

# Electrodynamics of Multiconductor Transmission-line Theory with Antenna Mode

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## 1. Introduction

In the modern life we depend completely on the electricity as the most useful form of energy. The technology on the use of electricity has been developed in all directions and also in very sophisticated manner. All the electric devices have to use electric power (energy) and they use both direct current (DC) and alternating current (AC). Today a powerful technology of manipulation of frequency and power becomes available due to the development of chopping devices as IGBT and other methods. This technology of manipulating electric current and voltage, however, unavoidably produces electromagnetic noise with high frequency. We are now filled with electromagnetic noise in our circumstance.

This situation seems to be caused by the fact that we do not have a theory to describe the electromagnetic noise and to take into account the effect of the circumstance in the design of electric circuit. We have worked out such a theory in one of our papers as "Three-conductor transmission-line theory and origin of electromagnetic radiation and noise" (Toki & Sato (2009)). In addition to the standard two-conductor transmission-line system, we ought to introduce one more transmission object to treat the circumstance. As the most simple object, we introduce one more line to take care of the effect of the circumstance. This third transmission-line is the place where the electromagnetic noise (electromagnetic wave) goes through and influences the performance of the two major transmission-lines. If we are able to work out the three-conductor transmission-line theory by taking care of unwanted electromagnetic wave going through the third line, we understand how we produce and receive electromagnetic noise and how to avoid its influence.

To this end, we had to introduce the coefficient of potential instead of the coefficient of capacity, which is used in all the standard multi-conductor transmission line theories (Paul (2008)). We are then able to introduce the normal mode voltage and current, which are usually considered in ordinary calculations, and at the same time the common mode voltage and current, which are not considered at all so far and are the sources of the electromagnetic noise (Sato & Toki (2007)). We are then able to provide the fundamental coupled differential equations for the TEM mode of the three-conductor transmission-line theory and solve the coupled equations analytically. As the most important consequence we obtain that the main two transmission-lines should have the same qualities and same geometrical shapes and their distances to the third line should be the same in order to decouple the normal mode from the common mode. The symmetrization is the key word to minimize the influence of the circumstance and hence the electromagnetic noise to the electric circuit. The symmetrization makes the normal mode decouple from the common mode and hence we are able to avoid the

influence of the common mode noise in the use of the normal mode (Toki & Sato (2009)). The symmetrization has been carried out at HIMAC (Heavy Ion Medical Accelerator in Chiba) (Kumada (1994)) one and half decade ago and at Main Ring of J-PARC recently (Kobayashi (2009)). Both synchrotrons are working well at very low noise level.

As the next step, we went on to develop a theory to couple the electric circuit theory with the antenna theory (Toki & Sato (2011)). This work is motivated by the fact that when the electromagnetic noise is present in an electric circuit, we observe electromagnetic radiation in the circumstance. In order to complete the noise problem we ought to couple the performance of electric circuit with the emission and absorption of electromagnetic radiation in the circuit. To this end, we introduce the Ohm's law as one of the properties of the charge and current under the influence of the electromagnetic fields outside of a thick wire. As a consequence of the new multi-conductor transmission-line theory with the antenna mode, we again find that the symmetrization is the key technology to decouple the performance of the normal mode from the common and antenna modes (Toki & Sato (2011)).

The Ohm's law is considered as the terminal solution of the equation of motion of massive amount of electrons in a transmission-line of a thick wire with resistance, where the collisions of electrons with other electrons and nuclei take place. This consideration is able to put the electrodynamics of electromagnetic fields and dynamics of electrons in the field theory. We are also able to discuss the skin effect of the TEM mode in transmission-lines on the same footing. In this paper, we would like to formulate the multi-conductor transmission-line theory on the basis of electrodynamics, which includes naturally the Maxwell equations and the Lorentz force.

This paper is arranged as follows. In Sect.2, we introduce the field theory on electrodynamics and derive the Maxwell equation and the Lorentz force. In Sect.3, we develop the multiconductor transmission-line (MTL) equations for the TEM mode. We naturally include the antenna mode by taking the retardation potentials. In Sect.4, we provide a solution of one antenna system for emission and absorption of radiation. In Sect.5, we discuss a three-conductor transmission-line system and show the symmetrization for the decoupling of the normal mode from the common and antenna modes. In Sect.6, we introduce a recommended electric circuit with symmetric arrangement of power supply and electric load for good performance of the electric circuit. Sect.7 is devoted to the conclusion of the present study.

## 2. Electrodynamics

We would like to work out the multiconductor transmission-line (MTL) equation with electromagnetic emission and absorption. To this end, we should work out fundamental equations for a multiconductor transmission-line system by using the Maxwell equation and the properties of transmission-lines. We shall work out electromagnetic fields outside of multi-conductor transmission-lines produced by the charges and currents in the transmission-lines. In this way, we are able to describe electromagnetic fields far outside of the transmission-line system so that we can include the emission and absorption of electromagnetic wave. For this purpose, we take the electrodynamics field theory, since a multiconductor transmission-line system is a coupled system of charged particles and electromagnetic fields. In this way, we are motivated to treat the scalar potential in the same way as the vector potential and find it natural to use the coefficients of potential instead of the coefficients of capacity as the case of the coefficients of inductance.

We discuss here the dynamics of charged particles with electromagnetic fields in terms of the modern electrodynamics field theory. For those who are not familiar to this theory, you can skip this paragraph and start with the equations (6) and (7). In the electrodynamics, we

have the gauge theory Lagrangian, where the interaction of charge and current of a Fermion (electron) field  $\psi$  with the electromagnetic field  $A_\mu$  is determined by the following Lagrangian,

$$L = \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \bar{\psi}(i\gamma_\mu D^\mu - m)\psi. \quad (1)$$

with  $D^\mu = \partial^\mu - ieA^\mu$ , where  $A^\mu$  is the electromagnetic potential. Here,  $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$  is the anti-symmetric tensor with the four-derivative defined as  $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{\partial}{c\partial t}, \nabla)$  and the four-coordinate as  $x^\mu = (ct, \mathbf{x})$ . Here, electrons are expressed by the Dirac field  $\psi$ , which possesses spin as the source of the permanent magnet and therefore we do not have to introduce the notion of the perfect conductor anymore (Maxwell (1876)). The vector current is written by using the charged field as  $j^\mu = \bar{\psi}\gamma^\mu\psi$ . The variation of the above Lagrangian with respect to  $A^\mu$  provides the Maxwell equation with a source term expressed in the covariant form (Maxwell (1876)).

$$\partial^\mu F_{\mu\nu}(x) = ej_\nu(x) \quad (2)$$

They are Maxwell equations, which become clear by writing explicitly the anti-symmetric tensor in terms of the electric field  $\mathbf{E}$  and magnetic field  $\mathbf{B}$ .

$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ -\frac{1}{c}E_x & 0 & -B_z & B_y \\ -\frac{1}{c}E_y & B_z & 0 & -B_x \\ -\frac{1}{c}E_z & -B_y & B_x & 0 \end{pmatrix} \quad (3)$$

Here,  $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$  and  $\mathbf{B} = \nabla \times \mathbf{A}$ . The two more equations are explicitly written as  $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon}q$  and  $\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu \mathbf{j}$  by using the above equation of motion (2).

It is convenient to write the Maxwell equation in the covariant form for the symmetry of the relevant quantities without worrying about the factors as  $c$ ,  $\mu$  and  $\epsilon$ . The four-vector potential is written by the scalar and vector potentials as  $A^\mu(x) = (V(x)/c, \mathbf{A}(x))$  and the four-current, which is a source term of the potentials, is given as  $ej^\mu = \mu(cq, \mathbf{j})$ . Here, the charge  $q$  and current  $\mathbf{j}$  are both charge and current densities. The contra-variant four vector  $x^\mu$  is related with the co-variant four vector  $x_\mu$  as  $x^\mu = g^{\mu\nu}x_\nu$ . Here, the metric is  $g^{\mu\nu} = 1$  for  $\mu = \nu = 0$  and  $g^{\mu\nu} = -1$  for  $\mu = \nu = 1, 2, 3$  and zero otherwise (Bjorken (1970)). The Maxwell equation (2) gives the following differential equation (Maxwell (1876)).

$$\partial^\mu \partial_\mu A_\nu(x) - \partial^\mu \partial_\nu A_\mu(x) = ej_\nu(x) \quad (4)$$

In order to simplify the differential equation and also to keep the symmetry among the scalar and vector potentials, we take the Lorenz gauge  $\partial^\mu A_\mu(x) = 0$  (Lorenz (1867); Jackson (1998)). In this case, we get a simple covariant equation for the potential with the source current.

$$\partial^\mu \partial_\mu A_\nu(x) = ej_\nu(x) \quad (5)$$

This expression based on the field theory shows the fact that the dynamics of the four-vector potential  $A_\nu$  is purely given by the corresponding source current  $j_\nu$ . This fact should be contrasted with the standard notion that the time-dependent electric and magnetic fields are the sources from each other through the Ampere-Maxwell's law and the Faraday's law in the Maxwell equation. When there is no source term  $j_\nu = 0$  in the space outside of the conductors, the four-vector potential satisfies the wave equation with the light velocity. In

the electrodynamics, the propagation of electromagnetic wave with the velocity of light is the property of a vector particle with zero mass.

We express now the four-vectors in the standard three-vector form. The scalar potential  $V(\mathbf{x}, t)$  and the vector potential  $\mathbf{A}(\mathbf{x}, t)$  should satisfy the following equations with sources in the Lorenz gauge.

$$\left( \frac{\partial^2}{c^2 \partial t^2} - \nabla^2 \right) V(\mathbf{x}, t) = \frac{1}{\epsilon} q(\mathbf{x}, t) \quad (6)$$

$$\left( \frac{\partial^2}{c^2 \partial t^2} - \nabla^2 \right) \mathbf{A}(\mathbf{x}, t) = \mu \mathbf{j}(\mathbf{x}, t) \quad (7)$$

These two second-order differential equations (6) and (7) clearly show that the charge and current are the sources of electromagnetic fields. For the propagation of electromagnetic power through a MTL system, we are interested in the electromagnetic fields outside of thick electric wires with resistance. In this case, we are able to solve the differential equations by using retardation charge and current (Lorenz (1867); Rieman (1867); Jackson (1998)).

$$V(\mathbf{x}, t) = \frac{1}{4\pi\epsilon} \int d\mathbf{x}' \frac{q(\mathbf{x}', t - \frac{|\mathbf{x} - \mathbf{x}'|}{c})}{|\mathbf{x} - \mathbf{x}'|} \quad (8)$$

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu}{4\pi} \int d\mathbf{x}' \frac{\mathbf{j}(\mathbf{x}', t - \frac{|\mathbf{x} - \mathbf{x}'|}{c})}{|\mathbf{x} - \mathbf{x}'|} \quad (9)$$

These expressions are valid for the scalar and vector potentials outside of the transmission-lines. The presence of the retardation effect in the time coordinate in the integrand is important for the production of electromagnetic radiation. The retardation terms generate a finite Poynting vector going out of a surface surrounding the MTL system not only at a far distance but also at a boundary.

This part is related with the derivation of the Lorentz force from the field theory. You may skip this part and directly move to the next section. It is important to derive the current conservation equation of the field theory, which is related with the behavior of charged particles. The current conservation is derived by writing an equation of motion for  $\psi$  using the above Lagrangian as

$$(i\gamma_\mu \partial^\mu + e\gamma_\mu A^\mu - m)\psi(x) = 0. \quad (10)$$

Using this Dirac equation together with the complex-conjugate Dirac equation, we obtain

$$\partial_\mu j^\mu(x) = 0, \quad (11)$$

which is the charge conservation law of the field theory. The electromagnetic potential for a charged particle is given from the above equation as  $e j_\mu A^\mu$ . From this expression, we are able to derive an electromagnetic force exerted on a charged particle. To write it explicitly, we ought to use a Lagrangian of a point particle with the electromagnetic potential  $e j_\mu A^\mu$ , where  $j^\mu = (c, \mathbf{v})$ .

$$L = \frac{1}{2} m \left( \frac{d\mathbf{x}}{dt} \right)^2 - eV(x) + e\mathbf{v} \cdot \mathbf{A}(x) \quad (12)$$

We use the Euler equation  $-\frac{\partial L}{\partial \mathbf{x}} + \frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}} = 0$ , we get

$$m \frac{d^2 \mathbf{x}}{dt^2} + e \nabla V(x) - e \nabla (\mathbf{A}(x) \cdot \mathbf{v}) + e \frac{d\mathbf{A}(x)}{dt} = 0. \quad (13)$$

Here,  $\mathbf{v} = \frac{d\mathbf{x}}{dt}$  is used. We have the relations

$$\frac{d\mathbf{A}(x)}{dt} = \frac{\partial \mathbf{A}(x)}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{A}(x), \quad (14)$$

and

$$\mathbf{B} \times \mathbf{v} = (\nabla \times \mathbf{A}) \times \mathbf{v} = (\mathbf{v} \cdot \nabla) \mathbf{A}(x) - \nabla (\mathbf{A}(x) \cdot \mathbf{v}). \quad (15)$$

Hence, the Lorentz force is written as

$$\mathbf{F}_L = e \mathbf{E}(x) + e \mathbf{v} \times \mathbf{B}(x). \quad (16)$$

with  $\mathbf{E}(x) = -\nabla V(x) - \frac{\partial \mathbf{A}(x)}{\partial t}$ . Charged particles are influenced by the electromagnetic field through the Lorentz force given above. In the present discussion, we use the phenomenological relation in terms of the Ohm's law for the relation of the current with the electromagnetic field. Because the total energy should be conserved, the summation of electromagnetic power of circuit, energy of emission and absorption of electromagnetic wave, and Joule's heat energy is kept constant in a multiconductor transmission-line system.

### 3. Multiconductor transmission-line theory with radiation

We start with the properties of transmission-lines, where the charge and current are present and they oscillate in space and time for the propagation of electromagnetic energy through the transmission-lines. We introduce  $N$  parallel lines numbered by  $i$  ( $= 1, \dots, N$ ) and its direction  $x$  with a round cross section of a thick wire with resistance. First of all, we have the charge conservation equation (11) of the field theory, which indicates the conservation of charge  $\partial q / \partial t + \nabla \cdot \mathbf{j} = 0$  and the continuity equation of the standard electromagnetism. We introduce  $i$ -th current and  $i$ -th charge by integrating  $\mathbf{j}$  and  $q$  over the cross section of each transmission-line at a space-time position  $x, t$  taking into account the skin effect in the transmission-line,  $I_i(x, t) = \int ds j_i^x(x, y, z, t)$  and  $Q_i(x, t) = \int ds q_i(x, y, z, t)$ , where  $ds = dydz$ .

$$\frac{\partial I_i(x, t)}{\partial x} = - \frac{\partial Q_i(x, t)}{\partial t} \quad (17)$$

Here,  $I_i(x, t)$  and  $Q_i(x, t)$  denote the conduction current and true electric charge of the  $i$ -th transmission-line at a position  $x$  and a time  $t$ . The subscript  $i$  indicates the charge and current of the  $i$ -th transmission-line. This equation indicates that a current goes through a transmission line while satisfying the continuity equation. Hence, the next natural equation for a transmission-line is the Ohm's law for a current due to an electric field. The Ohm's law relates the electric field  $E_i^x(x, t)$  at the inner surface of the resistive conductor in the direction of the current through the resistance  $R_i$  with the current  $I_i(x, t)$ .

$$R_i I_i(x, t) = E_i^x(x, t) \quad (18)$$

Here, the superscript  $x$  denotes the  $x$  component of the electric field of the  $i$ -th transmission-line. We note that the resistance  $R_i$  should depend on the wave-length of the

electromagnetic wave going through each transmission-line due to the skin effect (Takeyama (1983)). With a finite  $E^{\parallel}$  at the surfaces of transmission-lines together with  $B^{\theta}$  perpendicular to both  $E^{\perp}$  and  $E^{\parallel}$ , we have an electromagnetic wave in far distance. The boundary condition in the direction of the current even for the resistive conductor is identical to that for the perfect conductor so that  $E^{\parallel}$  is equivalent to  $E_i^x(x, t)$  as

$$E_i^{\parallel}(x, t) = E_i^x(x, t) . \quad (19)$$

The electric field  $E_i^{\parallel}(x, t)$  is expressed in terms of the scalar potential  $V_i(x, t)$  and the vector potential  $A_i(x, t)$  in the direction of the current  $I_i(x, t)$ .

$$E_i^{\parallel}(x, t) = -\frac{\partial V_i(x, t)}{\partial x} - \frac{\partial A_i(x, t)}{\partial t} \quad (20)$$

Hence, from Eqs. (18), (19) and (20) we have the following relation.

$$-\frac{\partial V_i(x, t)}{\partial x} - \frac{\partial A_i(x, t)}{\partial t} = R_i I_i(x, t) \quad (21)$$

It is very interesting to point out that this equation with  $R_i = 0$  corresponds to the expression of the electromagnetic potentials at the surface of the transmission-line for the TEM mode, which is worked out for the transverse electric and magnetic fields around the  $i$ -th conductor-line (Toki & Sato (2009); Paul (2008)). In this sense, we want to note again that the scalar and vector potentials here are those at the surface of the  $i$ -th conductor-line so that the TEM mode fields are obtained by using the Maxwell equation at the boundary and the outside of the conductor-line. The TEM mode fields are produced by the current and the charge in thick wires and the Ohm's law provides the effect of the TEM mode fields on these currents. We ought to solve the resulting coupled equations for the propagation of the TEM mode through a multiconductor transmission line system.

We consider now a MTL system consisting of many nearby parallel lines with circular cross sections numbered by  $i = 1, \dots, N$ . We relate then the scalar and vector potentials at the surface of each line with charges and currents in all the lines. The charges and currents are present in the transmission-lines and we express them as  $Q_i(x, t)$  and  $I_i(x, t)$ . The relations of the charge and current with the scalar and vector potentials have been worked out above as the properties of each transmission-line. We take the direction of the current in the  $x$  direction and the integral over  $x'$  is replaced by summation over parallel lines over  $j$  and integral in the direction  $x'$  of the parallel lines. Because the distance  $|x - x'|$  is given as  $((x - x')^2 + d_{ij}^2)^{1/2}$  where  $d_{ij}$  is a distance between two parallel  $ij$  lines, we can write the scalar and vector potentials at the surface of the  $i$ -th line as

$$V_i(x, t) = \frac{1}{4\pi\epsilon} \sum_{j=1}^N \int_0^l dx' \frac{Q_j(x', t - \sqrt{(x - x')^2 + d_{ij}^2}/c)}{\sqrt{(x - x')^2 + d_{ij}^2}} , \quad (22)$$

$$A_i(x, t) = \frac{\mu}{4\pi} \sum_{j=1}^N \int_0^l dx' \frac{I_j(x', t - \sqrt{(x - x')^2 + d_{ij}^2}/c)}{\sqrt{(x - x')^2 + d_{ij}^2}} . \quad (23)$$

The denominators of the above two equations indicate the distance of two points in two lines denoted by  $ij$ . For the diagonal case  $i = j$ ,  $d_{ij} = 0$  and as will be discussed the finite size effect

of each transmission-line is to be taken care by using the geometrical mean distance (GMD) in the same manner as the Neumann's formula (Takeyama (1983)). We also mention here that we define the scalar and vector potentials at the surface of each transmission-line. Hence, we consider that  $d_{ij}$  is of the order of the radius of each thick wire. We assume that the length of the wire  $l$  is much larger than the radius of each wire and the distance between two lines,  $l \gg d_{ij}$ . Hence, we consider the case where all the transmission-lines are packed together. These four equations; the continuity equation (17), the combined equation (21) of the Ohm's law (18) and the boundary condition (19), the scalar potential (22) and the vector potential (23), are the fundamental equations of the MTL system. We are able to know the performance of a MTL system by solving these four equations, which are now coupled integro-differential equations. Here, it is important to comment that the expressions for the scalar potential (22) and the vector potential (23) provide the electromagnetic fields outside of the wires and even at far distance if we introduce other coordinates  $y, z$  in addition to  $x$  to express the entire space. We further comment that the electromotive force (EMF) method for the input impedance of an antenna uses back the entire radiation energy to calculate the electromagnetic field at the surface of a wire (Stratton (1941)). Hence, these four equations are able to provide the behavior of electromagnetic wave even far outside of the MTL system. Therefore, when we solve these four equations we know not only the behavior of the MTL system but also the electromagnetic fields in the entire space outside of the thick wires. We are then able to include naturally emission and absorption of the EM waves. We comment here that the retardation charge and current in Eqs. (22) and (23) are responsible for a Poynting vector going out at far distance. We treat the retardation effect in the integral by considering that the coupled differential equations are linear and all the quantities have the time dependence as  $Q_i(x, t) = Q_i(x)e^{-j\omega t}$  and  $I_i(x, t) = I_i(x)e^{-j\omega t}$ . Inserting these expressions to the above equations, we get

$$V_i(x, t) = \frac{1}{4\pi\epsilon} \sum_{j=1}^N \int_0^l dx' \frac{Q_j(x', t) e^{j\omega \sqrt{(x-x')^2 + d_{ij}^2}/c}}{\sqrt{(x-x')^2 + d_{ij}^2}}, \quad (24)$$

$$A_i(x, t) = \frac{\mu}{4\pi} \sum_{j=1}^N \int_0^l dx' \frac{I_j(x', t) e^{j\omega \sqrt{(x-x')^2 + d_{ij}^2}/c}}{\sqrt{(x-x')^2 + d_{ij}^2}}. \quad (25)$$

These expressions for the scalar and vector potentials provide the right behaviors of electromagnetic fields far outside of the MTL system. Hence, these relations together with the continuity equation and the combined relation of the Ohm's law and the boundary condition provide a proper set of equations of electromagnetic waves with radiation. Since these integro-differential coupled equations are difficult to handle, we shall find an appropriate approximation.

In order to find out an appropriate approximation at a boundary of a thick wire, we study the property of the integrand with the retardation terms. The function  $1/\sqrt{(x-x')^2 + d_{ij}^2}$  has a strong peak at  $x' = x$  and drops rapidly as  $x'$  deviates from  $x$ . Furthermore, the real part of the factor  $e^{j\omega \sqrt{(x-x')^2 + d_{ij}^2}/c}$  behaves as  $\cos(\omega \sqrt{(x-x')^2 + d_{ij}^2}/c)$  and provides a further cutoff with  $|x-x'|$ . Hence, the integral has a dominant contribution in the narrow region close to the position  $x' = x$ . Hence, it is a good approximation to pull out the charge  $Q_i$  and current  $I_i$  from the integral by taking their arguments at  $x$ . This fact indicates that the electric field has the perpendicular component to the transmission-line and the magnetic field has the axial component produced by the current at the same coordinate. Hence, the

TEM mode propagates through the transmission-lines. We shall call this as the TEM mode approximation. It should be noted here as mentioned before that the real part of the scalar and vector potentials could satisfy the boundary condition of  $E^\perp$  for the resistive conductor due to the TEM mode approximation. The imaginary part behaves as  $\sin(\omega\sqrt{(x-x')^2+d_{ij}^2}/c)$  and together with the denominator, the integrand is the zero-th order spherical Bessel function  $j_0(\omega\sqrt{(x-x')^2+d_{ij}^2}/c)$  with some factor and drops rapidly with  $|x-x'|$  and oscillates at large  $\omega$ . We may take the TEM mode approximation for the imaginary part as well at large  $\omega$ , but it seems better to keep the charge and current in the integral. This is particularly the case when the angular velocity  $\omega$  is small. Hence, we write the scalar and vector potentials in the TEM mode approximation for the real part and keep the integral form for the imaginary part.

$$V_i(x, t) = \frac{1}{4\pi\epsilon} \sum_{j=1}^N \int_0^l dx' \frac{\cos(\omega\sqrt{(x-x')^2+d_{ij}^2}/c)}{\sqrt{(x-x')^2+d_{ij}^2}} Q_j(x, t) \quad (26)$$

$$+ j \frac{1}{4\pi\epsilon} \int_0^l dx' \frac{Q_t(x', t) \sin(\omega|x-x'|/c)}{|x-x'|},$$

$$A_i(x, t) = \frac{\mu}{4\pi} \sum_{j=1}^N \int_0^l dx' \frac{\cos(\omega\sqrt{(x-x')^2+d_{ij}^2}/c)}{\sqrt{(x-x')^2+d_{ij}^2}} I_j(x, t) \quad (27)$$

$$+ j \frac{\mu}{4\pi} \int_0^l dx' \frac{I_t(x', t) \sin(\omega|x-x'|/c)}{|x-x'|}.$$

It is very important to notice that the  $d_{ij}$  dependence is negligibly small when  $d_{ij} \ll \frac{c}{\omega}$  and we drop the  $ij$  dependence in the imaginary part. Hence, we can sum up over the wire number and write the total charge and current as  $Q_t(x', t) = \sum_j^N Q_i(x', t)$  and  $I_t(x', t) = \sum_j^N I_i(x', t)$ .

We write therefore the above relations as

$$V_i(x, t) = \sum_j P_{ij}(\omega) Q_j(x, t) + j M_e Q_t^I(l, x, t), \quad (28)$$

$$A_i(x, t) = \sum_j L_{ij}(\omega) I_j(x, t) + j M_m I_t^I(l, x, t).$$

Here, we have defined the integrated charge and current as

$$Q_t^I(l, x, t) = \int_0^l dx' \frac{Q_t(x', t) \sin(\omega|x-x'|/c)}{|x-x'|} \quad (29)$$

$$I_t^I(l, x, t) = \int_0^l dx' \frac{I_t(x', t) \sin(\omega|x-x'|/c)}{|x-x'|}.$$

with the coefficients defined as  $M_e = \frac{1}{4\pi\epsilon}$  and  $M_m = \frac{\mu}{4\pi}$ . It is very important to note that the integrals of the charge and current over the wire length generate the parallel component of the electric field at the wire surface. This is important for the radiation of the EM wave in the far distance.



We shall calculate now these coefficients  $P_{ij}$  and  $L_{ij}$  by using the Neumann's formula (Takeyama (1983)). To this end, we have to take into account the finite size effect of each transmission-line and also the skin effect. We shall study the finite size effect including the skin effect in a future publication (Sato & Toki (2011)). We first write the well known coefficient of inductance  $L_{ij}$  given by the Neumann's formula (Takeyama (1983)).

$$\begin{aligned} L_{ij}(\omega) &= \frac{\mu}{4\pi} \int_0^l dx' \frac{\cos(\omega \sqrt{(x-x')^2 + d_{ij}^2}/c)}{\sqrt{(x-x')^2 + d_{ij}^2}} \\ &\sim \frac{\mu}{4\pi l} \int_0^l dx \int_0^l dx' \frac{\cos(\omega \sqrt{(x-x')^2 + d_{ij}^2}/c)}{\sqrt{(x-x')^2 + d_{ij}^2}} \\ &= \frac{\mu}{2\pi} \left( \ln \frac{2\tilde{l}(\omega)}{d_{ij}} - 1 \right). \end{aligned} \quad (30)$$

The second line of this expression is the approximation of the Neumann's formula. We have to take into account further the finite size effect together with the skin effect. These effects are worked out by introducing the geometrical mean distance (GMD). The GMD is defined as (Takeyama (1983))

$$\ln \tilde{a}_{ij} = \frac{1}{S_i S_j} \iint \ln r(s_i, s_j : d_{ij}) ds_i ds_j, \quad (31)$$

where  $r(s_i, s_j : d_{ij})$  includes the distance between the two lines  $d_{ij}$  and the skin effect. Here,  $ds_i(ds_j)$  is a small area in a wire  $i(j)$  with the total area  $S_i(S_j)$ . With the GMD, we can finally write the coefficient of inductance as

$$L_{ij}(\omega) = \frac{\mu}{2\pi} \left( \ln \frac{2\tilde{l}(\omega)}{\tilde{a}_{ij}} - 1 \right). \quad (32)$$

We can work out the coefficients of potential  $P_{ij}$  exactly in the same way as those of inductance  $L_{ij}$  in the TEM mode approximation (Toki & Sato (2009)). This should be the case, because of the continuity equation, which forces the spatial distributions of the charge and current are the same.

$$P_{ij}(\omega) = \frac{1}{2\pi\epsilon} \left( \ln \frac{2\tilde{l}(\omega)}{\tilde{a}_{ij}} - 1 \right) \quad (33)$$

The usual coefficients used are the coefficients of capacity  $C$  in the MTL equations (Paul (2008)). This coefficient  $C_{ij}$  is the matrix inversion of the coefficients of potential  $C = P^{-1}$ . We mention that it is an essential feature to write  $P$  instead of  $C$  in the present derivation, because a capacitance per unit length is no longer an adequate quantity in the MTL theory (Toki & Sato (2009)).

We can work out the relations among the charges and currents and the scalar and vector potentials together with the electric resistances for the TEM mode. In order to use these

relations we take time derivatives of the above coupled equations.

$$\frac{\partial V_i(x, t)}{\partial t} = \sum_j^N P_{ij} \frac{\partial Q_j(x, t)}{\partial t} + jM_e \frac{\partial Q_t^I(l, x, t)}{\partial t}, \quad (34)$$

$$\frac{\partial A_i(x, t)}{\partial t} = \sum_j^N L_{ij} \frac{\partial I_j(x, t)}{\partial t} + jM_m \frac{\partial I_t^I(l, x, t)}{\partial t}. \quad (35)$$

Here, we have dropped writing  $\omega$  for the coefficients of all the terms for simplicity of writing. By replacing the charge  $Q_j(x, t)$  by the current using the continuity equation (17), the above equation (34) provides

$$\frac{\partial V_i(x, t)}{\partial t} = - \sum_j^N P_{ij} \frac{\partial I_j(x, t)}{\partial x} + jM_e \frac{\partial Q_t^I(l, x, t)}{\partial t}. \quad (36)$$

We use the combined equation (21) of the Ohm's law (18) and the boundary condition (19) in the above equation (35) in order to write the following equation in terms of the scalar potential as

$$\frac{\partial V_i(x, t)}{\partial x} = - \sum_j^N L_{ij} \frac{\partial I_j(x, t)}{\partial t} - jM_m \frac{\partial I_t^I(l, x, t)}{\partial t} - R_i I_i(x, t). \quad (37)$$

We consider Eqs. (36) and (37) as the fundamental equations for the TEM modes in the MTL system with emission and absorption. As we have seen the inclusion of the retardation terms with the use of the properties of transmission-lines of thick wires with resistance is a natural extension of the standard multiconductor transmission-line theory. We comment here that similar equations without the retardation terms for the case of one transmission line was derived by Kirchhoff (Kirchhoff (1857)). The development later of the Kirchhoff work is described in a book of Ohta (Ohta (2005)). We shall see that these two retardation terms provide naturally the emission and absorption of electromagnetic waves through the multiconductor transmission-line system. We emphasize here that electromagnetic waves go through a multiconductor transmission-line system in the TEM mode while making electromagnetic radiation.

#### 4. TEM mode of one line antenna

Since we have worked out the MTL equation including radiation, we would like to discuss an isolated system of one-conductor transmission-line so that we write explicitly how the electromagnetic energy is converted into Joule energy and radiation energy. In principle, we may have to consider the influence of the circumstance even for one-line antenna. However, for simplicity and also for the sake of understanding the antenna mode, we study the one-line antenna system using the new theory. Here, we have in mind the case of one transmission-line antenna and deal with the case that the current changes from its full value to the vanishing value. We write a set of the antenna mode equation using Eqs. (36) and (37) as

$$\frac{\partial V(x, t)}{\partial t} = -P \frac{\partial I(x, t)}{\partial x} + jM_e \frac{\partial Q^I(l, x, t)}{\partial t}. \quad (38)$$

and

$$\frac{\partial V(x,t)}{\partial x} = -L \frac{\partial I(x,t)}{\partial t} - jM_m \frac{\partial I^I(l,x,t)}{\partial t} - RI(x,t). \quad (39)$$

The quantities  $Q^I$  and  $I^I$  are those defined in Eqs. (29) by dropping the suffix  $t$  because here we treat one-line antenna. These expressions for the integrated charge and current together with those of the coefficients of potential  $P$  and inductance  $L$  remind us the functions of sine and cosine integrals  $S_i$  and  $C_i$  in the antenna theory (Stratton (1941)).

Hence, we write the one antenna equation as

$$\begin{aligned} \frac{\partial V(x,t)}{\partial t} &= -cZ \frac{\partial I(x,t)}{\partial x} + jMc \frac{\partial Q^I(l,x,t)}{\partial t}, \\ \frac{\partial V(x,t)}{\partial x} &= -\frac{Z}{c} \frac{\partial I(x,t)}{\partial t} - j\frac{M}{c} \frac{\partial I^I(l,x,t)}{\partial t} - RI(x,t). \end{aligned} \quad (40)$$

Here, we would like to write explicitly the characteristic impedance  $Z$ , the resistance  $R$  and the characteristic antenna mode coefficient  $M$  for an one-conductor transmission-line system. Since we are dealing with one line, the characteristic impedance is written as

$$Z = \frac{1}{2\pi} \sqrt{\frac{\mu}{\varepsilon}} \left( \ln \frac{2\tilde{l}}{\tilde{a}_{11}} - 1 \right). \quad (41)$$

This impedance is featured to include the length of the line-antenna explicitly. The resistance is simply the one of the transmission-line  $R = R_1$  and the characteristic antenna mode coefficient is  $M = \frac{1}{4\pi} \sqrt{\mu/\varepsilon}$ . The characteristic impedance  $Z$  resembles the coefficient of the antenna theory (Stratton (1941)).

The integrated quantities  $Q^I$  and  $I^I$  are those related with the charge and current integrated over the length, and we can fix the time dependence of the potential  $V(x,t) = V(x)e^{-j\omega t}$  and correspondingly for the current  $I(x,t) = I(x)e^{-j\omega t}$ . We can write then a coupled differential equation for a certain  $\omega$ .

$$\begin{aligned} \frac{dV(x)}{dx} &= j\frac{Z\omega}{c} I(x) - RI(x) - \frac{M\omega}{c} I^I(l,x), \\ -j\omega V(x) &= -Zc \frac{dI(x)}{dx} + Mc\omega Q^I(l,x). \end{aligned} \quad (42)$$

In order to proceed from here, we consider the case of long wave length. This approximation corresponds to the long wave length approximation in the antenna theory.

$$\begin{aligned} Q^I(l) &\sim \frac{\omega}{c} \int_0^l dx' Q(x') \\ I^I(l) &\sim \frac{\omega}{c} \int_0^l dx' I(x'). \end{aligned} \quad (43)$$

We insert the second equation to the first one of Eq. (42) and obtain a second order integro-differential equation for the current.

$$\frac{d^2 I(x)}{dx^2} = -\frac{\omega^2}{c^2} I(x) - j\frac{R\omega}{Zc} I(x) - j\frac{M\omega^2}{Zc^2} I^I(l). \quad (44)$$

We note that this equation is a second order linear differential equation with a constant, if we consider the last term is known. In this case, we can write a general solution as

$$I(x) = ie^{jkx} + i'e^{-jkx} - \frac{j}{1 + j\frac{Rc}{Z\omega}} \frac{M}{Z} I^I(l). \tag{45}$$

Here  $k = k_R + jk_I = \frac{\omega}{c} \sqrt{1 + j\frac{Rc}{Z\omega}}$ . By inserting this solution to the second equation (42) with the long wave length approximation, we get

$$V(x) = \frac{Zkc}{\omega} (ie^{jkx} - i'e^{-jkx}) + jMcQ^I(l). \tag{46}$$

We should keep in mind that these solutions for  $I(x)$  and  $V(x)$  are implicit solutions. Namely,  $I^I(l)$  and  $Q^I(l)$  are obtained by knowing the current  $I(x)$  and  $Q(x)$ . Of course, when the boundary conditions at the center and its ends of the transmission line are given, we are able to use the above solutions for any case of interest.

As the most interesting case, we consider the standard linear antenna which could operate for radiation-emission as a transmitter or for radiation-absorption as a receiver. For this purpose, a power supply or a passive lumped circuit element is placed in the middle of a transmission-line, respectively, and both ends are open. Hence, the boundary conditions in this case are

$$\begin{aligned} V(x = +\epsilon) &= V(0), \\ V(x = -\epsilon) &= -V(0), \\ I(x = l) &= 0, \\ I(x = -l) &= 0. \end{aligned} \tag{47}$$

These boundary conditions fix a relation of  $i$  and  $i'$  of Eq. (45) and hence the current  $I(x)$  and the potential  $V(x)$  in terms of  $i$ . We write the equation to fix  $i'$  in terms of  $i$  by using the condition that the current vanishes at the end of the antenna.

$$I(l) = ie^{jkl} + i'e^{-jkl} - \frac{j}{1 + j\frac{Rc}{Z\omega}} \frac{M}{Z} I^I(l) = 0 \tag{48}$$

With this relation and Eq. (45), we are able to write  $I(0)$  in a compact form.

$$I(0) = (1 - e^{jkl})i + (1 - e^{-jkl})i' \tag{49}$$

In the mean time, we get an implicit expression of  $I^I(l)$  by integrating  $I(x)$  of Eq. (45) from  $x = 0$  to  $x = l$  and find a more compact expression for  $I^I(l)$  in terms of  $i$  and  $i'$ .

$$I^I(l) = \frac{(1 + j\frac{Rc}{Z\omega})\frac{j\omega}{kc} [(1 - e^{jkl})i - (1 - e^{-jkl})i']}{1 + j\frac{Rc}{Z\omega} + j\frac{M\omega l}{Zc}} \tag{50}$$

We can then solve for  $i'$  in terms of  $i$  by using Eq. (48).

$$i' = -\frac{(1 + j\frac{Rc}{Z\omega} + j\frac{M\omega l}{Zc})e^{jkl} + \frac{M\omega}{Zkc}(1 - e^{jkl})}{(1 + j\frac{Rc}{Z\omega} + j\frac{M\omega l}{Zc})e^{-jkl} - \frac{M\omega}{Zkc}(1 - e^{-jkl})} i \tag{51}$$

Substituting this expression to Eq. (49), we get  $I(0)$  in terms of  $i$ .

$$I(0) = \frac{(1 + j\frac{Rc}{Z\omega} + j\frac{M\omega l}{Zc})(e^{-jkl} - e^{jkl}) - 2\frac{M\omega}{Zkc}(2 - e^{jkl} - e^{-jkl})}{(1 + j\frac{Rc}{Z\omega} + j\frac{M\omega l}{Zc})e^{-jkl} - \frac{M\omega}{Zkc}(1 - e^{-jkl})} i. \quad (52)$$

In order to obtain  $V(0)$ , we have to know  $Q^I(l)$ , which is obtained as  $Q^I(l) = -\frac{j}{c}(I(l) - I(0))$ . Since we take the boundary condition  $I(l) = 0$ , we find

$$\begin{aligned} Q^I(l) &= \frac{j}{c}I(0) \\ &= \frac{j}{c} \frac{(1 + j\frac{Rc}{Z\omega} + j\frac{M\omega l}{Zc})(e^{-jkl} - e^{jkl}) - 2\frac{M\omega}{Zkc}(2 - e^{jkl} - e^{-jkl})}{(1 + j\frac{Rc}{Z\omega} + j\frac{M\omega l}{Zc})e^{-jkl} - \frac{M\omega}{Zkc}(1 - e^{-jkl})} i. \end{aligned} \quad (53)$$

We can obtain also  $I^I(l)$  by using the above expressions.

$$\begin{aligned} I^I(l) &= \frac{(1 + j\frac{Rc}{Z\omega})\frac{j\omega}{kc}(e^{-jkl} + e^{jkl}) - 2}{(1 + j\frac{Rc}{Z\omega} + j\frac{M\omega l}{Zc})e^{-jkl} - \frac{M\omega}{Zkc}(1 - e^{-jkl})} i \\ &= \frac{(1 + j\frac{Rc}{Z\omega})\frac{j\omega}{kc}(e^{-jkl} + e^{jkl}) - 2}{(1 + j\frac{Rc}{Z\omega} + j\frac{M\omega l}{Zc})(e^{-jkl} - e^{jkl}) - 2\frac{M\omega}{Zkc}(2 - e^{jkl} - e^{-jkl})} I(0) \end{aligned} \quad (54)$$

We can obtain  $V(0)$  by using Eq. (46) with Eqs. (51), (52) and (53).

$$\begin{aligned} V(0) &= \frac{Zkc}{\omega}(i - i') + jMcQ^I(l) \\ &= \left( \frac{Zkc}{\omega} \frac{(1 + j\frac{Rc}{Z\omega} + j\frac{M\omega l}{Zc})(e^{-jkl} + e^{jkl}) + \frac{M\omega}{Zkc}(e^{-jkl} - e^{jkl})}{(1 + j\frac{Rc}{Z\omega} + j\frac{M\omega l}{Zc})(e^{jkl} - e^{-jkl}) - 2\frac{M\omega}{Zkc}(2 - e^{jkl} - e^{-jkl})} - M \right) I(0) \end{aligned} \quad (55)$$

We can obtain now the input impedance by taking the ratio of  $V(0)$  and  $I(0)$ .

$$\begin{aligned} Z_s &= \frac{2V(x=0)}{I(x=0)} \\ &= 2 \frac{Zkc}{\omega} \frac{(1 + j\frac{Rc}{Z\omega} + j\frac{M\omega l}{Zc})(e^{-jkl} + e^{jkl}) + \frac{M\omega}{Zkc}(e^{-jkl} - e^{jkl})}{(1 + j\frac{Rc}{Z\omega} + j\frac{M\omega l}{Zc})(e^{jkl} - e^{-jkl}) - 2\frac{M\omega}{Zkc}(2 - e^{jkl} - e^{-jkl})} - 2M \end{aligned} \quad (56)$$

Although lengthy, this expression does not depend on the initial input energy and is written in terms of  $Z$ ,  $R$ ,  $M$  and  $l$  for a given  $\omega$ , which are the properties of the transmission-line. It is the first time to obtain the input impedance of one resistive conductor antenna. This expression should be contrasted with the EMF method for the input impedance, which is obtained by assuming the expression for the current in a line antenna. Here, the current is obtained by solving the TEM mode wave equation with the boundary condition at the center and its ends of a line antenna.

We try to understand the meaning of the input impedance by setting  $M = 0$ . In this case, the input impedance is written as

$$\begin{aligned}
 Z_s &= 2 \frac{Zkc e^{-jkl} + e^{jkl}}{\omega (e^{jkl} - e^{-jkl})} \\
 &= 2 \frac{Z(k_R + jk_I)c (e^{-jk_{Rl}} + e^{jk_{Rl}})(e^{k_{I l}} + e^{-k_{I l}}) + (e^{-jk_{Rl}} - e^{jk_{Rl}})(e^{k_{I l}} - e^{-k_{I l}})}{\omega (e^{-jk_{Rl}} - e^{jk_{Rl}})(e^{k_{I l}} + e^{-k_{I l}}) + (e^{-jk_{Rl}} + e^{jk_{Rl}})(e^{k_{I l}} - e^{-k_{I l}})}
 \end{aligned}
 \tag{57}$$

We consider the case that  $R$  is small and write  $k = \frac{\omega}{c} + j\frac{R}{2Z} = k_R + jk_I$ . We set  $k_{I l} \ll 1$  and expand the exponent up to the first order.

$$\begin{aligned}
 Z_s &= 2 \frac{Z(k_R + jk_I)c k_{I l} + j\sin(k_{Rl})\cos(k_{Rl})(1 - k_{I l})}{\omega \sin^2(k_{Rl}) + \cos^2(k_{Rl})(k_{I l})^2} \\
 &\sim Rl \frac{1}{\sin^2(k_{Rl})} + j2Z \frac{\cos(k_{Rl})}{\sin(k_{Rl})}
 \end{aligned}
 \tag{58}$$

In the last step, we take the dominant terms for the case that  $\sin(k_{Rl})$  is not close to 0. The above expression indicates that the real part corresponds to the resistance and the imaginary part corresponds to the characteristic impedance for the TEM mode. We stress here that the TEM mode can exist even for one-line antenna in contrast to the standard understanding that the TEM mode is associated with at least two conductors. At the same time, the input impedance has a resonance structure around  $k_{Rl} = n\pi$  with  $n$  being an integer due to the sine-function in the denominator. The real part has a peak structure at this point, while the imaginary part changes sign and the small additional term makes the imaginary part to go through zero around this point.

With these expressions for the current and potential and the input impedance, we are able to calculate the electromagnetic power

$$P(x) = \frac{1}{4}(V(x)I^*(x) + V^*(x)I(x))
 \tag{59}$$

and the input power  $P(0)$ . We can then write all the power consumed by this one-line antenna system.

$$\begin{aligned}
 P_{total}(x=0) &= 2P(x=0) = \frac{1}{2}(V^*(0)I(0) + V(0)I^*(0)) \\
 &= \frac{1}{4}(Z_s^* + Z_s)|I(0)|^2 = \frac{1}{2}ReZ_s |I(0)|^2
 \end{aligned}
 \tag{60}$$

We note here that the power consumed by the one line antenna is not only used by the radiation but also by the resistance to heat up the one line antenna.

We can express the change rate of the EM power using the coupled differential equation.

$$\frac{dP(x)}{dx} = \frac{1}{4} \left( \frac{dV(x)}{dx} I^*(x) + \frac{dV^*(x)}{dx} I(x) + V^*(x) \frac{dI(x)}{dx} + V(x) \frac{dI^*(x)}{dx} \right)
 \tag{61}$$

Substituting Eq. (42) to this expression, we can calculate the change rate as a sum of the resistance and radiation terms.

$$\frac{dP(x)}{dx} = -\frac{1}{2}R|I|^2 - \frac{1}{4} \frac{M\omega}{c} (I^{I^*}(l)I + I^I(l)I^*) - j\frac{1}{4}Mc \left( Q^{I^*}(l) \frac{dI}{dx} - Q^I(l) \frac{dI^*}{dx} \right)
 \tag{62}$$

We can write the integrated change rate in a compact form.

$$P_t^{antenna} = \int_0^l \frac{dP^{antenna}(x)}{dx} dx = -\frac{1}{2}M|I^I(l)|^2 + \frac{1}{2}Mc^2|Q^I(l)|^2 \quad (63)$$

It is clear that the total change rate due to the antenna mode consists of the emission and absorption terms, which are indicated by the minus sign term and the plus sign term.

We would like to comment here which process as emission or absorption occurs. When a power supply is connected at the middle of one-conductor transmission-line, this antenna operates for radiation-emission as a transmitter because the emission term is larger than the absorption term. When a passive lumped circuit element is connected at the middle of one-conductor transmission line, this antenna operates for radiation-absorption as a receiver because the absorption term is larger than the emission term.

### 5. Three-conductor transmission-line system

We consider now the three-conductor transmission-line theory with emission and absorption through the antenna mode. This is a very interesting case where the two-conductor transmission-lines include the effect of the circumstance. In our previous publication (Toki & Sato (2009)), we have discussed the case where the total current is zero and hence the case without the antenna mode. The present situation with the antenna mode corresponds to the realistic case. In this case we introduce the normal, common and antenna modes. They are written with the currents and potentials of the three lines. Here, we consider that the lines 1 and 2 are the main lines and the line 3 denotes the circumstance.

$$\begin{aligned} I_n &= \frac{1}{2}(I_1 - I_2) \\ I_c &= \frac{1}{2}(I_1 + I_2 - I_3) \\ I_a &= \frac{1}{2}(I_1 + I_2 + I_3) \\ V_n &= V_1 - V_2 \\ V_c &= \frac{1}{2}(V_1 + V_2) - V_3 \\ V_a &= \frac{1}{2}(V_1 + V_2) + V_3 \end{aligned} \quad (64)$$

We work out the coupled integro-differential equations for the TEM mode with the retardation term treated explicitly. There is a factor two difference between the antenna mode current and the total current  $I_t = 2I_a$ . We write the results here for the normal, common and antenna modes. We use first the integro-differential equations for  $N = 3$  in Eq. (37) and express the equations in terms of various modes,

$$\begin{aligned} \frac{\partial V_n(x,t)}{\partial x} &= -L_n \frac{\partial I_n(x,t)}{\partial t} - L_{nc} \frac{\partial I_c(x,t)}{\partial t} - L_{na} \frac{\partial I_a(x,t)}{\partial t} - R_n I_n - R_{nc} I_c - R_{na} I_a \\ \frac{\partial V_c(x,t)}{\partial x} &= -L_{cn} \frac{\partial I_n(x,t)}{\partial t} - L_c \frac{\partial I_c(x,t)}{\partial t} - L_{ca} \frac{\partial I_a(x,t)}{\partial t} - R_{cn} I_n - R_c I_c - R_{ca} I_a \\ \frac{\partial V_a(x,t)}{\partial x} &= -L_{an} \frac{\partial I_n(x,t)}{\partial t} - L_{ac} \frac{\partial I_c(x,t)}{\partial t} - L_a \frac{\partial I_a(x,t)}{\partial t} - j2M_m \frac{\partial I_t^I(l,x,t)}{\partial t} \\ &\quad - R_{an} I_n - R_{ac} I_c - R_a I_a \end{aligned} \quad (65)$$

In the above equations all the coefficients are written as follows,

$$\begin{aligned} L_n &= L_{11} - 2L_{21} + L_{22} \\ L_c &= \frac{1}{4}(L_{11} + 2L_{12} + L_{22}) - (L_{13} + L_{23}) + L_{33} \\ L_a &= \frac{1}{4}(L_{11} + 2L_{12} + L_{22}) + L_{13} + L_{23} + L_{33} , \end{aligned} \quad (66)$$

for the diagonal coefficients and

$$\begin{aligned} L_{nc} &= \frac{1}{2}(L_{11} - L_{22}) - (L_{13} - L_{23}) \\ L_{na} &= \frac{1}{2}(L_{11} - L_{22}) + (L_{13} - L_{23}) \\ L_{ca} &= \frac{1}{4}(L_{11} + 2L_{12} + L_{22}) - L_{33} , \end{aligned} \quad (67)$$

for the non-diagonal coefficients. We get the resistance terms as

$$\begin{aligned} R_n &= R_1 + R_2 \\ R_c &= \frac{1}{4}(R_1 + R_2) + R_3 \\ R_a &= \frac{1}{4}(R_1 + R_2) + R_3 \\ R_{nc} &= \frac{1}{2}(R_1 - R_2) \\ R_{na} &= \frac{1}{2}(R_1 - R_2) \\ R_{ca} &= \frac{1}{4}(R_1 + R_2) - R_3 \end{aligned} \quad (68)$$

We obtain similar relations for transmission-line equations (36) including  $P_{ij}$  as written below.

$$\begin{aligned} \frac{\partial V_n(x,t)}{\partial t} &= -P_n \frac{\partial I_n(x,t)}{\partial x} - P_{nc} \frac{\partial I_c(x,t)}{\partial x} - P_{na} \frac{\partial I_a(x,t)}{\partial x} \\ \frac{\partial V_c(x,t)}{\partial t} &= -P_{cn} \frac{\partial I_n(x,t)}{\partial x} - P_c \frac{\partial I_c(x,t)}{\partial x} - P_{ca} \frac{\partial I_a(x,t)}{\partial x} \\ \frac{\partial V_a(x,t)}{\partial t} &= -P_{an} \frac{\partial I_n(x,t)}{\partial x} - P_{ac} \frac{\partial I_c(x,t)}{\partial x} - P_a \frac{\partial I_a(x,t)}{\partial x} + j2M_e \frac{\partial Q_t^I(l,x,t)}{\partial t} \end{aligned} \quad (69)$$

In the above equations all the coefficients are written as follows,

$$\begin{aligned} P_n &= P_{11} - 2P_{12} + P_{22} \\ P_c &= \frac{1}{4}(P_{11} + 2P_{12} + P_{22}) - (P_{13} + P_{23}) + P_{33} \\ P_a &= \frac{1}{4}(P_{11} + 2P_{12} + P_{22}) + P_{13} + P_{23} + P_{33} , \end{aligned} \quad (70)$$



for the diagonal coefficients and

$$\begin{aligned} P_{nc} &= \frac{1}{2}(P_{11} - P_{22}) - (P_{13} - P_{23}) \\ P_{na} &= \frac{1}{2}(P_{11} - P_{22}) + (P_{13} - P_{23}) \\ P_{ca} &= \frac{1}{4}(P_{11} + 2P_{12} + P_{22}) - P_{33} , \end{aligned} \quad (71)$$

for the non-diagonal coefficients. All the coefficients of potential  $P_{ij}$  are written in a compact form as the coefficients of inductance  $L_{ij}$ .

$$\begin{aligned} L_{ij} &= \frac{\mu}{2\pi} \left( \ln \frac{2\tilde{l}}{\tilde{a}_{ij}} - 1 \right) \\ P_{ij} &= \frac{1}{2\pi\epsilon} \left( \ln \frac{2\tilde{l}}{\tilde{a}_{ij}} - 1 \right) \end{aligned} \quad (72)$$

Using these coefficients, we can write all the coefficients associated with the normal, common and antenna modes. They are written as

$$\begin{aligned} P_n &= \frac{1}{2\pi\epsilon} \ln \frac{\tilde{a}_{12}^2}{\tilde{a}_{11}\tilde{a}_{22}} \\ P_c &= \frac{1}{8\pi\epsilon} \ln \frac{\tilde{a}_{13}^4 \tilde{a}_{23}^4}{\tilde{a}_{11}\tilde{a}_{22}\tilde{a}_{12}^2 \tilde{a}_{33}^4} \\ P_a &= \frac{1}{8\pi\epsilon} \left( \ln \frac{(2\tilde{l})^{16}}{\tilde{a}_{11}\tilde{a}_{22}\tilde{a}_{12}^2 \tilde{a}_{33}^4 \tilde{a}_{13}^4 \tilde{a}_{23}^4} - 16 \right) \end{aligned} \quad (73)$$

Only the antenna mode coefficient  $P_a$  contains the length of the transmission-lines explicitly and is appropriate for the antenna mode. The coupling terms are written as

$$\begin{aligned} P_{nc} &= \frac{1}{2\pi\epsilon} \ln \frac{\tilde{a}_{22}\tilde{a}_{13}}{\tilde{a}_{11}\tilde{a}_{23}} \\ P_{na} &= \frac{1}{2\pi\epsilon} \ln \frac{\tilde{a}_{22}\tilde{a}_{23}}{\tilde{a}_{11}\tilde{a}_{13}} \\ P_{ca} &= \frac{1}{8\pi\epsilon} \ln \frac{\tilde{a}_{33}^4}{\tilde{a}_{11}\tilde{a}_{22}\tilde{a}_{12}^2} \end{aligned} \quad (74)$$

We get similar expressions for  $L_i$ . They are related with  $P_i$  as  $L_i = P_i c^2$ . It is very interesting to note that the coefficient of capacity for the normal mode is written as  $C_n = 1/P_n = 2\pi\epsilon / \ln \frac{\tilde{a}_{12}^2}{\tilde{a}_{11}\tilde{a}_{22}}$ .

The coupled differential equations tell many interesting facts. When there is a symmetry between the lines 1 and 2 in their relations to the third line due to the symmetric arrangement, the coupling terms of the normal mode to both the common and antenna modes can be made zero. The normal mode decouples from the common and antenna modes. On the other hand, when the symmetry is lost between the lines 1 and 2, the normal mode couples not only with the common mode but also with the antenna mode. This coupling of three wave-type modes is considered to be the origin of EM noise, which can not be understood due to the

reflection and interference. It is therefore very important to take care of the symmetry of the lines 1 and 2. We repeat that if the third line represents the circumstance, it is impossible to make the coupling terms zero. Hence, the ordinary two-conductor transmission-line system is influenced by the circumstance and the electromagnetic emission and absorption take place. Therefore, we cannot avoid the noise problem. In addition, we comment that the common mode always couples with the antenna mode and the emission and absorption take place simultaneously with the generation of the EM noise in the circuit.

It is interesting to write the differential equation for the normal mode for the case of symmetrization, where the coupling terms of the normal mode to the common and antenna modes are zero. The TEM mode differential equations for the normal mode are written as

$$\begin{aligned} \frac{\partial V_n(x, t)}{\partial t} &= -P_n \frac{\partial I_n(x, t)}{\partial x} \\ \frac{\partial V_n(x, t)}{\partial x} &= -L_n \frac{\partial I_n(x, t)}{\partial t} - R_n I_n . \end{aligned} \tag{75}$$

These differential equations for the normal mode together with the coefficients  $L_n$ ,  $P_n$  and  $R_n$  agree with the two-conductor transmission-line equations (Paul (2008)). Usually the upper equation in Eq. (75) is written in terms of  $1/C_n$  in the place of  $P_n$ . The expression for  $C_n$  calculated by  $C_n = 1/P_n = 2\pi\epsilon/l_n \frac{\tilde{a}_{12}^2}{\tilde{a}_{11}\tilde{a}_{22}}$  agrees with the capacitance per unit length of the usual two line expression. We stress again the use of the coefficient of potential is essential for the formulation of the three-line system. Hence, the ordinary TEM mode propagation of the EM wave is achieved only when the symmetrization is introduced for the electric circuit in the circumstance.

We shall calculate the electromagnetic power of the three-conductor transmission-line system.

$$\begin{aligned} P(x) &= \frac{1}{4}(V_1^* I_1 + V_2^* I_2 + V_3^* I_3 + V_1 I_1^* + V_2 I_2^* + V_3 I_3^*) \\ &= \frac{1}{4}(V_n^* I_n + V_c^* I_c + V_a^* I_a + V_n I_n^* + V_c I_c^* + V_a I_a^*) \end{aligned} \tag{76}$$

It is interesting to calculate the change of the power with distance so that we can pick up only the terms which change with distance. We take the time dependence of all the modes as  $\exp(-j\omega t)$ . We can work out the change rate  $\frac{dP(x)}{dx}$  on the basis of Eqs. (65) and (69) in exactly the same way as the case of the one-conductor transmission line. We write only the final result.

$$\begin{aligned} \frac{dP(x)}{dx} &= -\frac{1}{2}(R_n |I_n(x)|^2 + R_c |I_c(x)|^2 + R_a |I_a(x)|^2 \\ &\quad + R_{nc}(I_n(x)I_c^*(x) + I_n^*(x)I_c(x)) + R_{na}(I_n(x)I_a^*(x) \\ &\quad + I_n^*(x)I_a(x)) + R_{ca}(I_c(x)I_a^*(x) + I_c^*(x)I_a(x)) \\ &\quad - \frac{1}{2} \frac{M\omega}{c} (I_t^*(l, x)I_a(x) + I_t^{l,x}(l)I_a^*(x)) \\ &\quad - j\frac{1}{2} Mc \left( Q_t^{I^*}(l, x) \frac{dI_a(x)}{dx} - Q_t^I(l, x) \frac{dI_a^*(x)}{dx} \right) \end{aligned} \tag{77}$$

This expression agrees with the one of the line-antenna (62), when the change rate is expressed with the total current by using the relation  $I_a(x) = \frac{1}{2}I_t(x)$ . It is interesting to point out that the change of the electric power is made by the resistance terms and the antenna mode terms.

There are two effects in the antenna mode terms. One is a term associated with emission and the other is a term associated with absorption.

Here, we would like to comment the strict symmetrization of the power supply. In a standard two-stage power supply in which two identical power supplies are connected in series but switchings are controlled alternately, a common mode current is unavoidable. Even when there is a symmetry between lines 1 and 2 in relation to the third line, coupling terms between the common and antenna modes do not vanish unavoidably. This implies that the radiation of EM wave occurs unless the common mode current is eliminated. The most effective method of eliminating the common and antenna modes is a strict symmetrization in which switchings of two-stage power supply should be synchronized in a symmetrized three-conductor transmission-line system.

## 6. Symmetrized electric circuit

In the previous section, we have discussed the performance of two transmission-lines in the circumstance and hence a three-conductor transmission-line system. The electromagnetic noise in the circumstance goes through the third line in the form of electromagnetic wave. In the standard two-stage power supply as mentioned above, the noise in the circumstance influences the main two-lines through the common mode. Since we are not able to control the circumstance, it is impossible to remove the electromagnetic noise in the case of the two line electric circuit in the circumstance. At the same time, the modern power supply and also both the inverter and converter use the chopping method and generate electromagnetic noise with high frequency. This noise goes through the standard two line circuit and at the same time goes out from the circuit in the form of electromagnetic wave.

The way out is to introduce a new third line to the main two line system in order to minimize the effect of the circumstance by asking the new third line to take care of the common mode effect. We are then able to control the whole electric circuit by arranging all the elements (conductor-lines, powers, loads etc.) so as to minimize the effect of noise. One very important thing is to introduce two identical power supplies and connect the third line to the middle point of the two power supplies. In this way, the common mode noise produced in the power supply system finds a way to go through the third line and does not go out from the electric circuit.

It is then important to decouple the normal mode from the common mode. This is achieved by using the same size and same quality transmission-lines for the main two lines and by arranging the geometrical distances of the two lines to the third line equal. At the same time, we have to arrange lumped-loads symmetrically around the third line. We have discussed why the normal mode decouples from the common mode in a symmetrical arrangement by calculating three-line lumped-circuit in our first publication (Sato & Toki (2007)), which reviewed the design principle of HIMAC synchrotron (Kumada (1994)) and provided a guide of alteration of magnet wiring of J-PARC MR (Kobayashi (2009)). The present new MTL theory with the antenna mode tells that the conditions of the normal mode to decouple from the common and antenna modes are to impose the symmetrization among the three-conductor transmission-lines in addition to the symmetrization of the lumped elements.

We show one example of electric circuit to use the normal mode current with largely reduced noise by the symmetric arrangement around the third line as shown in Fig.1. The present day power supply uses a chopping device and produces high frequency noise. We use a standard two-stage power supply and connect the third line at the middle point of the two-stage power supply denoted by  $P$  to confine high frequency noise produced by the power supply. The filtering device  $F$  should cut down high frequency noise and allows only low frequency noise to pass through the filter. It should be noted that the filtering device  $F$  consists of the common

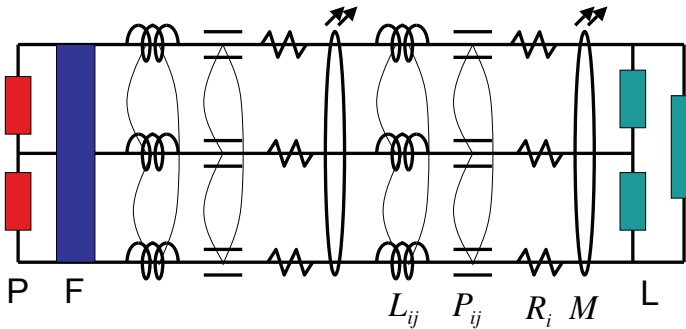


Fig. 1. A symmetrized electric circuit for the use of the normal mode decoupled from the common and antenna modes. Around the third line (middle line) two power supplies  $P$ , the filtering element  $F$  are placed in the left end and the electric loads  $L$  are placed in the right end. The connecting three-conductor transmission lines have the properties of self and mutual inductances  $L_{ij}$  denoted by coil, self and mutual coefficients of potential  $P_{ij}$  denoted by short parallel lines, resistance  $R_i$  and antenna mode coefficient  $M$ . The three-conductor transmission lines are coupled each other and their performance follows a coupled integro-differential equation with these coefficients.

mode filter in addition to the normal mode filter in order to cut down not only the normal mode noise but also the common mode noise. In the right end of the three-lines placed are electric loads  $L$  symmetrically. The arrangement of these lumped devices symmetrically is the requirement of the decoupling of the normal mode from the common mode.

Very important fact is now that these power-filter element  $P - F$  and the electric loads  $L$  are connected by transmission-lines, which are not just structureless lines, but contain several functions as inductance  $L_{ij}$ , coefficient of potential  $P_{ij}$ , resistance  $R_i$  and antenna coefficient  $M$ . There are self- and mutual-inductances  $L_{ij}$  and they are denoted by coils on lines and connections of coils as usual in Fig.1. We denote the coefficients of potential  $P_{ij}$ , which have both self- and mutual-coefficients, by two short parallel lines and connections of short parallel lines. We abandon here the concept of capacitor  $C_{ij}$  and use a similar but rotated symbol for  $P_{ij}$ . The resistances  $R_i$  are denoted by the standard symbol and are attached to each line. In addition, we have the antenna coefficient  $M$  for radiation, which is associated with the whole transmission-lines and use the connection symbol with two arrows indicating radiation. The performance of the transmission-lines is controlled then by a set of coupled integro-differential equations with these coefficients  $L_{ij}$ ,  $P_{ij}$ ,  $R_i$  and  $M$ . At a glance of Fig.1, readers might picture that the symmetric arrangement provides decoupling of normal mode from common and antenna modes because of  $L_{nc} = L_{na} = 0$  in Eq. (67),  $R_{nc} = R_{na} = 0$  in Eq. (68) and  $P_{nc} = P_{na} = 0$  in Eq. (71). Consequently, the symmetric arrangement of the transmission-lines and the lumped elements are the necessary step for the good performance of an electric circuit. We should on top consider that the noise is EM wave and goes through the transmission-lines in the TEM mode with loss of Joule and radiation energies.

### 7. Conclusion

We have constructed a new multi-conductor transmission-line (MTL) theory with the antenna mode. To this end, we have started from the electrodynamics field theory, which denotes that sources of electric and magnetic fields outside the conductors are true charge and conduction current inside the conductors. Based on the definite statement of the field theory, it is allowed to consider the dynamics of charge and current in resistive conductors of

transmission-lines of thick wires and their coupling to the electromagnetic fields surrounding the transmission-lines. We have used the continuity equation of the charge and current and the combined equation of the boundary condition at the surface and the Ohm's law with the resistance, which controls the movement of the charge and current. The Maxwell equation then relates the dynamics of the charge and current to the scalar and vector potentials surrounding the transmission-lines. Since we are interested in the performance of the electromagnetic fields outside of the MTL system, we solve the wave equations for the scalar and vector potentials in the Lorenz gauge with the retardation charge and current. The scalar and vector potentials are now expressed in the integral forms and they are called retarded potentials. These four equations are the fundamental equations for the MTL system with the antenna mode. The coupled integro-differential equations are to be solved for the propagation of the electromagnetic wave and the energy loss due to the Joule and radiation processes.

To proceed, we have analyzed the retardation potentials for each frequency mode. The retardation charge and current introduces the real part with a cosine function and the imaginary part with a sine function. We are then able to make the TEM mode approximation for the real part, but should keep the imaginary part in the integral form. This TEM mode approximation can relate the scalar and vector potentials at the surface of each transmission-line with the charge and current for the introduction of the coefficients of potential  $P_{ij}$  and inductance  $L_{ij}$ . In this process, we consider the retardation charge and current effect explicitly. Hence, we modify the coefficients of inductance  $L_{ij}$  and potential

$P_{ij}$  by including the  $\omega$  dependent term  $\cos(\omega \sqrt{(x-x')^2 + d_{ij}^2}/c)$  in the integrand. As for the imaginary terms, we have now the omega dependent term  $\sin(\omega|x-x'|/c)$  in the numerator and should keep the integral form. We call the newly added integral terms coming from the imaginary parts of the retardation charge and current as the antenna mode terms with antenna mode coefficients  $M_e$  and  $M_m$ . We are then able to express MTL integro-differential equations for the scalar potential and the current by eliminating the charge and the vector potentials by using the continuity equation which is equivalent to the current conservation equation of the field theory and the combined equation of the boundary condition and the Ohm's law equation.

We have worked out an one-conductor transmission-line system to discuss the standard line-antenna with the propagation of a TEM mode through the transmission-line. In this case, we use the long wavelength approximation for the antenna mode terms originating from the retardation terms. Due to the fact that we are able to calculate the coefficients of inductance and potential, we can write down coupled integro-differential equations for potential  $V$  and current  $I$  with  $L$ ,  $P$  and  $M_e$ ,  $M_m$  and  $R$ . We solve the coupled equations formally and work out the input impedance, which is now a function of the size, the length and the resistance of the transmission-line. We have explicitly worked out the case for one linear transmission-line antenna. We can provide the solution of the differential equation and give an expression of the input impedance  $Z_s$  for the first time with the long wave length approximation after the MTL equations are fixed for the TEM mode. We work out the power of the system at the origin, which is eventually consumed by the Joule energy and the radiation energy. We have provided the input impedance for a typical case of a line antenna of thick wire with resistance. We have studied also a three-conductor transmission-line system with emission and absorption. In addition to mathematical expressions, we propose a new circuit diagram of multi-conductors on the basis of coefficient of potential, coefficient of inductance, coefficient of antenna mode, and resistance. There appear three kinds of waves of normal, common, and antenna modes. All these modes propagate around the transmission-lines in the TEM mode waves. It is very interesting to point out if there is a symmetry between the lines 1 and 2 due to a symmetric arrangement, then the normal mode decouples from the common and

antenna modes simultaneously. On the other hand, when the symmetry between the lines 1 and 2 is lost, the normal mode couples with both the common and antenna modes. This is a realistic situation of the ordinary two-conductor transmission-line system with the inclusion of the circumstance. We have to introduce the third line to the main two line system, instead of circumstance of which electrical performance is unclear, and symmetrize the system in order to confine the electromagnetic fields within the three-line system with the help of the common mode filter. In the near future we shall work out the skin effect by taking into account the motion of the current to the radial direction of each transmission-line in the MTL theory with the antenna mode (Sato & Toki (2011)).

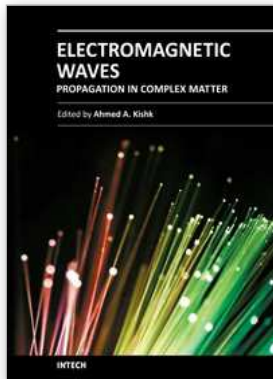
Finally we would like to comment on a distinction between the present MTL theory and the former standard two-conductor transmission-line theory from the view point of electromagnetism. We consider resistive conductors for transmission-lines and abandon the concept of perfect conductor. The TEM mode wave could exist even in the case of one-conductor transmission line so that the TEM mode approximation is useful in the present study. The coefficients of potential are important to determine not only the coupling impedance between mutual transmission lines but also the characteristic impedance of a single transmission line itself. Consequently, it is unnecessary for a transmission line theory to introduce capacitance per unit length between two transmission lines and the displacement current flowing through the capacitance for a transmission-line theory any more. The boundary conditions for the electromagnetic fields at the surface of the resistive conductor provide the propagation of the TEM mode and replaces the concept of the Kirchoff's current law between two lines due to displacement current.

## 8. Acknowledgment

The authors are grateful to Prof. H. Kobayashi and Prof. H. Horiuchi for fruitful discussions and encouragements.

## 9. References

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## **Electromagnetic Waves Propagation in Complex Matter**

Edited by Prof. Ahmed Kishk

ISBN 978-953-307-445-0

Hard cover, 292 pages

**Publisher** InTech

**Published online** 24, June, 2011

**Published in print edition** June, 2011

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