x \in [x_{j-1}, x_j], j = 1, 2, \dots, n\} \tag{29}$$

with $x_0 = 0$ and $x_n = 1$.

Consider the following heat conduction problem in an inhomogeneous bar of the unit length with a conductivity $a \in \Pi$:

$$\begin{cases} u_t = (a(x)u_x)_x, & (x, t) \in Q = (0, 1) \times (0, \infty), \\ u(0, t) = g(t), u(1, t) = 0, & t \in (0, \infty), \\ u(x, 0) = 0, & x \in (0, 1). \end{cases} \tag{30}$$

Suppose that the extra data $f(t) = a(0)u_x(0, t) \neq 0$, i.e., the heat flux through the left end of the bar, is known.

The inverse problem (IP) for (29)-(30) is:

IP: *Given $f(t)$ and $g(t)$ for all $t > 0$, find $a(x)$.*

In this Section we establish the identifiability for the IP. Additional details including a fast computational algorithm can be found in (Gutman & Ramm, 2010) and (Hoang & Ramm, 2009).

The main idea of the proof is to apply a "layer peeling" argument. Suppose that two conductivities $a, b \in \Pi$ satisfy (30) with the same data $f(t)$ and $g(t)$ for $t > 0$. Let both a and b have no discontinuities on an interval $[0, y]$, $0 < y \leq 1$. Then we can show that $a(x) = b(x)$ for $x \in [0, y]$. A repeated application of this argument shows that $a = b$ on the entire interval $[0, 1]$. See (Hoang & Ramm, 2009) for further refinements of this result, in particular for the data f, g available only on a finite interval $(0, T)$.

The main tool for the uniqueness proof is Property C (completeness of the products of solutions for (30)). We will use the following Property C result established in (Hoang & Ramm, 2009).

Theorem 9.1. *Let $PC[0, 1]$ be the set of piecewise-constant functions on $[0, 1]$. Let $q_1, q_2 \in PC[0, 1]$ be two positive functions. Suppose that $\psi_1(x, k)$ and $\psi_2(x, k)$ satisfy*

$$-\psi_j''(x, k) + k^2 q_j^2(x)\psi_j(x, k) = 0, \quad \psi_j(1, k) = 1, \quad \psi_j'(1, k) = 0, \quad j = 1, 2. \tag{31}$$

Then the set of products $\{\psi_1(x, k)\psi_2(x, k)\}_{k>0}$ is dense in $PC[0, 1]$. That is, if $h \in PC[0, 1]$ and

$$\int_0^1 h(x)\psi_1(x, k)\psi_2(x, k)dx = 0 \quad (32)$$

for any $k > 0$, then $h = 0$.

Theorem 9.2. Problem IP has at most one solution $a \in \Pi$.

Proof. Following Hoang & Ramm (2009), problem (30) is restated in terms of the Laplace transform

$$v(x, s; a) = (\mathcal{L}u)(x, s; a) = \int_0^\infty u(x, t; a)e^{-st} dt, \quad s > 0.$$

Let $G(s) = \mathcal{L}(g(t))$ and $F(s) = \mathcal{L}(f(t))$. Thus (30) with the extra condition $a(0)u_x(0, t) = f(t)$ becomes

$$\begin{aligned} (a(x)v')' - sv &= 0, \quad 0 < x < 1, \\ v(0, s; a) &= G(s), \quad a(0)v'(0, s; a) = F(s), \\ v(1, s; a) &= 0. \end{aligned} \quad (33)$$

Let

$$k = \sqrt{s}, \quad \psi(x, k) = a(x)v'(x, s; a), \quad \text{and} \quad q(x) = \sqrt{\frac{1}{a(x)}}.$$

Then, using $k^2v(x, s; a) = \psi'(x, k)$, system (33) is rewritten as

$$\begin{aligned} -\psi''(x, k) + k^2q^2(x)\psi(x, k) &= 0, \quad 0 < x < 1, \\ \psi(0, k) &= F(k^2), \quad \psi'(0, k) = k^2G(k^2), \quad \psi'(1, k) = 0. \end{aligned} \quad (34)$$

Let $\psi_1(x, k)$ and $\psi_2(x, k)$ be the solutions of (34) for two positive piecewise-constant functions $q_1(x)$ and $q_2(x)$ correspondingly. That is,

$$\begin{aligned} -\psi_1''(x, k) + k^2q_1^2(x)\psi_1(x, k) &= 0, \quad 0 < x < 1, \\ \psi_1(0, k) &= F(k^2), \quad \psi_1'(0, k) = k^2G(k^2), \quad \psi_1'(1, k) = 0, \end{aligned} \quad (35)$$

and

$$\begin{aligned} -\psi_2''(x, k) + k^2q_2^2(x)\psi_2(x, k) &= 0, \quad 0 < x < 1, \\ \psi_2(0, k) &= F(k^2), \quad \psi_2'(0, k) = k^2G(k^2), \quad \psi_2'(1, k) = 0. \end{aligned} \quad (36)$$

Multiply equation (35) by $\psi_2(x, k)$ and integrate it over $[0, 1]$. Then use an integration by parts and the boundary conditions in (35) and (36) to obtain

$$k^2 \int_0^1 q_1^2 \psi_1 \psi_2 dx = \psi_1' \psi_2|_0^1 - \int_0^1 \psi_1' \psi_2' dx = -k^2 G(k^2) F(k^2) - \int_0^1 \psi_1' \psi_2' dx. \quad (37)$$

Similarly,

$$k^2 \int_0^1 q_2^2 \psi_1 \psi_2 dx = -k^2 G(k^2) F(k^2) - \int_0^1 \psi_1' \psi_2' dx. \quad (38)$$

Subtracting (38) from (37) gives

$$\int_0^1 (q_1^2 - q_2^2) \psi_1 \psi_2 dx = 0$$

for any $k > 0$.

Given nonzero F and G , consider (35) as an initial value problem for ψ_1 at $x = 0$. Its solution $\psi_1(x, k)$ must satisfy $\psi_1(1, k) \neq 0$, because of the condition $\psi_1'(1, k) = 0$. The same goes for $\psi_2(x, k)$. Now we can conclude that the set of products $\{\psi_1(x, k)\psi_2(x, k)\}_{k>0}$ is dense in $PC[0, 1]$ by Theorem 9.1. Therefore $q_1 = q_2$. Thus (34) has a unique solution $q \in PC[0, 1]$. Consequently (33) has a unique solution $a \in \Pi$, and the Theorem is proved. \square

10. Computational algorithms

The main objective of this research is the development of a theoretical framework for the parameter identifiability described in previous sections. Nevertheless, from a practical perspective it is desirable to develop an algorithm for such an identifiability incorporating the new insights gained in the theoretical part. The main new element of it is the separation of the identification process into the following two parts. First, the observation data is used to recover the M -tuple $\mathcal{G}(a)$, i.e. the first eigenvalue of (5), and a multiple of the first eigenfunction at the observation points p_m , see (3). In the second step this input is used to recover the conductivity distribution. We emphasize that only one (first) eigenvalue and the eigenfunction are needed for the identification. For other methods for inverse heat conduction problems see (Beck et al., 1985) and the references therein.

Before considering noise contaminated observation data $z_m(t)$, let us assume that $z_m(t)$ are known precisely on an interval $I = (t_0, T)$, $t_0 \geq 0$. In case (i) of Theorem 7.1 the observations are given by the Dirichlet series

$$z_m(t) = \sum_{k=1}^{\infty} \langle g, \psi_k \rangle e^{-\lambda_k t} \psi_k(p_m). \tag{39}$$

We have not implemented yet other cases of Theorem 7.1.

In principle, functions $z_m(t)$ are analytic for $t > 0$. Therefore they can be uniquely extended to $(0, \infty)$ from I . Then the first eigenvalue λ_1 and the data sequence $\{G_m = \langle g, \psi_1 \rangle \psi_1(p_m)\}_{m=1}^{M-1}$ can be recovered from the Dirichlet series (39) representing $z_m(t)$ by

$$\lambda_1 = -\frac{1}{h} \lim_{t \rightarrow \infty} \ln \frac{z_m(t+h)}{z_m(t)}, \quad G_m = \lim_{t \rightarrow \infty} e^{\lambda_1 t} z_m(t), \tag{40}$$

where $h > 0$.

The second step of the algorithm, i.e. the identification of the conductivity a is accomplished by the Marching Algorithm. Numerical experiments show that it provides the perfect identification only if $\mathcal{G}(a)$ is known precisely. However, even for noiseless data $z_m(t)$, the numerical identification of $\mathcal{G}(a)$ from the Dirichlet series (39) representing $z_m(t)$ can only be accomplished with a significant error. This numerical evidence is presented in (Gutman & Ha, 2009).

Hence a different algorithm is needed for the practically important case of noise contaminated data. It should also take into account the severe ill-posedness of the identification of data from Dirichlet series, see (Acton, 1990). Our numerical experiments confirm that even the second eigenvalue of the associated Sturm-Liouville problem cannot be reliably identified even for

noiseless data. It is the distinct advantage of the proposed algorithm that it uses only the first eigenvalue λ_1 for the conductivity identification. In what follows *LMA* refers to the Levenberg-Marquardt algorithm for the nonlinear least squares minimization, and *BA* to the Brent algorithm for a single variable nonlinear minimization, see (Press et al., 1992) for details. First, consider a simple regression type algorithm for the identification of the M -tuple $\mathcal{G}(a)$. In step 1, for each observation data $z_m(t)$ we find λ and c to best fit $z_m(t)$ in the objective function $\Psi(\lambda, c; m)$ defined by (41). In step 2 the obtained eigenvalues $\lambda^{(m)}$ are averaged over the middle third of the observation points, since such data would presumably be less affected by noise. The result of the averaging is the sought eigenvalue λ_1 . In step 3, the averaged eigenvalue λ_1 is kept fixed, and the functions $\Psi(\lambda_1, c; m)$ are minimized in variable c only. The resulting values G_m form the M -tuple $\mathcal{G}(a)$.

Regression Algorithm for λ_1 identification.

Let the data consist of the observations $z_m(t_j)$, $j = 1, 2, \dots, J$, $m = 1, 2, \dots, M - 1$.

(i) Let $\lambda, c \in \mathbb{R}$ and

$$\Psi(\lambda, c; m) = \sum_{j=1}^J (ce^{-\lambda t_j} - z_m(t_j))^2. \quad (41)$$

Let

$$\Psi(\lambda, c_m(\lambda); m) = \min_{c \in \mathbb{R}} \Psi(\lambda, c; m).$$

Note that such a minimizer $c_m(\lambda)$ can be found directly by

$$c_m(\lambda) = \frac{\sum_{j=1}^J z_m(j) e^{-\lambda t_j}}{\sum_{j=1}^J e^{-2\lambda t_j}}.$$

For each $m = 1, \dots, M - 1$ apply *BA* to find a $\lambda^{(m)}$ such that

$$\Psi(\lambda^{(m)}, c_m(\lambda^{(m)}); m) = \min_{\lambda \in \mathbb{R}} \Psi(\lambda, c_m(\lambda); m).$$

(ii) Let $k = \text{card}\{[M/3], \dots, [2M/3]\}$ and

$$\lambda_1 = \frac{1}{k} \sum_{m=[M/3]}^{[2M/3]} \lambda_1^{(m)}.$$

(iii) Keep λ_1 fixed. For each $m = 1, \dots, M - 1$ find $G_m = c_m(\lambda_1)$ such that

$$\Psi(\lambda_1, G_m; m) = \min_{c \in \mathbb{R}} \Psi(\lambda_1, c; m).$$

(iv) Let $\mathcal{G}(a) = \{\lambda_1, G_1, \dots, G_{M-1}\}$.

One may assume that fitting the data $z_m(t)$ using two exponents as in (43) could result in a better estimate for the eigenvalue λ_1 . To examine this assumption let us consider a more complicated algorithm which we call the *LMA Algorithm for λ_1 identification*. This algorithm proceeds as follows (see details below).

(i). This step is the same as step (i) in the regression algorithm above, i.e. we minimize the functions $\Psi(\lambda, c; m)$ in both λ and c for $m = 1, \dots, M - 1$. Call the minimizers by $\mu^{(m)}$ and $c_m(\mu^{(m)})$ respectively.

(ii). Apply the *LMA* to minimize $\Phi(\mu, \nu, c, b; m)$ defined in (43). Use the initial guess $\mu^{(m)}, 4\mu^{(m)}, c_m(\lambda), 0$ for the variables μ, ν, c, b correspondingly. Call the results of these minimizations for the variable μ by $\lambda_1^{(m)}$. The initial value $4\mu^{(m)}$ for the second eigenvalue is used because of Theorem 2.2(iii). A direct application of the *LMA* without the initial values obtained in Step (i) did not produce consistent results. Now the data $z_m(t)$ is approximated by the first two terms of the Dirichlet series (39). Thus, for each m there is an estimate $\lambda_1^{(m)}$ for the first eigenvalue λ_1 .

(iii). Let λ_1 be an average of the computed values $\lambda_1^{(m)}$. We used the middle third of the indices m since the maximum of our initial data $g(x)$ was attained in the middle of the interval $[0, 1]$. Hence these observations were relatively less affected by the noise.

(iv-v). Repeat the minimizations of Steps (i) and (ii), but keep λ_1 frozen. Let G_m be the values of the coefficients c that minimize $\Phi(\lambda_1, \nu, c, b; m)$. This is the best fit to the data $z_m(t)$ by the first two terms of the Dirichlet series (39) with the fixed first eigenvalue λ_1 . By now the first part of the identification algorithm is completed, since we have recovered the first eigenvalue λ_1 and a multiple G_m of the first eigenfunction $\psi_1(p_m), m = 1, 2, \dots, M - 1$.

LMA Algorithm for λ_1 identification.

Let the data consist of the observations $z_m(t_j), j = 1, 2, \dots, J, m = 1, 2, \dots, M - 1$.

(i) Let $\lambda, c \in \mathbb{R}$ and

$$\Psi(\lambda, c; m) = \sum_{j=1}^J (ce^{-\lambda t_j} - z_m(t_j))^2. \tag{42}$$

Let

$$\Psi(\lambda, c_m(\lambda); m) = \min_{c \in \mathbb{R}} \Psi(\lambda, c; m).$$

Note that such a minimizer $c_m(\lambda)$ can be found directly by

$$c_m(\lambda) = \frac{\sum_{j=1}^J z_m(j)e^{-\lambda t_j}}{\sum_{j=1}^J e^{-2\lambda t_j}}.$$

For each $m = 1, \dots, M - 1$ apply *BA* to find a $\mu^{(m)}$ such that

$$\Psi(\mu^{(m)}, c_m(\mu^{(m)}); m) = \min_{\lambda \in \mathbb{R}} \Psi(\lambda, c_m(\lambda); m).$$

(ii) Let

$$\Phi(\mu, \nu, c, b; m) = \sum_{j=1}^J (ce^{-\mu t_j} + be^{-\nu t_j} - z_m(t_j))^2. \tag{43}$$

Apply the *LMA* to minimize $\Phi(\mu, \nu, c, b; m)$ using the initial guess $\mu^{(m)}, 4\mu^{(m)}, c_m(\mu^{(m)}), 0$ for the variables μ, ν, c, b correspondingly. Let

$$\Phi(\lambda_1^{(m)}, \nu_m, c_m, b_m; m) = \min_{\mu, \nu, c, b} \Phi(\mu, \nu, c, b; m).$$

(iii) Let $k = \text{card}\{[M/3], \dots, [2M/3]\}$ and

$$\lambda_1 = \frac{1}{k} \sum_{m=[M/3]}^{[2M/3]} \lambda_1^{(m)}.$$

(iv) Find $c_m(\lambda_1)$, $m = 1, 2, \dots, M$ (as in Step 1) such that

$$\Psi(\lambda_1, c_m(\lambda_1); m) = \min_{c \in \mathbb{R}} \Psi(\lambda_1, c; m).$$

(v) Apply the LMA to minimize $\Phi(\lambda_1, v, c, b; m)$ in variables v, c, b using the initial guess $4\lambda_1, c_m(\lambda_1), 0$ for the variables v, c, b correspondingly. Let

$$\Phi(\lambda_1, v_m, G_m, b_m; m) = \min_{v, c, b} \Phi(\lambda_1, v, c, b; m).$$

(vi) Let $\mathcal{G}(a) = \{\lambda_1, G_1, \dots, G_{M-1}\}$.

The second part of the algorithm identifies the conductivity \bar{a} from the M -tuple $\mathcal{G}(a)$. As we have already mentioned the Marching Algorithm provides a perfect identification for noiseless data, otherwise one has to find \bar{a} by a nonlinear minimization.

Identification of piecewise constant conductivity.

The data is the M -tuple $\mathcal{G}(a) = \{\lambda_1, G_1, \dots, G_{M-1}\}$.

(i) Fix $N > 0$. Form the objective function $\Pi(a)$ by

$$\Pi(a) = \min_{c \in \mathbb{R}} \sum_{m=1}^M (cG_m - \psi_1(p_m; a))^2, \quad (44)$$

for the conductivities $a \in A_N \subset A_{\text{ad}}$ having at most $N - 1$ discontinuity points on the interval $[0, 1]$.

(ii) Use Powell's minimization method in $K = 2N - 1$ variables ($N - 1$ discontinuity points and N conductivity values) to find

$$\Pi(\bar{a}) = \min_{a \in A_N} \Pi(a).$$

The minimizer \bar{a} is the sought conductivity.

The function $\psi_1(p_m; a)$ in step (i) of the above algorithm is the first normalized eigenfunction of the Sturm-Liouville problem (5) corresponding to the conductivity $a \in A_N$. Powell's minimization method, a shooting method for the computation of the eigenvalues and the eigenfunctions, and numerical experiments are presented in (Gutman & Ha, 2009).

11. Conclusions

While in most parameter estimation problems one can hope only to achieve the best fit to data solution, sometimes it can be shown that such an identification is unique. In such case it is said that the sought parameter is identifiable within a certain class. In our recent work (Gutman & Ha, 2007; 2009) we have shown that piecewise constant conductivities $a \in \mathcal{PC}(\sigma)$ are identifiable from observations $z_m(t; a)$ of the heat conduction process (2) taken at finitely many points p_m .

Let $\mathcal{G}(a) = \{\lambda_1(a), G_1(a), \dots, G_{M-1}(a)\}$, where the values $G_m(a)$ are a constant nonzero multiple of the first eigenfunction $\psi_1(a)$. In principle, if $\mathcal{G}(a)$ is known, then the identification of the conductivity a can be accomplished by the Marching Algorithm. Theorem 7.1 shows under what conditions the M -tuple $\mathcal{G}(a)$ can be extracted from the observations $z_m(t)$, thus assuring the identifiability of a .

It is shown in Theorem 8.2 that the Marching Algorithm not only provides the unique identification of the conductivity a , but that the identification is also continuous (stable). This result is based on the continuity of eigenvalues, eigenfunctions, and the solutions with respect to the $L^1(0, 1)$ topology in the set of admissible parameters A_{ad} , see Section 4.

Numerical experiments show that, because of the ill-posedness of the identification of eigenvalues from a Dirichlet series representation, one can only identify $\mathcal{G}(a)$ with some error. Thus the Marching Algorithm would not be practically useful. In Section 10 we presented algorithms for the identification of conductivities from noise contaminated data. Its main novel point is, in agreement with the theoretical developments, the separation of the identification process into two separate parts. In part one the first eigenvalue and a multiple of the first eigenfunction are extracted from the observations. In the second part a general minimization method is used to find a conductivity which corresponds to the recovered eigenfunction.

The first eigenvalue and the eigenfunction in part one of the algorithm are found from the Dirichlet series representation of the solution of the heat conduction process. The numerical experiments in (Gutman & Ha, 2009) confirm that even for noiseless data the second eigenvalue cannot be reliably found. These experiments showed that in our tests a simple regression type algorithm identified λ_1 better than a more complex Levenberg-Marquardt algorithm. The last part of the algorithm employs Powell's nonlinear minimization method because it does not require numerical computation of the gradient of the objective function. The numerical experiments show that the conductivity identification was achieved with a 15-18% relative error for various noise levels in the observations.

12. References

- Acton, F. S. (1990). *Numerical methods that work. Rev. and updated ed.*, Washington: The Mathematical Association of America.
- Beck, J. V., Blackwell, B. & Clair, C. R. (1985). *Inverse heat conduction. Ill-posed problems.*, A Wiley-Interscience Publication. New York etc.: John Wiley and Sons, Inc. XVII.
- Benabdallah, A., Dermenjian, Y. & Rousseau, J. L. (2007). Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient and applications to controllability and an inverse problem, *Journal of Mathematical Analysis and Applications* 336(2): 865 – 887.
- Birkhoff, G. & Rota, G.-C. (1978). *Ordinary Differential Equations*, 3rd edn, Wiley, New York.
- Cannon, J. R. (1984). *The one-dimensional heat equation. Foreword by Felix E. Browder.*, Encyclopedia of Mathematics and Its Applications, Vol. 23. Menlo Park, California etc.: Addison-Wesley Publishing Company; Cambridge etc.: Cambridge University Press.
- Courant, R. & Hilbert, D. (1989). *Methods of mathematical physics. Volume I. Transl. and rev. from the German Original. Reprint of the 1st Engl. ed. 1953.*, Wiley Classics Edition; A Wiley-Interscience Publication. New York etc.: John Wiley & Sons.
- Demir, A. & Hasanov, A. (2008). Identification of the unknown diffusion coefficient in a linear parabolic equation by the semigroup approach, *Journal of Mathematical Analysis and Applications* 340(1): 5 – 15.

- Duchateau, P. (1995). Monotonicity and invertibility of coefficient-to-data mappings for parabolic inverse problems, *SIAM Journal on Mathematical Analysis* 26(6): 1473–1487. URL: <http://link.aip.org/link/?SJM/26/1473/1>
- Elayyan, A. & Isakov, V. (1997). On uniqueness of recovery of the discontinuous conductivity coefficient of a parabolic equation, *SIAM Journal on Mathematical Analysis* 28(1): 49–59. URL: <http://link.aip.org/link/?SJM/28/49/1>
- Evans, L. C. (2010). *Partial differential equations. 2nd ed.*, Graduate Studies in Mathematics 19. Providence, RI: American Mathematical Society.
- Gelfand, I. M. & Levitan, B. M. (1955). On the determination of a differential equation from its spectral function, *Amer. math. Soc. Transl. Ser. 2*(2): 253–304.
- Gutman, S. & Ha, J. (2007). Identifiability of piecewise constant conductivity in a heat conduction process, *SIAM Journal on Control and Optimization* 46(2): 694–713. URL: <http://link.aip.org/link/?SJC/46/694/1>
- Gutman, S. & Ha, J. (2009). Parameter identifiability for heat conduction with a boundary input., *Math. Comput. Simul.* 79(7): 2192–2210.
- Gutman, S. & Ramm, A. G. (2010). Inverse problem for a heat equation with piecewise-constant conductivity, *J. Appl. Math. and Informatics* 28(3–4): 651–661.
- Hoang, N. S. & Ramm, A. G. (2009). An inverse problem for a heat equation with piecewise-constant thermal conductivity, *Journal of Mathematical Physics* 50(6): 063512. URL: <http://link.aip.org/link/?JMP/50/063512/1>
- Isakov, V. (2006). *Inverse problems for partial differential equations. 2nd ed.*, Applied Mathematical Sciences 127. New York, NY: Springer.
- Kitamura, S. & Nakagiri, S. (1977). Identifiability of spatially-varying and constant parameters in distributed systems of parabolic type., *SIAM J. Control Optimization* 15: 785–802.
- Kohn, R. V. & Vogelius, M. (1985). Determining conductivity by boundary measurements ii. interior results, *Communications on Pure and Applied Mathematics* 38(5): 643–667. URL: <http://dx.doi.org/10.1002/cpa.3160380513>
- Lions, J. (1971). *Optimal control of systems governed by partial differential equations.*, Berlin-Heidelberg-New York: Springer-Verlag.
- Nakagiri, S.-i. (1993). Review of Japanese work of the last ten years on identifiability in distributed parameter systems., *Inverse Probl.* 9(2): 143–191.
- Orlov, Y. & Bentsman, J. (2000). Adaptive distributed parameter systems identification with enforceable identifiability conditions and reduced-order spatial differentiation., *IEEE Trans. Autom. Control* 45(2): 203–216.
- Pierce, A. (1979). Unique identification of eigenvalues and coefficients in a parabolic problem., *SIAM J. Control Optimization* 17: 494–499.
- Press, W. H., Teukolsky, S. A., Vetterling, W. T. & Flannery, B. P. (1992). *Numerical recipes in FORTRAN. The art of scientific computing. 2nd ed.*, Cambridge: Cambridge University Press.
- Ramm, A. G. (2005). *Inverse Problems. Mathematical and analytical techniques with applications to engineering.*, New York, NY: Springer.
- Saks, S. & Zygmund, A. (1965). *Analytic functions*, Warsaw: PWN - Polish Scientific Publishers. IX.
- Titchmarsh, E. (1962). *Introduction to the theory of Fourier integrals*, Oxford University Press.



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