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An Iterative Approach to the Fixed-Order Robust H_∞ Control Problem Using a Sequence of Infeasible Controllers

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1. Introduction

It is well known that the robust disturbance attenuation against uncertainties can be achieved by the robust H_∞ controllers and some practical situations make us use the fixed-order controllers. These facts imply that the fixed-order robust H_∞ controllers are important for practical control problems. However it is difficult to design such robust controllers, because the robust H_∞ control problems include an infinite number of matrix inequality constraints, in other words, they are described by Robust Semi-Definite Programming (RSDP) problems. For obtaining a feasible solution of the RSDP problems coming from the robust control problems with state feedback controllers or full-order controllers, many numerical methods have been proposed. Classically, the quadratic stability theory, i.e. a common constant Lyapunov function for the entire uncertain set is used for reducing the infinite constraints to the finite ones at the expense of conservatism (Boyd et al. 1994). Recently, parameter dependent Lyapunov functions are used to improve the conservatism (Chesi et al. 2005) - (Ichihara et al. 2003), (Kami et al. 2009) - (Shaked 2001), (Xie 2008) and some one-shot type approaches using extended LMI conditions, which allows to use the affine parameter dependent Lyapunov functions, have been proposed (Pipeleers et al. 2009), (Shaked 2001), (Xie 2008). However these methods can not always produce the robust controller, because common additional variables are required and these methods can not be used for designing fixed-order controllers. In this sense, an iterative type approach may be useful to the problems such that these one-shot type approaches can not be applied.
In the field of the numerical optimization, there are two types of iterative approaches for finding feasible or locally optimal solutions of the optimization problems: one is an interior-point approach which needs an initial feasible solution to be carried out and the other is an exterior-point approach which does not need it. From these facts, exterior-point approach can be efficient for obtaining the solutions of the problems such that feasible solutions are difficult to be found. However, there are no exterior-point approaches except those in (Iwasaki & Skelton 1995), (Kami & Nobuyama 2004), (Kami et al. 2009), (Vanbierviet 2009) for control problems to our knowledge.

In this paper, we deal with the fixed-order robust H_∞ controller synthesis problem against time invariant polytopic uncertainties, which can be described by parameter dependent bilinear matrix inequality (PDBMI) problems. The purpose of this paper is to propose an iterative approach which is like an exterior-point one. To do that, we introduce an `axis-shifted system' which is obtained by shifting the imaginary axis of the complex plane so that all perturbing closed-poles are included in the LHS of the shifted imaginary axis. Our approach constructs a sequence of infeasible controller variables on which the shifted imaginary axis returns to the original position while the H_∞ norm of the axis-shifted system is less than the prescribed H_∞ norm bound. The advantage of our approach is to be able to use any controller variables as an initial point. The efficiency of our approach is shown by a numerical example.

In this paper, the following notations are used. $\mathbf{R}$, $\mathbf{R}^{n\times m}$ and $\mathbf{S}^n$ are the sets of real scalars, $n\times m$ real matrices and $n\times n$ real symmetric matrices, respectively. He$\{Z\}$, $\begin{bmatrix} A & * \\ B^T & C \end{bmatrix}$ and $\sigma(\cdot)$ denote $Z+Z^T$, the block symmetric matrix $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ and a set of eigenvalues, respectively. Moreover, Ω denotes a hyper-rectangle and vert Ω indicates the set of vertices of Ω.

2. Problem formulation

In this paper, we consider the following plant $P(\theta)$ with a time invariant uncertain parameter $\theta := [\theta_1 \ \cdots \ \theta_N]$:

$$P(\theta) := \begin{cases} \dot{x}(t) = A(\theta)x(t) + Bu(t) + B_w w(t) \\ z(t) = Cx(t) + Du(t) \\ y(t) = Ex(t) \end{cases} \tag{1}$$

$$A(\theta) := A_0 + \sum_{i=1}^{N} \theta_i A_i \tag{2}$$

where $x(t)$ is the plant state, $w(t)$ is any exogenous input, $u(t)$ is the control input, $z(t)$ is the performance output, $y(t)$ is the measurement output and $\theta := [\theta_1 \ \cdots \ \theta_N] \in \Omega$ is an uncertain parameter vector whose elements satisfy

$$\theta_i \in [\underline{\theta}_i \ \ \overline{\theta}_i], i = 1, \cdots, N. \tag{3}$$

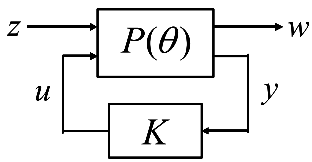

Fig. 1. Control system.

Moreover, we have the following assumptions:
1. $(A(\theta),B)$ is controllable for all $\theta\in\Omega$.
2. $(A(\theta),B_w,C)$ is controllable and observable for all $\theta\in\Omega$.

For this system let us consider the following fixed-order controller Σ_d or the static state feedback controller Σ_s:

$$\Sigma_d:\begin{cases}\dot{x}_c(t)=A_cx_c(t)+B_cy(t)\\ u(t)=C_cx_c(t)+D_cy(t)\end{cases} \tag{4}$$

$$\Sigma_s:u(t)=Kx(t) \tag{5}$$

where $x_c(t)\in\mathbf{R}^r$ is the controller state and r is the prescribed integer which achieves $0<r<n$. Note that Σ_d and Σ_s become state feedback controllers in the case that $E=I$ holds.
Via the controller Σ_d and Σ_s the closed-loop system can be described by

$$\begin{cases}\dot{x}_{cl}(t)=A_{cl}(\bar{K},\theta)x_{cl}(t)+B_{clw}w(t)\\ z(t)=C_{cl}(\bar{K})x_{cl}(t)\end{cases}. \tag{6}$$

$$A_{cl}(\bar{K},\theta):=\bar{A}(\theta)+\bar{B}\bar{K}\bar{E},C(\bar{K}):=\bar{C}+\bar{D}\bar{K}\bar{E} \tag{7}$$

For the controller Σ_d $x_{cl}(t)$ and the coefficient matrices in (7) are given by

$$x_{cl}(t)=\begin{bmatrix}x(t)\\ x_{cl}(t)\end{bmatrix},\bar{A}(\theta)=\bar{A}_0+\sum_{i=1}^{N}\theta_i\bar{A}_i,\bar{A}_i=\begin{bmatrix}A_i&0\\0&0\end{bmatrix}(i=0,1,\cdots,N), \tag{8}$$

$$\bar{B}=\begin{bmatrix}B&0\\0&I\end{bmatrix},\bar{E}=\begin{bmatrix}E&0\\0&I\end{bmatrix},\bar{C}=\begin{bmatrix}C&0\end{bmatrix},\bar{D}=\begin{bmatrix}D&0\end{bmatrix},\bar{K}=\begin{bmatrix}D_c&C_c\\B_c&A_c\end{bmatrix}. \tag{9}$$

For the controller Σ_s $x_{cl}(t)$ and the coefficient matrices are given by

$$x_{cl}(t)=x(t),\bar{A}(\theta)=A(\theta),\bar{B}=B,\bar{E}=E,\bar{C}=C,\bar{D}=D,\bar{K}=K \tag{10}$$

For the closed-loop system (6) the control problem to be solved in this paper is defined as follows:
Robust H_∞ synthesis problem:
Given an H_∞ norm bound γ_p, find $\bar{K}$ which achieves

$$\left\|T_{zw}(\bar{K},\theta)\right\|_\infty<\gamma_p \tag{11}$$

where $T_{zw}(\bar{K},\theta)$ is the transfer function from w to z of the closed-loop system (6) and $\|\cdot\|_\infty$ denotes the H_∞ norm.
For the control problem (11) the following lemma holds (Boyd et al. 1994):

Lemma 1 $\|T_{zw}(\bar{K},\theta)\|_\infty < \gamma_p$ holds if there exists a parameter dependent Lyapunov function

$$P^\infty(\theta) := P_0^\infty + \sum_{i=1}^{N} \theta_i P_i^\infty > 0 \tag{12}$$

which satisfies

$$\begin{bmatrix} P^\infty(\theta)A_{cl}(\bar{K},\theta) + A_{cl}(\bar{K},\theta)^T P^\infty(\theta) & * & * \\ B_{clw}^T P^\infty(\theta) & -\gamma_p I & * \\ C_{cl}(\bar{K}) & 0 & -\gamma_p I \end{bmatrix} < 0. \tag{13}$$

This lemma implies that the robust H_∞ synthesis problem (11) can be described as PDBMI problem, which has an infinite number of BMI constraints corresponding to all points on Ω. Hence it is difficult to obtain the feasible controller variables $\bar{K}$ achieving (13). One well known classical method for obtaining Σ_s in the case that $E = I$ is to use quadratic (parameter independent constant) Lyapunov functions (Boyd et al. 1994). i.e., defining

$$P^\infty(\theta)^{-1} := X, \; W := KX \tag{14}$$

to get the controller variables from

$$K = WX^{-1} \tag{15}$$

where X and W are the solutions of the next inequalities:

$$\begin{bmatrix} A(\theta)X + XA(\theta)^T + BW + W^T B^T & * & * \\ B_w^T & -\gamma_p I & * \\ CX + DW & 0 & -\gamma_p I \end{bmatrix} < 0, \forall \theta \in \text{vert}\,\Omega \tag{16}$$

However the quadratic Lyapunov functions X do not always exist and even if they exist the obtained controller includes a high conservatism. Moreover, this method can be only used in the case that $E = I$.
Recently, various studies with parameter dependent Lyapunov functions have been reported to reduce the conservatism (Chesi et al. 2005) - (Ichihara et al. 2003), (Kami et al. 2009) - (Shaked 2001), (Xie 2008). Especially, some interesting one-shot type approaches for designing static state feedback controllers or full-order controllers with extended matrix inequality conditions have been proposed (Pipeleers et. al., 2009), (Shaked 2001), (Xie 2008). However these methods do not always produce the feasible controllers in some cases, because some additional common matrix variables are required and this method can not be used in the case that $E \neq I$. In this paper, we propose an iterative approach to the fixed-order robust H_∞ synthesis problem, which can be used if $E \neq I$. The features of our approach are to constructs a controller sequence from the infeasible region to the feasible one and to be able to use any matrix as an initial point.

3. Multi-convex relaxation method

In this section, let us consider the next PDMI problem

$$\text{find } z \text{ s.t. } M(z,\theta) := M_0(z) + \sum_{i=1}^{N} \theta_i M_i(z) + \sum_{i=1}^{N}\sum_{j=i}^{N} \theta_i \theta_j M_{ij}(z) < 0, \forall \theta \in \Omega \tag{17}$$

where $z := \begin{bmatrix} z_1 & \cdots & z_N \end{bmatrix}^T$ $(z_i \in \mathbf{R})$ is a vector of decision variables, $\theta := \begin{bmatrix} \theta_1 & \cdots & \theta_N \end{bmatrix}^T \in \Omega$ is a parameter vector whose elements $\theta_i \in \mathbf{R}$ are in the given range $\theta_i \in \begin{bmatrix} \underline{\theta_i} & \overline{\theta_i} \end{bmatrix}$ and $M_0(z)$, $M_i(z)$ and $M_{ij}(z)$ are symmetric matrices with appropriate sizes. It is well known that feasible solutions of the PDMI problem (17) are difficult to be obtained, because this problem has an infinite number of constraints corresponding to all points on Ω. In this section, we show the multi-convex relaxation method (Ichihara et al. 2003) which is used for reducing the infinitely constrained problem to a finitely constrained one for obtaining a feasible solution of (17).

3.1 Multi-convex function

In this subsection, we review the definition and the properties of the multi-convex function.

Definition 1: If the function $f(\theta), \theta = \begin{bmatrix} \theta_1 & \cdots & \theta_N \end{bmatrix}$ becomes a multi-convex function with respect to any θ_j in the case that $\theta_i (i = 1, \cdots, j-1, j+1, \cdots, N)$ are fixed then the function $f(\theta)$ is said as a multi-convex function.

From the definition the multi-convex function has the next properties:

Lemma 2 The next statements hold:

1. The function $f(\theta)$ is the multi-convex function if and only if $\frac{\partial f(\theta)}{\partial \theta_i} \geq 0$ hold $\forall i = 1, \cdots, N$.
2. The maximum of the multi-convex function $f(\theta)$ on $\theta \in \Omega$ is on the vertex of Ω (See Fig. 2).

Using these properties the relaxation method for obtaining the feasible solution of (17) is shown in the next subsection.

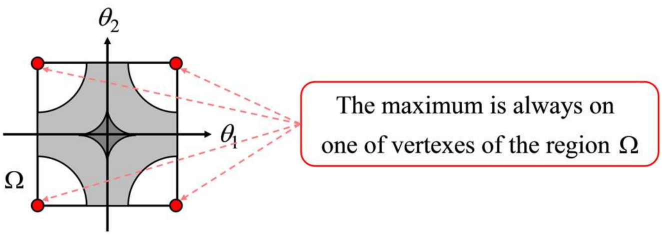

Fig. 2. The concept of the multi-convex functions.

3.2 Multi-convex relaxation

In this subsection, we show a relaxation method with multi-convex function (Ichihara et al. 2003) which is needed to derive our approach. The key idea of this method is to make the multi-convex upper bound of $M(z,\theta)$.

The multi-convex relaxation method can be described as the next lemma:
Lemma 3 z is a feasible solution of the PDMI problem (17) if there exist z, $Q_i \in \mathbf{R}^{n\times n}$ and $R_i \in \mathbf{S}^n (i = 1,\cdots,N)$ which achieve

$$\begin{bmatrix} M(z,\theta) & * & * & * \\ Q_1 - \theta_1 R_1 & -R_1 & * & * \\ \vdots & 0 & \ddots & * \\ Q_N - \theta_N R_N & 0 & 0 & -R_N \end{bmatrix} < 0, \forall \theta \in \text{vert}\,\Omega \tag{18}$$

$$M_{ii}(z) + R_i \ge 0 (i = 1,\cdots N). \tag{19}$$

Proof $M(z,\theta) < 0, \forall \theta \in \Omega$ holds iff we have

$$f(z,\theta) := x^T M(z,\theta) x < 0, \forall \theta \in \Omega, \forall x (\neq 0) \in \mathbf{R}^n. \tag{20}$$

Now, let us define $f_1(z,\theta)$ and $f_2(\theta)$ as

$$f_1(z,\theta) := x^T M(z,\theta) x + \sum_{i=1}^{N} \theta_i^2 x^T R_i x, \tag{21}$$

$$f_2(\theta) := \sum_{i=1}^{N} \theta_i^2 x^T R_i x, \tag{22}$$

respectively, where $R_i \in \mathbf{S}^n$ achieve

$$x^T (M_{ii}(z) + R_i) x \ge 0, i := 1,\cdots N \tag{23}$$

which is the necessary and sufficient condition for $f_1(z,\theta)$ to be multi-convex function with respect to θ. Then the function $\overline{f}(z,\theta)$ given by (24) becomes a multi-convex upper bound function of $f(z,\theta)(= f_1(z,\theta) - f_2(\theta))$:

$$\overline{f}(z,\theta) := f_1(z,\theta) - \overline{f}_2(\theta), \tag{24}$$

$$\overline{f}_2(\theta) := \sum_{i=1}^{N} \left(\theta_i \mathrm{B}_i^T R_i + \theta_i R_i \mathrm{B}_i - \mathrm{B}_i^T R_i \mathrm{B}_i \right). \tag{25}$$

This is because $-f_2(\theta) \le -\overline{f}_2(\theta)$ holds from

$$-\theta_i^2 R_i \le -(\theta_i \mathrm{B}_i^T R_i + \theta_i R_i \mathrm{B}_i - \mathrm{B}_i^T R_i \mathrm{B}_i), \forall \mathrm{B}_i \in \mathbf{R}^{n\times n}. \tag{26}$$

Then, from the property of the multi-convex functions $\overline{f}(z,\theta) < 0$ holds iff we have

$$M(z,\theta) + \sum_{i=1}^{N} (\theta_i I - \mathrm{B}_i)^T R_i (\theta_i I - \mathrm{B}_i) < 0, \forall \theta \in \text{vert}\,\Omega \tag{27}$$

and the inequality (27) can be transformed into

$$\begin{bmatrix} M(z,\theta) & * & * & * \\ R_1\mathrm{B}_1 - \theta_1 R_1 & -R_1 & * & * \\ \vdots & 0 & \ddots & * \\ R_N\mathrm{B}_N - \theta_N R_N & 0 & 0 & -R_N \end{bmatrix} < 0, \forall \theta \in \mathrm{vert}\,\Omega. \tag{28}$$

Therefore, z is a feasible solution of $M(\theta,z) < 0$ if there exist z, B_i and R_i which achieve (28) for all $\theta \in \mathrm{vert}\,\Omega$ and replacing $R_i\mathrm{B}_i$ by Q_i in (28) we have (18).
Using this lemma the problem (17) with an infinite number of constraints can be reduced into that with a finite number of constraints.

4. Iterative approach to the robust H_∞ synthesis problems

In this section, we propose an iterative approach to the robust H_∞ control problem (11) using Lemma 3. To do that, we introduce an `axis-shifted system' which is obtained by shifting the imaginary axis so that all perturbing poles are located in the LHS of the imaginary axis. The key idea of our approach is to return the shifted imaginary axis to the original position while the H_∞ norm of the axis-shifted system is less than γ_p. The feature of our approach is to be able to use any controller variables as an initial point.
Firstly, we add the practical assumption for the closed-loop system (6) such that the poles of the system (6) do not exist infinitely far from the imaginary axis on the RHS of the complex plane, i.e., there always exists a finite scalar β which achieves:

$$\max_{\lambda \in \sigma(A_{cl}(\bar{K},\theta))} \mathrm{Re}[\lambda] < \beta, \forall \theta \in \Omega \tag{29}$$

and we introduce the following system using β, which is needed to derive our iterative approach:

$$\begin{cases} \dot{x}_{cl}(t) = (A_{cl}(\bar{K},\theta) - \beta I)x_{cl}(t) + B_{clw}w(t) \\ z(t) = C_{cl}(\bar{K})x_{cl}(t) \end{cases}. \tag{30}$$

This system has the next property.
Lemma 4 The system (30) is robustly stable for the parameter $\theta \in \Omega$.
Proof It is obvious from (29).
Remark 1. In this paper, we interpret the meaning of " $A_{cl}(\bar{K},\theta) - \beta I$ " as shifting the imaginary axis of the complex plane to the right by β (See Fig. 3). In this sense, the system (30) is called as `axis-shifted system' in this paper.
Now, letting $\bar{T}_{zw}(\bar{K},\theta,\beta)$ be a transfer function of the system (30) from w to z the next lemma holds for the H_∞ norm condition

$$\left\| \bar{T}_{zw}(\bar{K},\theta,\beta) \right\|_\infty < \gamma_p. \tag{31}$$

Lemma 5 (31) holds if there exists a parameter dependent Lyapunov function

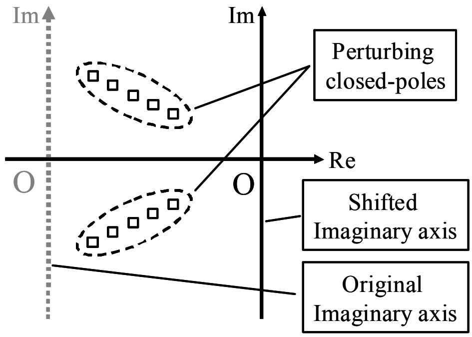

Fig. 3. Concept of complex plane of the axis-shifted system.

$$P(\theta) := P_0^{\infty} + \sum_{i=1}^{N} \theta_i P_i^{\infty} > 0 \tag{32}$$

which achieves

$$M^{\infty}(P^{\infty}(\theta), \bar{K}, \beta, \theta, \gamma_p) < 0 \tag{33}$$

where

$$M^{\infty}(P^{\infty}(\theta), \bar{K}, \beta, \theta, \gamma_p) := \begin{bmatrix} P^{\infty}(\theta)(A_{cl}(\bar{K},\theta) - \beta I) + (A_{cl}(\bar{K},\theta) - \beta I)^T P^{\infty}(\theta) & * & * \\ B_{clw}^T P^{\infty}(\theta) & -\gamma_p I & * \\ C_{cl}(\bar{K}) & 0 & -\gamma_p I \end{bmatrix}. \tag{34}$$

Proof It is obvious from Lemma 1.

Remark 2. If $A_{cl}(\bar{K},\theta)$ is robustly stable $\forall \theta \in \Omega$ we can let $\beta = 0$ and in this case $\|\bar{T}_{zw}(\bar{K},\theta,0)\|_{\infty} = \|T_{zw}(\bar{K},\theta)\|_{\infty}$ holds.

Now, the inequality (33) can be described as

$$M_0^{\infty} + \sum_{i=1}^{N} \theta_i M_i^{\infty} + \sum_{i=1}^{N} \sum_{j=i}^{N} \theta_i \theta_j M_{ij}^{\infty} < 0 \tag{35}$$

$$M_0^{\infty} = \begin{bmatrix} P_0^{\infty}(\bar{A}_0 + \bar{B}\bar{K}\bar{E}) + (\bar{A}_0 + \bar{B}\bar{K}\bar{E})^T P_0^{\infty} & * & * \\ B_{clw}^T P_0^{\infty} & -\gamma_p I & * \\ \bar{C} + \bar{D}\bar{K}\bar{E} & 0 & -\gamma_p I \end{bmatrix} \tag{36}$$

$$M_i^{\infty} = \begin{bmatrix} P_0^{\infty}\bar{A}_i + \bar{A}_i^T P_0^{\infty} + P_i^{\infty}(\bar{A}_0 + \bar{B}\bar{K}\bar{E}) + (\bar{A}_0 + \bar{B}\bar{K}\bar{E})^T P_i^{\infty} & * & * \\ B_{clw}^T P_i^{\infty} & 0 & * \\ 0 & 0 & 0 \end{bmatrix} \tag{37}$$

$$M_{ij} = \begin{cases} \begin{bmatrix} P_i^\infty A_j + A_j^T P_i^\infty + P_j^\infty A_i + A_i^T P_j^\infty & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix} & i = j \\ \begin{bmatrix} P_i^\infty A_i + A_i^T P_i^\infty & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix} & i \neq j \end{cases} \tag{38}$$

Then we can get the next the next lemma with Lemma 3 which is needed to derive our iterative approach.

Lemma 6 $\bar{K}$ and P_i^∞ achieve the H_∞ norm constraint (31) if there exist Q_i and $R_i (i := 1, \cdots, N), \forall \theta \in \mathrm{vert}\,\Omega$, which achieve

$$\bar{M}^\infty(P^\infty(\theta), \bar{K}, \beta, \gamma_p, \theta) := \begin{bmatrix} M^\infty(P^\infty(\theta), \bar{K}, \beta, \gamma_p, \theta) & * & \cdots & * \\ Q_1^\infty - \theta_1 R_1^\infty & -R_1 & & * \\ \vdots & & \ddots & \vdots \\ Q_N^\infty - \theta_N R_N^\infty & 0 & \cdots & -R_N \end{bmatrix} < 0, \tag{39}$$

$$M_{ii}^\infty + R_i \geq 0, \forall i. \tag{40}$$

Moreover, we have the next lemma with respect to the existence of β which achieves $\left\|\bar{T}_{zw}(\bar{K}, \theta, \beta)\right\|_\infty < \gamma_p, \forall \theta \in \Omega$ for given controller variables $\bar{K}_k$.

Lemma 7 For a given $\bar{K}_k$ achieving the condition (29) there always exists β achieving

$$\left\|\bar{T}_{zw}(\bar{K}, \theta, \beta)\right\|_\infty < \gamma_p, \forall \theta \in \Omega \tag{41}$$

Proof: Let us consider the next matrix:

$$A_{cl}(\bar{K}_k, \theta) + A_{cl}(\bar{K}_k, \theta)^T + \begin{bmatrix} B_{clw} & C_{cl}(\bar{K})^T \end{bmatrix} \begin{bmatrix} \gamma_p & 0 \\ 0 & \gamma_p \end{bmatrix}^{-1} \begin{bmatrix} B_{clw}^T \\ C_{cl}(\bar{K}) \end{bmatrix} \tag{42}$$

Then, from (29), we can choose β which is larger than the maximum eigenvalue of the next symmetric matrix:

$$\frac{1}{2}\left(A_{cl}(\bar{K}_k, \theta) + A_{cl}(\bar{K}_k, \theta)^T + \begin{bmatrix} B_{clw} & C_{cl}(\bar{K})^T \end{bmatrix} \begin{bmatrix} \gamma_p & 0 \\ 0 & \gamma_p \end{bmatrix}^{-1} \begin{bmatrix} B_{clw}^T \\ C_{cl}(\bar{K}) \end{bmatrix} \right), \tag{43}$$

which implies that there exists β which achieves

$$2\beta I > A_{cl}(\bar{K}_k, \theta) + A_{cl}(\bar{K}_k, \theta)^T + \begin{bmatrix} B_{clw} & C_{cl}(\bar{K})^T \end{bmatrix} \begin{bmatrix} \gamma_p & 0 \\ 0 & \gamma_p \end{bmatrix}^{-1} \begin{bmatrix} B_{clw}^T \\ C_{cl}(\bar{K}) \end{bmatrix}, \theta \in \mathrm{vert}\,\Omega. \tag{44}$$

This inequality can be transformed into the next inequality:

$$\begin{bmatrix} A_{cl}(\bar{K},\theta)-\beta I+(A_{cl}(\bar{K},\theta)-\beta I)^T & * & * \\ B_{clw}^T & -\gamma_p I & * \\ C_{cl}(\bar{K}) & 0 & -\gamma_p I \end{bmatrix} < 0, \theta \in \text{vert}\,\Omega \tag{45}$$

and this inequality can be obtained by substituting the common constant Lyapunov function $P^{\infty}(\theta)=I$ into (33). Hence $\left\|\bar{T}_{zw}(\bar{K},\theta,\beta)\right\|_{\infty} < \gamma_{p}, \forall\theta\in\Omega$ holds.

Using Lemmas 6 and 7, we propose the following iterative approach to obtain a feasible solution of the problem (11):

Algorithm

Step 1: Find any $\bar{K}_1$ and let β_1 and μ_1 be scalars which achieve

$$\beta_1 > \max_{\lambda\in\sigma(A_{cl}(\bar{K}_1,\theta))} \text{Re}[\lambda],\ \left\|\bar{T}_{zw}(\bar{K}_1,\theta,\mu_1)\right\|_{\infty} < \gamma_p,\ \beta_1 \le \mu_1 \tag{46}$$

respectively, for $\theta\in\text{vert}\,\Omega$. For example, let $\mu_1=\beta_1$ where μ_1 can be chosen as the solution of the LMI's (45). Let $k:=1$ and choose ω from 0 to 1.

Step 2: If $\beta_k \le 0$ then let $K^* := K_k$ and exit. Otherwise let

$$\mu_{k+1} := \omega\beta_k + (1-\omega)\mu_k \tag{47}$$

and go to the next step.

Step 3: Find $P_i^{\infty}(i:=0,\cdots,N)$ which satisfy

$$\bar{M}^{\infty}(P^{\infty}(\theta),K_k,\mu_{k+1},\gamma_p,\theta)<0,\forall\theta\in\text{vert}\,\Omega \tag{48}$$

and let them be P_{ik}^{∞} and define

$$P_k^{\infty}(\theta) := P_{0k}^{\infty} + \sum_{i=1}^{N}\theta_i P_{ik}^{\infty} \tag{49}$$

Step 4: Find K and β_t which are the solutions of

$$\min_{K,\beta_t} \beta_t \text{ s.t. } \beta_t < \mu_{k+1}, \tag{50}$$

$$\bar{M}^{\infty}(P_k^{\infty}(\theta),K,\beta_t,\gamma_p,\theta)<0,\forall\theta\in\text{vert}\,\Omega \tag{51}$$

and let $K_{k+1}:=K, \beta_{k+1}:=\beta_t$ and $k:=k+1$ and go to Step 2.

Theorem 1 The next statements hold for our algorithm.

1. μ_k is an upper bound of β_k, i.e., $\mu_k \ge \beta_k$ holds.
2. μ_k is monotonically decreasing, i.e., $\mu_k > \mu_{k+1}$ holds.
3. $\left\|\bar{T}_{zw}(\bar{K}_k,\theta,\mu_k)\right\|_{\infty} < \gamma_{p}, \forall\theta\in\Omega$ holds for all k.

Proof 1. and 2. From (46) and (50), $\mu_k \geq \beta_k$ holds. Moreover, from (47) we have

$$\beta_k < \mu_{k+1} < \mu_k . \tag{52}$$

3. From Step 4 of the algorithm and the fact that $\mu_k \geq \beta_k$ holds, we have

$$\begin{aligned} 0 &> \bar{M}^\infty(P_{k-1}^\infty(\theta), K_k, \beta_k, \gamma_p, \theta) \\ &\geq \bar{M}^\infty(P_{k-1}^\infty(\theta), K_k, \mu_k, \gamma_p, \theta) \\ &\geq M^\infty(P_{k-1}^\infty(\theta), K_k, \mu_k, \gamma_p, \theta) , \end{aligned} \tag{53}$$

which implies K_k achieves $\left\|\bar{T}_{zw}(\bar{K}_k, \theta, \mu_k)\right\|_\infty < \gamma_p, \forall \theta \in \Omega$. Hence Theorem 1 holds.

Remark 1: The key idea of our approach is to decrease μ_k so as to approach β_k to 0, i.e., the shifted imaginary axis approach the original position while the H_∞ norm constraint $\left\|\bar{T}_{zw}(\bar{K}_k, \theta, \mu_k)\right\|_\infty < \gamma_p, \forall \theta \in \Omega$ is achieved(See Fig.4). This fact implies that the controller K_k is updated from a non robust H_∞ controller for the original system to a robust H_∞ one as k increases. In this sense, this approach can be an exterior-point approach.

Remark 2: Unfortunately, our approach can not always produce a robust H_∞ controller, in other words, there does not exist the efficient ways of choices of K_1, β_1, μ_1 and ω so that a feasible robust controller is always obtained. Hence a condition for detecting an infeasibility for obtaining a robust feasible H_∞ controller may be needed. Moreover, $\beta_k \leq 0$ is a sufficient condition for $\left\|T_{zw}(\bar{K}_k, \theta)\right\|_\infty < \gamma_p, \forall \theta \in \Omega$. Hence we may also need a efficient criterion for K_k to be a feasible solution of the problem (11).

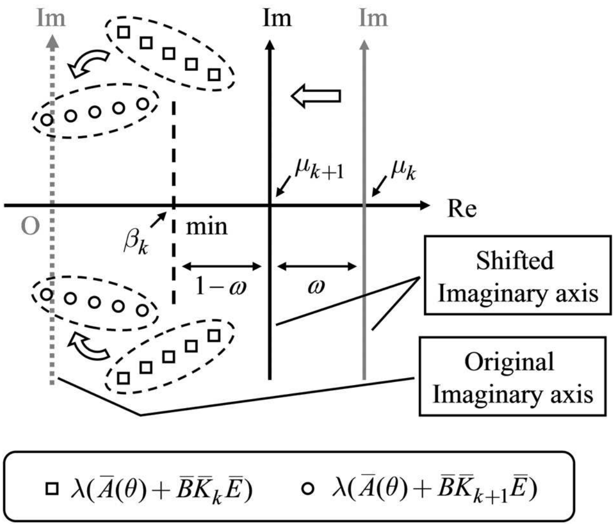

Fig. 4. Concept of our exterior-point approach.

5. Numerical example

To demonstrate the efficiency of our approach let us consider the following matrices:

$$A_0 = \begin{bmatrix} -9 & 1 & 2 \\ 6 & -8 & -11 \\ -1 & 4 & -7 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 1 \\ -5 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & -3 & -2 \\ 0 & -1 & 5 \\ -3 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}, B_w = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{54}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, D = 0.1, E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \gamma_p = 0.1 \tag{55}$$

Note that the one-shot type methods (Pipeleers et al. 2009), (Shaked 2001), (Xie 2008) can not use for designing the robust H_∞ controller because of $E \neq I$.
For this numerical example, we set the initial condition for carrying out our approach as follows:

$$K_1 = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right], \omega = 0.3, \beta_1 = \mu_1 = 8.1027 \tag{56}$$

where $\beta_1 (= \mu_1)$ is given as the solution of the LMI's (45).
Fig. 5 shows locations of eigenvalues of $\overline{A}(\theta) + \overline{B}\overline{K}_1\overline{E}$, i.e., the perturbations of poles of the uncertain closed-loop system via initial controller variables $\overline{K}_1$. This figure shows that $\overline{K}_1$ is not a robust stabilizing controller.
After 10 iterations the next controller variables are given from our approach:

$$\overline{K}^* = \left[\begin{array}{cc|cc} -1.5195 & -3.6942 & -8.3794 & -2.6309 \\ \hline 35.6459 & -43.3047 & -270.7538 & 85.3833 \\ -40.8834 & -1.7382 & 131.5127 & -91.5248 \end{array}\right]. \tag{57}$$

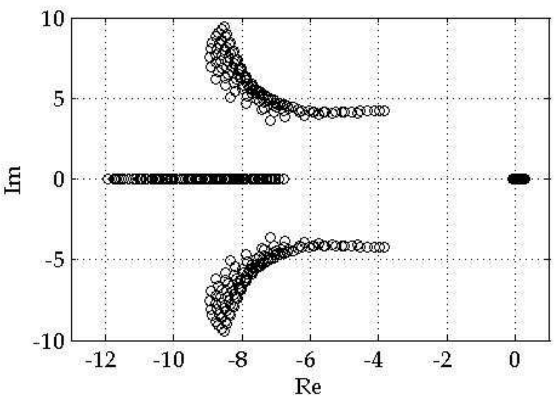

Fig. 5. Placement of the closed-poles via $\overline{K}_1$.

Fig. 6 and 7 show locations of eigenvalues of $\bar{A}(\theta)+\bar{B}\bar{K}^*\bar{E}$, i.e., the perturbations of poles of the uncertain closed-loop system via controller variables $\bar{K}^*$ and the contour plot of $\|T_{zw}(\bar{K}^*,\theta)\|_\infty$ on Ω, respectively. From these figures, K^* is a feasible solution of the problem (11).

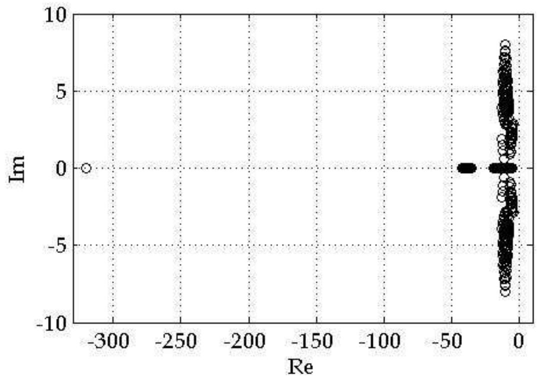

Fig. 6. Placement of the closed-poles via $\bar{K}^*$.

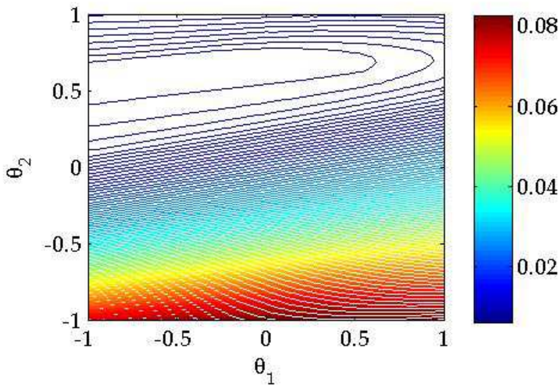

Fig. 7. The contour plot of $\|T_{zw}(\bar{K}^*,\theta)\|_\infty$ on Ω.

Fig. 8 shows behaviours of μ_k and β_k as a function of iteration number k. This figure shows that μ_k is an upper bound of β_k and monotonically decreasing, which implies that the controller variables $\bar{K}_k$ is updated from a non robust stabilizing controller to a robust H_∞ controller.

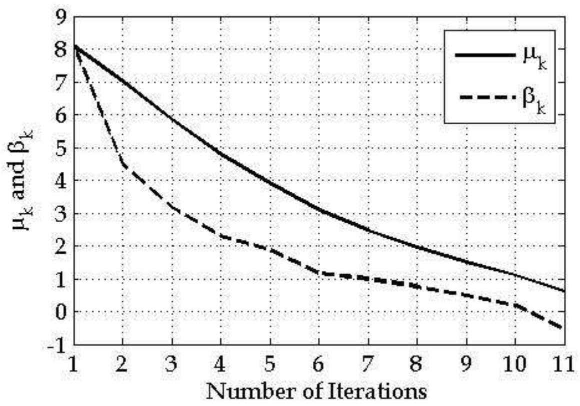

Fig. 8. Behaviours of μ_k and β_k.

6. Conclusions

In this paper, we have considered the robust H_∞ control problem against time invariant uncertainties. Firstly, we show the relaxation method for obtaining a feasible solution of the PDMI problem with multi-convex functions. Secondly, we introduce the axis-shifted system and show that this system can be constructed so as to achieve the H_∞ norm constraint. Next, we propose an iterative approach using the axis-shifted system and multi-convex relaxation method for obtaining the robust H_∞ controllers. The property of our approach is to construct a controller sequence on which the shifted imaginary axis approaches the original position with the H_∞ norm constraint achieved and to be able to choose any controller variables as an initial point. Finally we have given a numerical example which shows the efficiency of our approach.

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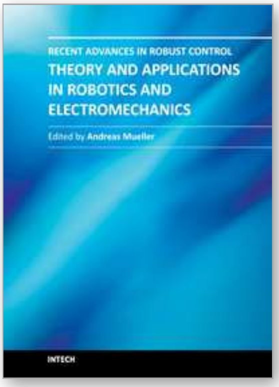

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Robust control has been a topic of active research in the last three decades culminating in H_2/H_\infty and \mu design methods followed by research on parametric robustness, initially motivated by Kharitonov's theorem, the extension to non-linear time delay systems, and other more recent methods. The two volumes of Recent Advances in Robust Control give a selective overview of recent theoretical developments and present selected application examples. The volumes comprise 39 contributions covering various theoretical aspects as well as different application areas. The first volume covers selected problems in the theory of robust control and its application to robotic and electromechanical systems. The second volume is dedicated to special topics in robust control and problem specific solutions. Recent Advances in Robust Control will be a valuable reference for those interested in the recent theoretical advances and for researchers working in the broad field of robotics and mechatronics.

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