

A Sum of Squares Optimization Approach to Robust Control of Bilinear Systems

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1. Introduction

Robust control problems for nonlinear systems are usually formulated as L_2 -induced norm minimization problems and those problems are reduced to the solvability of the so-called "Hamilton-Jacobi equation" (see, for example, van der Schaft (1996) and references therein). However, in the case of bilinear systems the usual L_2 -induced norm minimization problem leads to an obvious solution (the zero input is optimal!). To avoid the obvious solution Shimizu et al. (1997) introduced nonlinear weights on the evaluated signal and proposed a design method using linearization of the state-dependent matrix Riccati inequality derived from the Hamilton-Jacobi equation. In contrast to this, the purpose of this paper is to propose a new design method using SOS (Sum-of-Squares) optimization without linearization. It is known that the Hamilton-Jacobi equality coming from the L_2 -induced norm minimization problem is reduced to the solvability of an inequality condition of quadratic form, i.e.,

$$h^T(x)M(x)h(x) \geq 0, \quad \forall x \quad (1)$$

and this inequality is moreover reduced to the following matrix positive semi-definiteness condition:

$$M(x) \succeq 0, \quad \forall x \quad (2)$$

where $M(x)$ is a Riccati-type matrix including the state variables. This matrix inequality is usually called "a state-dependent matrix Riccati inequality" derived from the L_2 -induced norm optimization problem. Most papers have tried to find a solution to the matrix inequality (2) so far. See, for example, Ichihara (2009); Prajna et al. (2004) and the references therein. However, it should be noted that the condition (2) is just a sufficient condition for (1) unless $h(x)$ is independent of x , because $M(x)$ includes x . In most L_2 -induced norm optimization cases, $h(x)$ includes x (in our case $h(x) = P^{-1}x$ as shown in the later section) and hence the methods of Ichihara (2009); Prajna et al. (2004) and other papers based upon the condition (2) can have significant conservativeness. Note that Ichihara (2009) proposed a redesign method for reducing the conservativeness; however, it has to find a solution to (2) before applying the redesign method. Hence, the redesign method cannot be applied if the matrix inequality (2) does not have a solution.

In the present paper, to avoid the conservativeness we propose a new method for finding a solution to (1) directly without finding a solution to (2). A key idea of our method is to treat the dependency of $h(x)$ with x as an equality condition and formulate the problem to be concerned as an SOS (Sum of Squares) optimization problem with an equality constraint. After that we apply SOS optimization technique to the problem to propose an iterative algorithm for finding a robust feedback controller.

This paper is organized as follows: In Section 2, the plant to be concerned is described and a robust control problem is formulated after introducing nonlinear weights. Moreover, an inequality condition of quadratic form and the corresponding state-dependent matrix Riccati inequality are derived without using the Hamilton-Jacobi equality. In Section 3, some definitions and basic properties of SOS polynomials and SOS matrices are given. In Section 4, a new iterative method is proposed for finding a solution to the inequality condition of quadratic form. In Section 5, a numerical example is demonstrated to show the efficiency of our method, and in Section 6 this paper is concluded.

In this paper, the following notations are used:

\mathbf{R} the set of real numbers.

\mathbf{Z} the set of integers.

\mathbf{Z}_+ the set of non-negative integers.

$\mathbf{R}[x]$ the set of polynomials in x . ($\mathbf{R}[x]$ is also written as $\mathbf{R}[x_1 \cdots x_n]$ for $x = [x_1 \cdots x_n]^T$.)

M^T the transpose of the matrix M .

\otimes the Kronecker product.

Σ the set of SOS polynomials. In particular, Σ_x denotes the set of SOS polynomials in x .

I an identity matrix of appropriate size. In particular, I_r denotes the $r \times r$ identity matrix.

Moreover, for a square matrix M , $M \succ 0$ and $M \succeq 0$ imply that M is positive definite and positive semi-definite, respectively.

2. Problem statement

2.1 Plant and nonlinear weights

Consider the following bilinear systems:

$$\begin{aligned}\dot{x}_p(t) &= A_p x_p(t) + B_{p1} w(t) + \sum_{i=1}^{n_q} B_{p2i} x_p(t) u_i(t) \\ &= A_p x_p(t) + B_{p1} w(t) + B_{p2}(x) u(t)\end{aligned}\quad (3)$$

$$z_p(t) = C_p x_p(t) \quad (4)$$

where $x_p \in \mathbf{R}^{n_p}$ is the state variable, $u := [u_1 \cdots u_{n_q}]^T \in \mathbf{R}^{n_q}$ is the input, $w \in \mathbf{R}^{n_w}$ is the exogenous input, $z_p \in \mathbf{R}^r$ is the output to be evaluated, and the matrices A_p , B_{p1} , B_{p2i} ($i = 1, \dots, n_q$) are real matrices of appropriate sizes with

$$B_{p2}(x) := \sum_{i=1}^{n_q} B_{p2i} x.$$

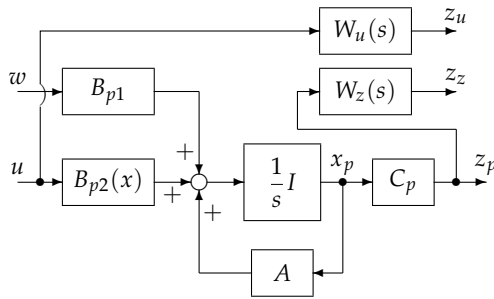


Fig. 1. Plant with frequency weights

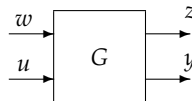


Fig. 2. Generalized plant

In this paper, we consider to evaluate the L_2 -induced norm from w to z_p and u with the following frequency weights $W_z(s)$ and $W_u(s)$, respectively (see Fig. 1):

$$W_z(s) : \begin{cases} \dot{x}_z(t) = A_z x_z(t) + B_z z_p(t), \\ z_z(t) = C_z x_z(t) + D_z z_p(t), \end{cases} \tag{5}$$

$$W_u(s) : \begin{cases} \dot{x}_u(t) = A_u x_u + B_u u(t), \\ z_u(t) = C_u x_u(t) + D_u u(t) \end{cases} \tag{6}$$

where $x_z \in \mathbf{R}^{n_z}$, $x_u \in \mathbf{R}^{n_u}$ and the matrices in (5) and (6) are real matrices of appropriate size. Here, we assume that the state variable is available for feedback. Then the plant with the frequency weights in Fig. 1 can be represented as a generalized plant G in Fig. 2 which is given by

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2(x)u(t) \tag{7}$$

$$z(t) = \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ D_{12} \end{bmatrix} u(t) \tag{8}$$

$$y(t) = x(t) \tag{8}$$

where $y(t)$ is the output for feedback and

$$x(t) = \begin{bmatrix} x_p(t) \\ x_z(t) \\ x_u(t) \end{bmatrix}, z(t) = \begin{bmatrix} z_z(t) \\ z_u(t) \end{bmatrix}, \tag{9}$$

$$A = \begin{bmatrix} A_p & 0 & 0 \\ B_z C_p & A_z & 0 \\ 0 & 0 & A_u \end{bmatrix}, B_1 = \begin{bmatrix} B_{p1} \\ 0 \\ 0 \end{bmatrix}, B_2(x) = \begin{bmatrix} B_{p2}(x) \\ 0 \\ B_u \end{bmatrix}, \tag{10}$$

$$C_{11} = [D_z C_p \ C_z \ 0], C_{12} = [0 \ 0 \ C_u], D_{12} = D_u. \tag{11}$$

Let n ($:= n_p + n_z + n_u$) denote the dimension of x .

The purpose of this paper is to find a feedback controller which reduces the effect of w on z . For this purpose, the problem to minimize the L_2 -induced norm from w to z defined by

$$\sup_{w \neq 0} \frac{\|z\|_2}{\|w\|_2} \tag{12}$$

is usually considered where $\|\cdot\|_2$ denotes the L_2 norm. However, the bilinear system (3) is uncontrollable for $x = 0$ because of $B_2(0) = 0$, so that the effect of w cannot be reduced around $x = 0$. Moreover, it is known that the zero input $u = 0$ is optimal for the problem of minimizing the L_2 -induced norm (12) when the evaluated variable z is affine in x and u . Hence, the minimization problem with respect to the L_2 -induced norm (12) is no use for our purpose.

Although the system is uncontrollable at $x = 0$, the system behavior can be improved by some proper controllers except at $x = 0$. To formulate the problem of finding such controllers Shimizu et al. (1997) introduce nonlinear weights on z . It is shown by them that the obvious solution (the zero input) can be avoided by introducing the nonlinear weights.

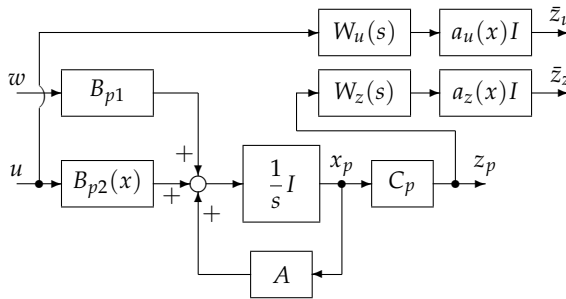


Fig. 3. Plant with nonlinear weights

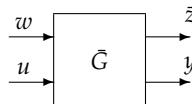


Fig. 4. Generalized plant including nonlinear weights

As in the paper of Shimizu et al. (1997) we will also introduce nonlinear weights on z as shown in Fig. 3 where $a_z(x)$ and $a_u(x)$ are the nonlinear weights which are functions of x .

With the introduction of the nonlinear weights the new generalized plant \bar{G} shown in Fig. 4 is represented as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1w(t) + B_2(x)u(t), \\ \bar{z}(t) &= \begin{bmatrix} a_z(x)C_{11} \\ a_u(x)C_{12} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ a_u(x)D_{12} \end{bmatrix} u(t), \\ y(t) &= x(t)\end{aligned}\quad (13)$$

where

$$\begin{aligned}\bar{z}(t) &:= \begin{bmatrix} \bar{z}_z(t) \\ \bar{z}_u(t) \end{bmatrix} \\ &= \begin{bmatrix} a_z(x)z_z(t) \\ a_u(x)z_u(t) \end{bmatrix}.\end{aligned}$$

Then the problem to be considered in this paper is formulated as the one of finding the feedback controller which minimizes the L_2 -induced norm from w to \bar{z} defined by

$$\sup_{w \neq 0} \frac{\|\bar{z}\|_2}{\|w\|_2}. \quad (14)$$

The next theorem is shown by Ohsaku et al. (1998); Shimizu et al. (1997) using linealization.

Theorem 1. Consider the bilinear system (13) with $C_{12} = 0$. For given $\gamma > 0$ suppose that there exists a positive definite symmetric matrix P which satisfies

$$PA + A^T P + \frac{1}{\gamma^2} P B_1 B_1^T P + C_{11}^T C_{11} < 0 \quad (15)$$

and the nonlinear weights $a_z(x)$ and $a_u(x)$ satisfy the condition

$$\frac{1}{a_u^2(x)} x^T P B_2(x) B_2^T(x) P x + (1 - a_z^2(x)) x^T C_{11}^T C_{11} x \geq 0. \quad (16)$$

Then the L_2 -induced norm from w to \bar{z} is less than or equal to γ via the feedback control

$$u(t) = -\frac{1}{a_u^2(x)} B_2^T(x) P x(t). \quad (17)$$

This theorem gives a method for choosing the nonlinear weights after P is obtained; however, this means that the nonlinear weights cannot be chosen before obtaining P and the condition (16) restricts the choice of the nonlinear weights. In contrast to this, in our method given below the nonlinear weights can be chosen a priori and the condition which they have to satisfy is just that $a_z^2(x) \in \mathbf{R}[x]$ and $1/a_u^2(x) \in \mathbf{R}[x]$.

A typical choice of the nonlinear weights is as follows:

$$\begin{aligned} a_z(x) &= \sqrt{1 + x^T R_z x}, \\ a_u(x) &= \frac{1}{\sqrt{1 + x^T R_u x}} \end{aligned} \tag{18}$$

where $R_x \succeq 0$ and $R_u \succeq 0$. Fig. 5 shows an example in the case of $x \in \mathbf{R}$. The weight $a_z(x)$ shown in Fig. 5 is utilized for suppressing the effect of w on z_z and $a_u(x)$ is for allowing large input values except at $x = 0$.

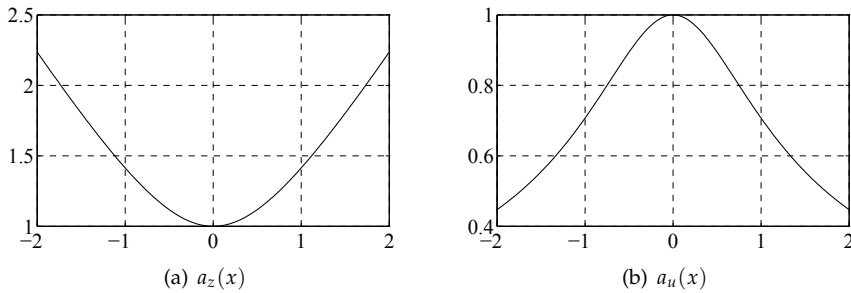


Fig. 5. Example of nonlinear weights

2.2 Derivation of state-dependent inequalities

In the sequel, we assume $D_{12} = I$ for simplicity. Then we have the next theorem.

Theorem 2. *Suppose that for given $\gamma > 0$ there exists a positive definite symmetric matrix P which satisfies the following state-dependent inequality:*

$$\begin{aligned} \phi(x, P) := & x^T \left[-P^{-1}(A + B_2(x)C_{12}) - (A + B_2(x)C_{12})^T P^{-1} \right. \\ & \left. - P^{-1} \left(\frac{1}{\gamma^2} B_1 B_1^T - \frac{1}{a_u^2(x)} B_2(x) B_2^T(x) \right) P^{-1} - a_z^2(x) C_{11}^T C_{11} \right] x > 0, \\ & \forall x (\neq 0) \in \mathbf{R}^n \end{aligned} \tag{19}$$

Then by the feedback

$$u(t) = - \left(\frac{1}{a_u^2(x)} B_2^T(x) P^{-1} + C_{12} \right) x(t) \tag{20}$$

the closed-loop system is asymptotically stable and the L_2 -induced norm (14) is less than or equal to γ , i.e.,

$$\sup_{w \neq 0} \frac{\|z\|_2}{\|w\|_2} \leq \gamma. \tag{21}$$

Proof: First, we will show that the closed-loop system via the feedback (20) with $w(t) = 0$ is stable when (19) holds. To show this we adopt $V(t) = x^T(t)P^{-1}x(t)$ as a Lyapunov function candidate. Then we have

$$\begin{aligned} \frac{d}{dt}V(t) &= \dot{x}^T P^{-1}x + x^T P^{-1}\dot{x} \\ &= (Ax + B_2(x)u)^T P^{-1}x + x^T P^{-1}(Ax + B_2(x)u) \\ &= \left[Ax - B_2(x) \left(\frac{1}{a_u^2(x)} B_2^T(x)P^{-1} + C_{12} \right) x \right]^T P^{-1}x \\ &\quad + x^T P^{-1} \left[Ax - B_2(x) \left(\frac{1}{a_u^2(x)} B_2^T(x)P^{-1} + C_{12} \right) x \right] \quad \text{from (20)} \\ &= x^T \left[P^{-1}(A + B_2(x)C_{12}) + (A + B_2(x)C_{12})^T P^{-1} - \frac{2}{a_u^2} P^{-1}B_2(x)B_2^T(x)P^{-1} \right] x \\ &< -x^T \left[\frac{1}{\gamma^2} P^{-1}B_1B_1^T P^{-1} + a_z^2 C_{11}^T C_{11} + \frac{1}{a_u^2} P^{-1}B_2(x)B_2^T(x)P^{-1} \right] x \quad \text{from (19)} \\ &\leq 0, \quad \text{for } x \neq 0. \end{aligned} \tag{22}$$

This shows the closed-loop system is asymptotically stable.

Next, from (13) we have

$$\gamma^2|w|^2 - |z|^2 = \gamma^2|w|^2 - a_z^2 x^T C_{11}^T C_{11} x - a_u^2 x^T C_{12}^T C_{12} x - a_u^2 u^T u - 2a_u^2 x^T C_{12}^T u \tag{23}$$

and the following identity holds:

$$\begin{aligned} 0 &= 2x^T P^{-1}\dot{x} - 2x^T P^{-1}\dot{x} \\ &= 2x^T P^{-1}\dot{x} - 2x^T P^{-1}Ax - 2x^T P^{-1}B_1w - 2x^T P^{-1}B_2(x)u. \end{aligned} \tag{24}$$

By adding the both-sides of (24) to (23) and completing the square we have

$$\begin{aligned} \gamma^2|w|^2 - |z|^2 &= 2x^T P^{-1}\dot{x} + \gamma^2 w^T w - 2x^T P^{-1}B_1w \\ &\quad - a_u^2 \left[u^T u + 2 \left(\frac{1}{a_u^2} x^T P^{-1}B_2(x) + x^T C_{12}^T \right) u \right] \\ &\quad - x^T \left[2P^{-1}A + a_z^2 C_{11}^T C_{11} + a_u^2 C_{12}^T C_{12} \right] x \\ &= 2x^T P^{-1}\dot{x} + \gamma^2 \tilde{w}^T \tilde{w} - \frac{1}{\gamma^2} x^T P^{-1}B_1B_1^T P^{-1}x \\ &\quad - a_u^2 \tilde{u}^T \tilde{u} + a_u^2 x^T \left(\frac{1}{a_u^2} P^{-1}B_2(x) + C_{12}^T \right) \left(\frac{1}{a_u^2} B_2^T(x)P^{-1} + C_{12} \right) x \\ &\quad - x^T \left[2P^{-1}A + a_z^2 C_{11}^T C_{11} + a_u^2 C_{12}^T C_{12} \right] x \\ &= 2x^T P^{-1}\dot{x} + \gamma^2 \tilde{w}^T \tilde{w} - a_u^2 \tilde{u}^T \tilde{u} + \phi(x, P) \end{aligned} \tag{25}$$

where

$$\begin{aligned}\tilde{w} &= w - \frac{1}{\gamma^2} B_1^T P^{-1} x, \\ \tilde{u} &= u + \left(\frac{1}{a_u^2} B_2^T(x) P^{-1} + C_{12} \right) x.\end{aligned}$$

Then from (19) and (20)

$$\gamma^2 |w|^2 - |z|^2 \geq 2x^T P^{-1} \dot{x} + \gamma^2 \tilde{w}^T \tilde{w},$$

and hence

$$\begin{aligned}\int_0^\tau (\gamma^2 |w|^2 - |z|^2) dt &\geq \int_0^\tau (2x^T P^{-1} \dot{x} + \gamma^2 \tilde{w}^T \tilde{w}) dt \\ &= x^T(\tau) P^{-1} x(\tau) - x^T(0) P^{-1} x(0) + \int_0^\tau (\gamma^2 \tilde{w}^T \tilde{w}) dt.\end{aligned}$$

Here, let $x(0) = 0$ and $\tau \rightarrow \infty$ then

$$\gamma^2 \|w\|^2 - \|z\|^2 = \int_0^\infty (\gamma^2 |w|^2 - |z|^2) dt = \int_0^\infty (\gamma^2 \tilde{w}^T \tilde{w}) dt \geq 0, \tag{26}$$

which leads to (21). Note that to derive (26) we use $\lim_{\tau \rightarrow \infty} x(\tau) = 0$ which holds because the closed-loop is asymptotically stable. Q.E.D.

Note that $\phi(x, p)$ in (19) can be represented as

$$\phi(x, P) = (P^{-1} x)^T M(x, P) (P^{-1} x) \tag{27}$$

where

$$\begin{aligned}M(x, P) &:= - (A + B_2(x) C_{12}) P - P (A + B_2(x) C_{12})^T \\ &\quad - \left(\frac{1}{\gamma^2} B_1 B_1^T - \frac{1}{a_u^2(x)} B_2(x) B_2^T(x) \right) - a_z^2(x) P C_{11}^T C_{11} P.\end{aligned} \tag{28}$$

From this we have the next corollary.

Corollary 1. *Suppose that for given $\gamma > 0$ there exists a positive definite symmetric matrix P which satisfies the following state-dependent inequality:*

$$M(x, P) \succ 0, \quad \forall x \in \mathbf{R}^n \tag{29}$$

Then by the feedback (20) the closed-loop system is asymptotically stable and the L_2 -induced norm (14) is less than or equal to γ .

Proof: It is obvious from (27) that (19) is satisfied if (29) holds. Hence we obtain this corollary from Theorem 2. Q.E.D.

The inequality (29) is called “a state-dependent matrix Riccati inequality” and equivalent to

$$\begin{aligned}
 & h^T [-(A + B_2(x)C_{12})P - P(A + B_2(x)C_{12}) \\
 & \quad - \left(\frac{1}{\gamma^2} B_1 B_1^T - \frac{1}{a_u^2(x)} B_2(x) B_2^T(x) \right) - a_z^2(x) P C_{11}^T C_{11} P] h > 0, \\
 & \forall x \in \mathbf{R}^n, h (\neq 0) \in \mathbf{R}^n.
 \end{aligned}
 \tag{30}$$

Note that h is independent of x in (30), whereas there is a relationship of $h = P^{-1}x$ between h and x in (19). This means that the condition (30) can be very conservative compared with the condition (19). As mentioned in Introduction, most papers have tried to find P which satisfies a matrix state-dependent inequality like (29) (or (30)). In contrary to this, we will try to find P which satisfies (19) (instead of (30)) to reduce the conservativeness.

3. Sum of squares

In this section, we briefly survey the so-called “SOS (Sum of Squares) optimization.”

3.1 Definitions and basic properties

A monomial in $x = [x_1 \cdots x_n]^T$ is represented as $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$. This is also written as $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The degree of a monomial x^α , denoted by $\deg(x^\alpha)$, is defined by $\deg(x^\alpha) := \sum_{i=1}^n \alpha_i$ and the degree of a polynomial $f(x) \in \mathbf{R}[x]$, denoted by $\deg(f)$, is defined by the degree of the monomial which has the highest degree among all the monomials included in $f(x)$. For a polynomial matrix $F(x) \in \mathbf{R}^{q \times r}$ the degree of $F(x)$, denoted by $\deg(F)$, is defined by $\deg(F) := \max_{ij} \deg(F_{ij})$ where F_{ij} denotes the (i, j) element of $F(x)$.

A real polynomial $f(x) \in \mathbf{R}[x]$ is said to be an SOS (Sum of Squares) polynomial if it can be represented as a sum of squares of some polynomials, i.e., there exist some polynomials $g_i(x) \in \mathbf{R}[x]$ ($i = 1, \dots, p$) such that

$$f(x) = \sum_{i=1}^p g_i^2(x).
 \tag{31}$$

Moreover, a polynomial symmetric matrix $F(x) \in \mathbf{R}^{r \times r}[x]$ is said to be an SOS matrix if it can be represented as

$$F(x) = L^T(x)L(x)$$

for some polynomial matrix $L(x)$ of appropriate size. In this paper, we denote the set of SOS polynomials by Σ , and the set of $r \times r$ SOS matrices by $\Sigma^{r \times r}$.

From the definitions it is obvious that

$$f(x) \in \Sigma \Rightarrow f(x) \geq 0 \quad (\forall x \in \mathbf{R}^n),
 \tag{32}$$

$$F(x) \in \Sigma^{r \times r} \Rightarrow F(x) \succeq 0 \quad (\forall x \in \mathbf{R}^n).
 \tag{33}$$

Here, for a positive integer d let $v_d(x)$ be a polynomial vector in x of size $n+d$ defined by

$$v_d(x) := [1 \ x_1 \ \cdots \ x_n \ x_1^2 \ x_1 x_2 \ \cdots \ x_n^2 \ \cdots \ x_1^d \ \cdots \ x_n^d]^T,
 \tag{34}$$

which contains all monomials whose degrees are less than or equal to d where ${}_a C_\beta = a! / (\beta!(a - \beta)!)$ for positive integers a and β with $a \geq \beta$. Then the next lemmas are known.

Lemma 1. (Parrilo (2003)) Let $\deg(f) = 2d$ where $f(x) \in \mathbf{R}[x]$. Then the following (i) and (ii) are equivalent:

(i) $f(x) \in \Sigma$.

(ii) There exists a positive semi-definite symmetric matrix of appropriate size such that $f(x) = v_d^T(x)Qv_d(x)$.

Lemma 2. (Scherer & Hol (2006)) Let $\deg(F) = 2d$ where $F(x) \in \mathbf{R}[x]^{r \times r}$. Then the following (i) and (ii) are equivalent:

(i) $F(x) \in \Sigma^{r \times r}$.

(ii) There exists a positive semi-definite symmetric matrix of appropriate size Q such that $F(x) = (v_d(x) \otimes I_r)^T Q (v_d(x) \otimes I_r)$.

Using these lemmas, the problem of determining whether a polynomial $f(x)$ (or a polynomial matrix $F(x)$) is an SOS polynomial (or an SOS matrix) or not is reduced to an SDP (Semi-Definite Programming) problem, which can be solved numerically, of checking the positive semi-definiteness of the corresponding matrix Q .

3.2 SOS polynomials with equality constraints

Let us consider the following equality constraints:

$$f_j(x) = 0, \quad j = 1, \dots, p \tag{35}$$

where $f_j(x) \in \mathbf{R}[x]$ and their feasible set is defined by

$$S := \{x \in \mathbf{R}^n \mid f_j(x) = 0, j = 1, \dots, p\}. \tag{36}$$

Here we consider the problem of determining whether a given polynomial $f_0(x) \in \mathbf{R}$ is non-negative or not for all $x \in \mathbf{R}$ in the feasible set, i.e., the following condition holds or not:

$$f_0(x) \geq 0, \quad \forall x \in S. \tag{37}$$

For this problem, define a generalized Lagrange function $L(x, \lambda)$ by

$$L(x, \lambda) := f_0(x) - \sum_{j=1}^p \lambda_j(x) f_j(x) \tag{38}$$

where $\lambda_j(x) \in \mathbf{R}[x]$ and let

$$\lambda(x) := [\lambda_1(x) \cdots \lambda_p(x)]^T \in \mathbf{R}[x]^p.$$

Then if for given $\lambda[x] \in \mathbf{R}[x]^p$

$$L(x, \lambda) \geq 0, \quad \forall x \in \mathbf{R}^n \tag{39}$$

holds, the condition (37) is satisfied. In fact, (39) implies

$$f_0(x) \geq \sum_{j=1}^p \lambda_j(x) f_j(x) = 0, \quad \forall x \in S. \tag{40}$$

Moreover, if $L(x, \lambda) \in \Sigma$ then (39) holds from Lemma 1 and hence (37) holds. These facts are summarized in the following lemma.

Lemma 3. *If the following (i) or (ii) holds, the condition (37) holds.*

(i) *There exists $\lambda(x) \in \mathbf{R}[x]^p$ such that $L(x, \lambda) \geq 0 (\forall x \in \mathbf{R}^n)$.*

(ii) *There exists $\lambda(x) \in \mathbf{R}[x]^p$ such that $L(x, \lambda) \in \Sigma$.*

4. Proposed method

Theorem 2 implies that the state feedback (20) will stabilize the closed-loop system and (25) is satisfied if we can obtain a positive definite symmetric matrix P satisfying (19). In this section, we propose an SOS optimization method to find such P .

To this end, let us introduce sufficiently small $\epsilon > 0$ and define

$$\tilde{M}(x, P) := M(x, P) - \epsilon I, \tag{41}$$

$$\tilde{\phi}(x, P) := (P^{-1}x)^T \tilde{M}(x, P) (P^{-1}x). \tag{42}$$

Then it is easy to see

$$\tilde{M}(x, P) \succeq 0, \quad \forall x \in \mathbf{R}^n \quad \Rightarrow \quad M(x, P) \succ 0, \quad \forall x \in \mathbf{R}^n, \tag{43}$$

$$\tilde{\phi}(x, P) \geq 0, \quad \forall x \in \mathbf{R}^n \quad \Rightarrow \quad \phi(x, P) > 0, \quad \forall x(\neq 0) \in \mathbf{R}^n. \tag{44}$$

Hence, for obtaining the feedback (20) it suffices to find $P \succ 0$ such that

$$\tilde{\phi}(x, P) \geq 0, \quad \forall x \in \mathbf{R}^n. \tag{45}$$

From (42), the condition (45) can be written as

$$h^T \tilde{M}(x, P) h \geq 0, \quad \forall (x, h) \in \mathbf{R}^{2n} \quad \text{such that} \quad h = P^{-1}x \tag{46}$$

and moreover this can be written as

$$h^T \tilde{M}(x, P) h \geq 0, \quad \forall (x, h) \in \tilde{S} \tag{47}$$

where

$$\tilde{S} := \{(x, h) \in \mathbf{R}^{2n} \mid x - Ph = 0\}. \tag{48}$$

By this, the condition (45) is represented as the condition (47) including the equality constraint $x - Ph = 0$. For the condition (47) we define a generalized Lagrange function as in Section 3.2 as follows:

$$L(x, h, \lambda; P) := h^T \tilde{M}(x, P)h - \lambda^T(x, h)(x - Ph) \tag{49}$$

where $\lambda(x, h) \in \mathbf{R}[x, h]^n$. Then, from Lemma 3, (47) is satisfied if there exist λ and $P (> 0)$ which satisfies

$$L(x, h, \lambda; P) \geq 0, \quad \forall(x, h) \in \mathbf{R}^{2n} \tag{50}$$

Here, suppose the degree of λ is given, say m , then λ can be written as

$$\lambda(x, h) = H v_m(x, h)$$

where $v_m(x, h)$ is a vector of size ${}_{2n+m}C_m$ which contains all monomials in x and h whose degrees are less than or equal to m , and H is an $n \times ({}_{2n+m}C_m)$ real matrix. From this, (50) is reduced to

$$L_m(x, h; H, P) := h^T \tilde{M}(x, P)h - v_m^T(x, h)H^T(x - Ph) \geq 0, \quad \forall(x, h) \in \mathbf{R}^{2n}, \tag{51}$$

and the problem to be concerned becomes the one for finding matrices P and H which satisfies (51).

Note that $L_m(x, h; H, P)$ includes the product of H and P in the last term. Hence, we consider an iterative algorithm which repeats a step of finding H for fixed P and a step of finding P for fixed H .

First, suppose P is fixed. In this case, $L_m(x, h; H, P)$ can be written as

$$L_m(x, h; H, P) = v_{d_1}^T(x, h)Q_1(H)v_{d_1}(x, h) \tag{52}$$

where $d_1 = \text{deg}(L_m)$, the degree of L_m as a polynomial in x and h , $v_{d_1}(x, h)$ is a vector of size ${}_{2n+d_1}C_{d_1}$ which contains all monomials in x and h whose degrees are less than or equal to d_1 , and $Q_1(H)$ is a $({}_{2n+d_1}C_{d_1}) \times ({}_{2n+d_1}C_{d_1})$ symmetric matrix. Then $Q_1(H)$ is affine in H because so is L_m . Hence, in the case of fixed P , the problem to be concerned is reduced to an SDP problem to find H such that $Q_1(H) \succeq 0$, because (51) is satisfied if $L_m \in \Sigma_{(x, h)}$ which is equivalent to the existence of $Q_1(H) \succeq 0$ by Lemma 1.

Next, suppose H is fixed. In this case, L_m is not affine in P , but by Schur complement (51) is equivalent to

$$G(x, h; H, P) := \begin{bmatrix} G_{11}(x, h; H, P) & a_z^2(x)h^T P C_{11}^T \\ a_z^2(x)C_{11} P h & a_z^2(x)I_{n_z} \end{bmatrix} \succeq 0, \quad \forall(x, h) \in \mathbf{R}^{2n}, \tag{53}$$

which is affine in P , where

$$G_{11}(x, h; H, P) := h^T \left[-(A + B_2(x)C_{12})P - P(A + B_2(x)C_{12})^T - \bar{\gamma}B_1B_1^T + \bar{a}_u^2(x)B_2(x)B_2^T(x) \right] h - v_m^T(x, h)H^T(x - Ph) \tag{54}$$

and

$$\bar{\gamma} := \frac{1}{\gamma^2}, \quad \bar{a}_u(x) := \frac{1}{a_u(x)}. \tag{55}$$

Since G is affine in P , it can be written as

$$G(x, h; H, P) = (v_{d_2}(x, h) \otimes I_{n_z+1})^T Q_2(P) (v_{d_2}(x, h) \otimes I_{n_z+1}) \tag{56}$$

where $d_2 = \text{deg}(G)$, the degree of G as a polynomial matrix in x and h , $v_{d_2}(x, h)$ is a vector of size $2n+d_2 C_{d_2}$ which contains all monomials in x and h whose degrees are less than or equal to d_2 , and $Q_2(P)$ is a real symmetric matrix of appropriate size. Hence, in the case of fixed H , the problem to be concerned is reduced to an SDP problem to find P such that $Q_2(P) \succeq 0$, because (51) is satisfied if $G \in \Sigma_{(x,h)}^{(n_z+1) \times (n_z+1)}$ which is equivalent to the existence of $Q_2(H) \succeq 0$ by Lemma 2.

Note that L_m and G are also affine in $\bar{\gamma}$ and hence we can consider to maximize $\bar{\gamma} = 1/\gamma^2$ (i.e., minimize γ).

Now, let us summarize our method as an algorithm.

Algorithm 1. Step. 0 Choose an initial value $P_0 \succ 0$ and small ϵ . Let $k := 0$ and $\bar{\gamma}_0 = 0$.

Step 1 Let $P = P_k$ and get the optimal value $\bar{\gamma}^*$ and its optimizer H^* by solving numerically the next SDP problem

$$\max_{\bar{\gamma}, H} \bar{\gamma} \quad \text{s.t.} \quad Q_1(H) \succeq 0, \tag{57}$$

and let $H_k := H^*$.

Step 2 Let $H = H_k$ and get the optimal value $\bar{\gamma}^*$ and its optimizer $P^* \succ 0$ by solving numerically the next SDP problem

$$\max_{\bar{\gamma}, P \succ 0} \bar{\gamma} \quad \text{s.t.} \quad Q_2(P) \succeq 0, \tag{58}$$

and let $\bar{\gamma}_{k+1} := \bar{\gamma}^*$ and $P_{k+1} = P^*$.

Step 3 If $|\bar{\gamma}_{k+1} - \bar{\gamma}_k|$ is sufficiently small (i.e., $\bar{\gamma}_k$ is convergent), then return P_{k+1} as a solution and exit; otherwise, let $k := k + 1$ and go to Step 1.

Note that a feasible solution to $\bar{M}(0, P) \succeq 0$ for large γ can be used as an initial value P_0 , because $\bar{M}(0, P) \succeq 0$ is a usual Riccati inequality and has a feasible solution for large γ .

5. Numerical example

In this section, we give a numerical example. The bilinear system to be concerned is the semi-active suspension system for automobiles introduced by Ohsaku et al. (1998); Sampei et al. (1999).

The motion equation of the suspension system is given by

$$M_b \ddot{x}_b = C_s(\dot{x}_w - \dot{x}_b) + C_v(\dot{x}_w - \dot{x}_b) + K_s(x_w - x_b) \tag{59}$$

$$M_w \ddot{x}_w = -C_s(\dot{x}_w - \dot{x}_b) - C_v(\dot{x}_w - \dot{x}_b) - K_s(x_w - x_b) + K_t(x_r - x_w) \tag{60}$$

where

x_b is the displacement of the car body,

x_w is the displacement of the car wheel,

x_r is the displacement of the road,

M_b is the mass of the car body,
 M_w is the mass of the car wheel,
 K_s is the spring constant of the suspension,
 K_t is the elastic coefficient of the tire,
 C_s is the fixed damping coefficient of the suspension,
 C_v is the variable damping coefficient of the suspension,
 and C_v is the input and \dot{x}_r is the disturbance.

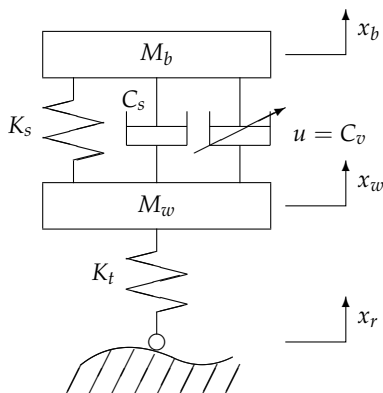


Fig. 6. Semi-active suspension system

Then the state-space representation of the generalized system with $W_z = I$, $W_u = 1$ and nonlinear weights $a_z(x)$, $a_u(x)$ is given by

$$\dot{x}(t) = Ax(t) + B_1w(t) + B_2(x)u(t) \tag{61}$$

$$\bar{z}(t) = a_z(x)C_{11}x(t) + a_u(x)u(t) \tag{62}$$

where

$$x(t) = \begin{bmatrix} x_w(t) - x_b(t) \\ \dot{x}_b(t) \\ x_r(t) - x_w(t) \\ \dot{x}_w(t) \end{bmatrix}, u(t) = C_v(t), w(t) = \dot{x}_r(t)$$

$$A = \begin{bmatrix} 0 & -1 & 0 & 1 \\ \frac{K_s}{M_b} & -\frac{C_s}{M_b} & 0 & \frac{C_s}{M_b} \\ 0 & 0 & 0 & -1 \\ -\frac{K_s}{M_w} & \frac{C_s}{M_w} & \frac{K_t}{M_w} & -\frac{C_s}{M_w} \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, B_2(x) = \begin{bmatrix} 0 \\ \frac{\dot{x}_w - \dot{x}_b}{M_b} \\ 0 \\ -\frac{\dot{x}_w - \dot{x}_b}{M_w} \end{bmatrix},$$

$$C_{11} = [0 \ 1 \ 0 \ 0]$$

and the nonlinear weights are given by

$$\begin{aligned} a_z(x) &= \sqrt{1 + x^T R_z x}, \\ a_u(x) &= \frac{1}{\sqrt{1 + x^T R_u x}}, \\ R_z &= m_z I, \\ R_u &= m_u I, \end{aligned} \quad (63)$$

with some positive numbers m_z, m_u . Note that this nonlinear weights do not satisfy the condition (16) in Theorem 1 in general. Hence, the method by Ohsaku et al. (1998); Shimizu et al. (1997) cannot be applied to this example with this weights.

The objective of the robust control to be concerned is to minimize the effect of the disturbance (the road roughness) on the velocity of the car body, which is formulated as the problem of minimizing the following L_2 -induced norm:

$$\sup_{w \neq 0} \frac{\|\bar{z}\|_2}{\|w\|_2}.$$

Fig. 7 shows the disturbance $w(t) = \dot{x}_r(t)$ and Fig. 8 shows the simulation results where the dashed red line shows the velocity of the car body without feedback control and the solid blue line shows the one with feedback control designed by our method. It can be seen that the amplitude of the body velocity by our method is suppressed compared with that without feedback control. This means that the effect of the disturbance on the body velocity is reduced by our method, which shows the efficiency of our method.

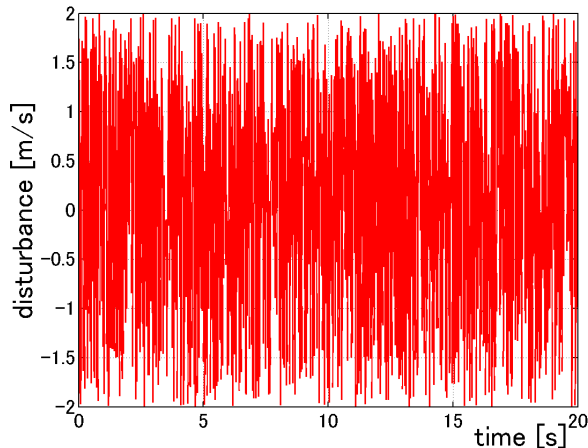


Fig. 7. Disturbance from the road surface

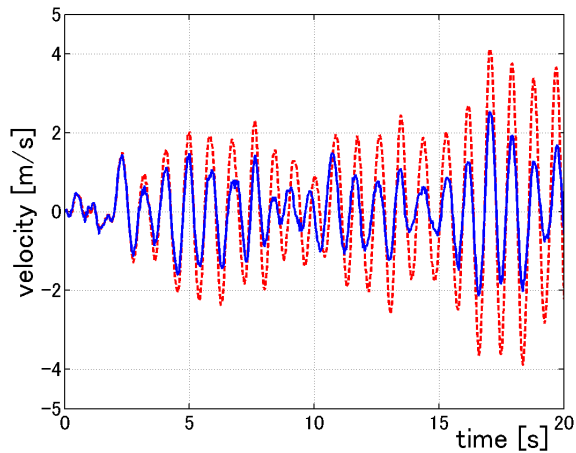


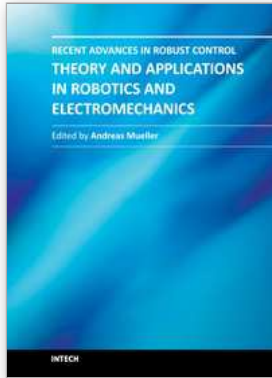
Fig. 8. Velocities of the car body (dashed red: open-loop, solid blue: by our method)

6. Conclusions

In this paper, first we have derived an inequality condition of quadratic form for the robust control problem of bilinear systems with nonlinear weights, and then proposed an iterative method for finding a solution to the inequality condition. Finally, we have given a numerical example to show the effectiveness of our method.

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