

# Variable Property Effects in Momentum and Heat Transfer

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## 1. Introduction

“Variable property effects” is a theoretical construct since a real fluid always is subject to variable properties when changes in temperature or pressure occur. Their influence compared to a corresponding situation but with artificially constant properties may be small and thus neglected in a first approximation. Those artificial “constant property results” may then be corrected with respect to the initially neglected effects due to the variability of the fluid properties. This concept assumes small variable property effects and therefore is not applicable when the flow itself is basically generated by a variable property (like natural convection, generated by density variations) or strongly affected by it (like strongly compressible flow, determined by density variations).

Therefore we define “variable property effects” as those (small) artificial effects that would be present if fluid properties could change from constant to variable. They have to be added to a solution which is gained under the assumption of constant properties in order to account for the fact that real fluids always have properties which are temperature and pressure dependent. The variable property correction of a constant property solution can be accomplished in different ways. Basically there are three methods which are widely used in this context. They are

### (1) Property ratio method

Results in terms of the nondimensional friction factor  $f$  and Nusselt number  $Nu$  are gained by multiplying the constant property results  $f_{cp}$  and  $Nu_{cp}$  with a property ratio correction factor, i.e.

$$f = f_{cp} \left( \frac{a_1}{a_2} \right)^{m_a}, \quad Nu = Nu_{cp} \left( \frac{a_1}{a_2} \right)^{n_a} \quad (1)$$

Here  $a_1, a_2$  are properties ( $\rho, \mu, k, c_p$ ) at two different temperatures and  $m_a, n_a$  are empirical exponents. There may be more than one factor, depending on which properties are assumed to contribute to the variable property effects. Studies with this approach are Li et al. (2007) and Mahmood et al. (2003), for example.

### (2) Reference temperature method

The constant property results in terms of  $f$  and  $Nu$  are evaluated at a certain temperature for the properties that appear in  $f$  and  $Nu$ . This so-called reference temperature,

$$T_R = T_1 + j(T_2 - T_1) \quad (2)$$

between two characteristic temperatures  $T_1$  and  $T_2$  of the problem must be chosen such that  $f$  and  $Nu$  determined under the assumption of constant properties (and with  $T_1$  or  $T_2$  as reference temperature) give the results for variable properties. For that purpose the factor  $j$  in equ (2) must be determined properly. Basically,  $j$  is an empirical parameter like the exponents in equ (1) are in the first method. Studies using this method are Jayari et al. (1999) and Debreštan & Anderson (1994), for example.

### (3) Asymptotic method

Since the effects of variable properties are assumed to be small the variable property solution can be taken as a perturbation of the constant property solution. In a systematic approach the problem is treated as a regular perturbation problem with a (small) perturbation parameter  $\varepsilon$  linked to the transfer rate. The constant property solution is that for  $\varepsilon = 0$  and variable property effects are described as 1<sup>st</sup>, 2<sup>nd</sup>, ... n<sup>th</sup> order effects of an asymptotic expansion with respect to the perturbation parameter  $\varepsilon$ . This approach in an early paper has been suggested by Carey and Mollendorf (1980) and afterwards has been adopted by the second author for a wide variety of problems, see for example Herwig and Wickern (1986), Herwig and Bauhaus (1986), Herwig et al. (1989) and Herwig and Schäfer (1992).

The advantage of this method over the first two methods to account for variable properties is its systematic generalization of the correction terms that appear within this approach. These terms hold for all Newtonian fluids and all (small) transfer rates that are the reason for variable property effects. Furthermore, with this approach the empirical parameters in the first two method ( $m_\alpha$ ,  $n_\alpha$  and  $j$ ) can be embedded in the expansion approach and thus be interpreted on a sound physical basis, as is demonstrated in Herwig and Wickern (1986), for example.

For a long time all three methods have been applied to certain standard situations like pipe, channel or boundary layer flows. Only recently it was shown how the asymptotic method can also be applied to complex problems, such as turbulent mixed convection in a room that is ventilated and heated by several heating elements, see Bünger and Herwig (2009), Jin and Herwig (2010) and Jin and Herwig (2011).

This extension to complex convective heat transfer problems will be presented and illustrated by some examples in the following.

## 2. The asymptotic approach: State of the art

*Variable properties* means that all fluid properties involved in a problem depend on temperature and pressure. Whenever temperature and pressure variations occur in a problem these variabilities affect the solution. Assuming the effects to be small, a perturbation approach is straight forward. Since pressure effects almost always are negligibly small, we only account for the temperature impact. The starting point for the temperature dependence is the Taylor series expansion of all properties with  $a^*$  representing  $\rho^*$  (density),  $\mu^*$  (viscosity),  $k^*$  (thermal conductivity), and  $c_p^*$  (specific heat capacity), which is with  $a_R^* = a^*(T_R^*)$

$$a = \frac{a^*}{a_R^*} = 1 + \varepsilon K_{a1} \theta + \frac{1}{2} \varepsilon^2 K_{a2} \theta^2 + \dots + \frac{1}{n!} \varepsilon^n K_{an} \theta^n + O(\varepsilon^{n+1}) \quad (3)$$

$$\varepsilon := \frac{\Delta T_R^*}{T_R^*} \quad (4)$$

$$\theta := \frac{T^* - T_R^*}{\Delta T_R^*} \quad (5)$$

$$K_{a1} := \left[ \frac{\partial a^* T^*}{\partial T^* a^*} \right]_R \quad (6)$$

$$K_{a2} := \left[ \frac{\partial^2 a^* T^{*2}}{\partial T^{*2} a^*} \right]_R \quad (7)$$

$$K_{an} := \left[ \frac{\partial^n a^* T^{*n}}{\partial T^{*n} a^*} \right]_R \quad (8)$$

Here,  $\varepsilon$  is a small quantity which basically represents the strength of heat transfer ( $\varepsilon = 0$ : isothermal situation) and formally serves as perturbation parameter. The dimensionless temperature is  $\theta$  which then is an  $O(1)$  quantity. The fluid behavior is covered by  $K_{aj}$  with  $j = 1, 2, \dots, n$ , which are  $O(1)$  quantities, representing the partial derivatives of  $a^*$  with respect to temperature.

The same expansion can be made with respect to the pressure  $p^*$ . Since, however, for most cases the corresponding  $\hat{K}_{an}$  values  $\left( \left( \frac{\partial^n a^*}{\partial p^{*n}} \right) / \left( p^{*n} / a^* \right) \right)$  and/or pressure variations are very small, we restrict ourselves to the temperature effects.

In order to get solutions in terms of expansions which correspond to the Taylor-series expansions of the properties, all dependent variables  $V^*$  are expanded as

$$V := \frac{V^*}{V_R^*} = 1 + \varepsilon \left[ K_{\rho 1} V_{\rho 1} + K_{\mu 1} V_{\mu 1} + K_{k 1} V_{k 1} + K_{c_p 1} V_{c_p 1} \right] + O(\varepsilon^2) \quad (9)$$

Here,  $V^*$  represents the velocity components  $u^*$ ,  $v^*$ ,  $w^*$  and the temperature  $T^*$  which appear in the basic equations of a problem. For many problems these are the Navier-Stokes and the thermal energy equations, here shown for a steady flow  $\partial/\partial t = 0$ , i.e:

$$\nabla \cdot (\boxed{\rho} \bar{u}) = 0 \quad (10)$$

$$(\boxed{\rho} \bar{u} \cdot \nabla) \bar{u} = \frac{1}{\text{Re}_R} \nabla \cdot \left( \boxed{\mu} (\nabla \bar{u} + \nabla \cdot \bar{u}^T) - \frac{2}{3} \nabla \cdot \bar{u} I \right) - \nabla p + \boxed{\rho} \bar{g} \quad (11)$$

$$\boxed{\rho c_p} (\bar{u} \cdot \nabla) T = \frac{1}{\text{Re}_R \text{Pr}_R} \nabla \cdot (\boxed{k} \nabla T) + p \nabla \cdot \bar{u} \quad (12)$$

In the thermal energy equation (12) viscous dissipation is neglected since it contributes very little to the energy balance. Eqs (10)-(12) are nondimensional equations in which all fluid properties  $a = a^*/a_R^*$  are marked by a box (note that  $a = 1$  for constant properties).

The nondimensional groups are

$$\text{Re}_R = \frac{\rho_R^* u_R^* L^*}{\mu_R^*}, \quad \text{Pr}_R = \frac{\mu_R^* c_{pR}^*}{k_R^*} \quad (13)$$

It is an important point that here they are formed with the reference fluid properties (index  $R$ ). When turbulent flows are treated by the RANS-approach, equs (10) – (12) must be time-averaged and subject to turbulence modeling. Then, with a  $k-\varepsilon$  model, for example, two more differential equations have to be solved in which fluid properties appear as well.

In the asymptotic method to account for variable property effects, all properties  $a$  (i.e.  $\rho$ ,  $\mu$ ,  $k$ ,  $c_p$ ) and all variables  $v$  (i.e.  $\bar{u}$ ,  $T$ ) are replaced by their expansions (3), (9) in the set of equs (10) – (12). Then two different ways to proceed are possible which will be explained in the next two subsections.

### 2.1 Higher order equations method (HOE)

In this method the equations are subject to the expansion process, i.e. all terms of  $O(1)$ ,  $\varepsilon K_{\rho 1}$ ,  $\varepsilon K_{\mu 1}$ , ...,  $\varepsilon^2 K_{\rho 2}$ ,  $\varepsilon^2 K_{\rho 1} K_{\rho 1}$ , ... are collected separately and form the zero order, first order, second order, ... sets of equations. In the complete approach with all four properties involved there is one zero order set of equations (corresponding to constant properties), four first order sets and fourteen sets of second order. Only the zero order set of equations is a nonlinear one, all subsequent equation sets have linear equations. All equations have to be solved sequentially starting with the zero order equations for constant properties.

The final result in terms of the friction factor  $f$  and the Nusselt number  $\text{Nu}$  gains an asymptotic form as well. For example, the Nusselt number for the thermal boundary condition  $T_w = \text{const}$ , reads, with  $\text{Nu}_{cp}$  for the constant property case:

$$\frac{\text{Nu}}{\text{Nu}_{cp}} = 1 + \varepsilon \left[ K_{\rho 1} \frac{T'_{\rho 1}}{T'_0} + K_{\mu 1} \frac{T'_{\mu 1}}{T'_0} + K_{k1} \frac{T'_{k1}}{T'_0} + K_{c_{p1}} \frac{T'_{c_{p1}}}{T'_0} \right] + O(\varepsilon^2) \quad (14)$$

Here,  $T'_{a1}$  are the temperature wall gradients since  $\dot{q}_w^*$  in the Nusselt number is  $\dot{q}_w^* = -(k^* T'^*)'_w$ .

In the second order 14 additional terms in equ (14) appear so that altogether 19 sets of equations have to be solved when second order accuracy is required. Then, however, the result (14) is quite general, since it holds for all Newtonian fluids (different values for  $K_{\rho 1}$ ,  $K_{\mu 1}$ , ... in (14)) and all small heating rates (different values for  $\varepsilon$  in equ (14)).

This method is straight forward for all laminar flows and has been applied in Herwig and Wickern (1986) and Herwig and Bauhaus (1986), for example.

When turbulent flows are considered, the higher order equation method needs higher order turbulence model equations. That may be a problem which does not occur in the alternative method described next.

### 2.2 Higher order coefficients method (HOC)

Instead of solving higher order equations from which the influence of the single properties can be deduced in the final results, like in equ (14) for example, the final results are taken in its asymptotic form as the starting point. Again, shown for the Nusselt number, it reads up to the second order:

$$\begin{aligned} \frac{\text{Nu}}{\text{Nu}_{cp}} = & 1 + \varepsilon \left[ K_{\rho 1} A_{\rho} + K_{\mu 1} A_{\mu} + K_{k 1} A_k + K_{c_p 1} A_{c_p} \right] + \varepsilon^2 \left[ K_{\rho 2} A_{\rho 2} + K_{\mu 2} A_{\mu 2} \right. \\ & + K_{k 2} A_{k 2} + K_{c_p 2} A_{c_p 2} + K_{\rho 1}^2 A_{\rho \rho} + K_{\mu 1}^2 A_{\mu \mu} + K_{k 1}^2 A_{kk} + K_{c_p 1}^2 A_{c_p c_p} \\ & + K_{\rho 1} \left( K_{\mu 1} A_{\rho \mu} + K_{k 1} A_{\rho k} + K_{c_p 1} A_{\rho c_p} \right) + K_{\mu 1} \left( K_{k 1} A_{\mu k} + K_{c_p 1} A_{\mu c_p} \right) \\ & \left. + K_{k 1} K_{c_p 1} A_{kc_p} \right] + O(\varepsilon^3) \end{aligned} \quad (15)$$

with the A-values  $A_{\rho}$ ,  $A_{\mu}$  ... as the crucial part of the result. Up to the second order there are  $4 + 14 = 18$  different A-values.

With the nondimensional property  $a$  according to equ (3) is rewritten as

$$a = 1 + h_{a1} \theta + \frac{1}{2} h_{a2} \theta^2 + \dots + \frac{1}{n!} h_{an} \theta^n + O(\varepsilon^{n+1}) \quad (16)$$

introducing

$$h_{aj} := \varepsilon^j K_{aj} = \left[ \frac{\partial^j a^* \Delta T^{*j}}{\partial T^{*j} a^*} \right]_R, \quad j = 1, 2, \dots, n. \quad (17)$$

one immediately recognizes the A-values to be

$$A_a := \frac{1}{\text{Nu}_{cp}} \left[ \frac{\partial \text{Nu}}{\partial h_{a1}} \right]_{h=0} \quad (18)$$

$$A_{aa} := \frac{1}{2 \text{Nu}_{cp}} \left[ \frac{\partial^2 \text{Nu}}{\partial h_{a1}^2} \right]_{h=0} \quad (19)$$

$$A_{a2} := \frac{1}{\text{Nu}_{cp}} \left[ \frac{\partial \text{Nu}}{\partial h_{a2}} \right]_{h=0} \quad (20)$$

$$A_{ab} := \frac{1}{\text{Nu}_{cp}} \left[ \frac{\partial^2 \text{Nu}}{\partial h_{a1} \partial h_{b1}} \right]_{h=0} \quad (21)$$

where  $a, b \in \{\rho, \mu, k, c_p\}$ ,  $a \neq b$ .

According to equs (18) – (21) the 18 A-values in (15) can be determined as first and second derivatives of Nu with respect to the h-values in (16). These h-values quite generally can be interpreted as combinations of  $\varepsilon$  and fluid properties ( $K_{\rho 1}, \dots$ ). Since the method holds for all Newtonian fluids it also holds for arbitrarily chosen **fictitious fluids** for which only one property is temperature dependent (in some arbitrary way) with all other properties being constant. Such fluids serve as “dummy fluids” from which the general A-values can be determined.

This is done by solving the full equations (10) – (12) or their turbulent version for these dummy fluids with certain values for  $h_{a1}$ ,  $h_{b1}$  and  $h_{a2}$ . Then the A values can be determined by the following numerical approximations of (18) – (21):

$$A_a = \frac{1}{\text{Nu}_{cp}} \lim_{s \rightarrow 0} \left\{ \frac{\text{Nu}(h_{a1} = s) - \text{Nu}(h_{a1} = -s)}{2s} \right\}_0 \quad (22)$$

$$A_{aa} = \frac{1}{2\text{Nu}_{cp}} \lim_{s \rightarrow 0} \left\{ \frac{\text{Nu}(h_{a1} = s) - 2\text{Nu}_{cp} + \text{Nu}(h_{a1} = -s)}{s^2} \right\}_0 \quad (23)$$

$$A_{a2} = \frac{1}{\text{Nu}_{cp}} \lim_{s \rightarrow 0} \left\{ \frac{\text{Nu}(h_{a2} = s) - \text{Nu}(h_{a2} = -s)}{2s} \right\}_0 \quad (24)$$

$$A_{ab} = \frac{1}{\text{Nu}_{cp}} \lim_{s \rightarrow 0} \frac{1}{4s^2} \left\{ \text{Nu}(h_{a1} = h_{b1} = s) + \text{Nu}(h_{a1} = h_{b1} = -s) - \text{Nu}(h_{a1} = -h_{b1} = s) - \text{Nu}(h_{a1} = h_{b1} = -s) \right\}_0. \quad (25)$$

Here the limiting process  $\lim_{s \rightarrow 0} \dots$  means that  $s$  must be small enough to avoid a higher order influence, but not too small since otherwise truncation errors appear. The index 0 at the right curly brackets indicates that all other variables  $h_{c1}$ ,  $h_{c2}$  are set to zero. Each of the eight terms

- $\text{Nu}(h_{a1} = s)_0, \text{Nu}(h_{a1} = -s)_0$
- $\text{Nu}(h_{a2} = s)_0, \text{Nu}(h_{a2} = -s)_0$
- $\text{Nu}(h_{a1} = h_{b1} = s)_0, \text{Nu}(h_{a1} = h_{b1} = -s)_0,$
- $\text{Nu}(h_{a1} = -h_{b1} = s)_0, \text{Nu}(h_{a1} = -h_{b1} = -s)_0$

corresponds to exactly one CFD-simulation. Hence, the calculation of (22), (23), and (24) needs two CFD-simulations for a fixed small value of  $s$ , while the determination of  $A_{ab}$  according to (25) needs four CFD-simulations. With one further CFD-calculation the constant property solution  $\text{Nu}_{cp}$  is determined.

In order to determine the complete set of all 18 A-values (second order accuracy of the result) thus 40 variable property solutions and the one for constant properties are needed. Since no higher order equations are determined, all kinds of turbulence modeling can be incorporated.

Details of the approach and an application of the HOC-method to a complex geometry and flow can be found in Bünger and Herwig (2009) and Jin and Herwig (2010), respectively. In Fig. 1, as an example taken from Jin and Herwig (2010), the influence of variable properties on the Nusselt number at two heating elements in a complex geometry is shown, determined with the HOC-method.

There can be, however, a further improvement of the HOC-method which considerably reduces the amount of numerical solutions that are needed for the determination of all A-values. This will be described next.

### 3. A more efficient (asymptotic) approach

For a more efficient method, we first rewrite the basic equations (10)-(12). Instead of referring the properties  $\rho^*$ ,  $\mu^*$ ,  $k^*$  and  $c_p^*$  to their values at a reference temperature  $T_R^*$  we incorporate them in the nondimensional groups  $\text{Re}$  and  $\text{Pr}$  (which then are no longer  $\text{Re}_R$  and  $\text{Pr}_R$  since they now are local, variable quantities).

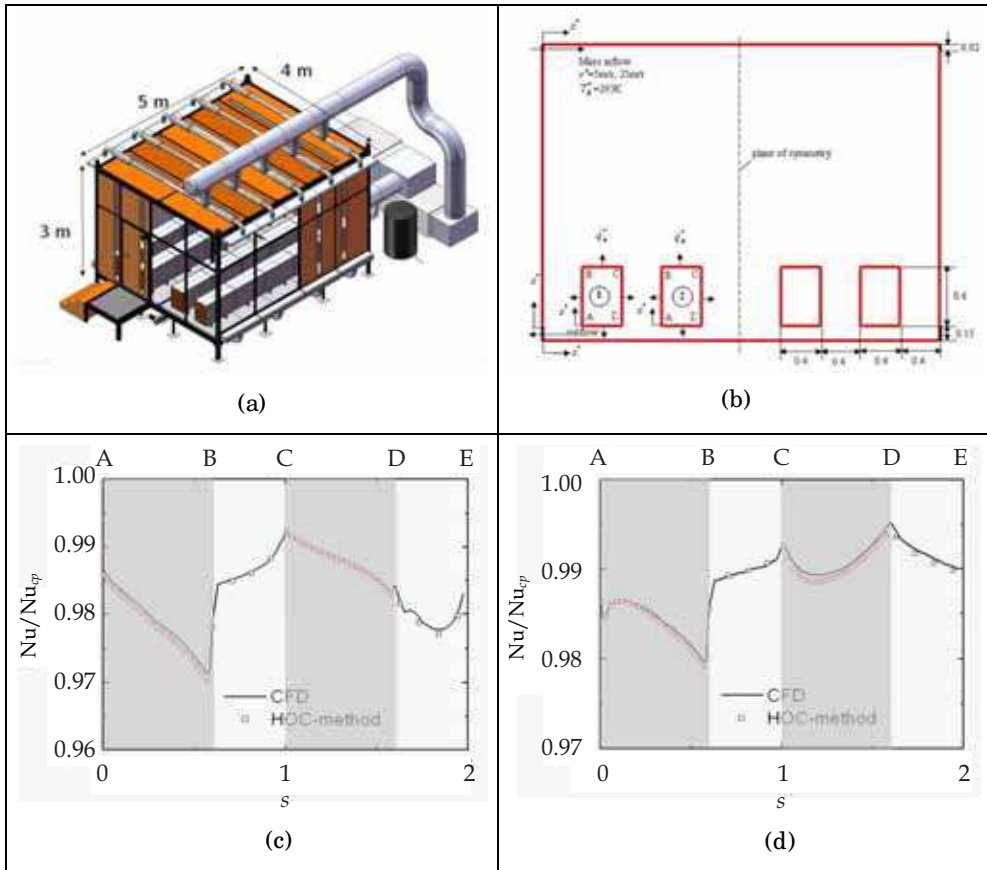


Fig. 1. Complex benchmark geometry, example from (Jin & Herwig, 2010); (a) 3D room for experiments, (b) 2D approximation for simulations, ①, ②: heating elements ( $\dot{q}_w^* = 300 \text{ W/m}^2$ ,  $\text{Re} = 1.64 \times 10^6$ )  $s^*$ : coordinate around the heating elements and along the lower, upper, and side walls A, B, C, D: corners of the heating elements (c) Heating element ① (d) Heating element ② CFD: numerical solution, based on eqs (10)-(12) HOC: asymptotic result, eq. (15)

Since  $\rho^*$ ,  $\mu^*$ , and  $k^*$  appear within a spatial derivative in (10)-(12), additional terms appear according to the product rule of differentiation. Instead of (10)-(12) one gets:

$$\nabla \cdot (\bar{u}) + P_\rho \nabla T \cdot \bar{u} = 0 \tag{26}$$

$$(\bar{u} \cdot \nabla) \bar{u} = \left( \frac{1}{\text{Re}} \nabla \cdot + P_\mu \nabla T \cdot \right) \left( (\nabla \bar{u} + \nabla \bar{u}^T) - \frac{2}{3} \nabla \cdot \bar{u} I \right) - \frac{\nabla p}{1 + (\rho - 1)} + \bar{g} \tag{27}$$

$$\left( \bar{u} - \frac{P_k}{\text{PrRe}} \nabla T \right) \cdot \nabla T = \frac{1}{\text{PrRe}} \nabla \cdot (\nabla T) \tag{28}$$

with the parameters  $P_\rho$ ,  $P_\mu$ ,  $P_k$  as

$$P_a = \frac{\Delta T^*}{a^*} \frac{\partial a^*}{\partial T^*}; \quad a^* = \rho^*, \mu^*, k^* \tag{29}$$

The only property that is left explicitly is  $\rho$  in the term  $-\frac{\nabla p}{1+(\rho-1)}$  in equ (27), all other properties are “hidden” in  $Re$ ,  $Pr$  and  $P_a$ . Asymptotically the parameters  $P_a$  are of order  $O(\varepsilon)$ , since

$$P_a = \varepsilon K_{a1} + O(\varepsilon) \tag{30}$$

accounting for the fact that  $a^*$  in  $K_{a1}$  is  $a^*(T_R^*)$  but  $a^*$  in  $P_a$  is  $a^*(T^*)$ . Based on equs (26)-(28) there are two simplified approaches to account for variable properties which are motivated by the fact that the dimensionless groups account for the variability of the properties leaving additional effects in the asymptotically small terms with  $P_a$  and  $(\rho-1)$ .

**3.1 High Reynolds number HOC-method (HR-HOC)**

For high Reynolds numbers the energy equation (28) can be approximated by setting  $P_k = 0$ , since  $P_k/RePr \rightarrow 0$  for  $Re \rightarrow \infty$ . This reduces the number of A-values in the final result for the Nusselt number. Instead of equ (15) one now has for the Nusselt number with

$$K_{\sigma 1} = K_{k1} - K_{c_p 1}, \quad K_{\sigma 2} = K_{k2} - K_{c_p 2} - 2K_{k1}K_{c_p 1} + 2K_{c_p 1}^2 \tag{31}$$

$$\frac{Nu}{Nu_{cp}} = 1 + \varepsilon [K_{\rho 1}A_{\rho} + K_{\mu 1}A_{\mu} + K_{\sigma 1}A_{\sigma}] + \varepsilon^2 [K_{\rho 2}A_{\rho 2} + K_{\mu 2}A_{\mu 2} + K_{\sigma 2}A_{\sigma 2} + K_{\rho}^2 A_{\rho\rho} + K_{\mu}^2 A_{\mu\mu} + K_{\sigma}^2 A_{\sigma\sigma} + K_{\rho\mu}A_{\rho\mu} + K_{\rho\sigma}A_{\rho\sigma} + K_{\mu\sigma}A_{\mu\sigma}] + O(\varepsilon^3) \tag{32}$$

The new  $h_{aj}$ -values (c.f. equ (17)) for the determination of the A-values are

$$h_{\sigma 1} = \varepsilon K_{\sigma 1} = \varepsilon (K_{k1} - K_{c_p 1}) \tag{33}$$

$$h_{\sigma 2} = \varepsilon^2 K_{\sigma 2} = \varepsilon^2 (K_{k2} - K_{c_p 2} - 2K_{k1}K_{c_p 1} + 2K_{c_p 1}^2) \tag{34}$$

Table 1 shows the reduction in terms and numerical solutions necessary for the determination of the A-values in equ (32), compared to those for equ (15).

Method	A-values		Numerical solutions	
	1. order	2. order	1. order	2. order
HOC	4	14	8	32
HR-HOC	3	9	6	18

Table 1. Number of A-values and numerical solutions

The complex benchmark geometry in Fig. 1 is now treated with the simplified HR-HOC approach. Fig. 2 shows that the quality of the results is as good as in Fig. 1 for the full HOC-method.



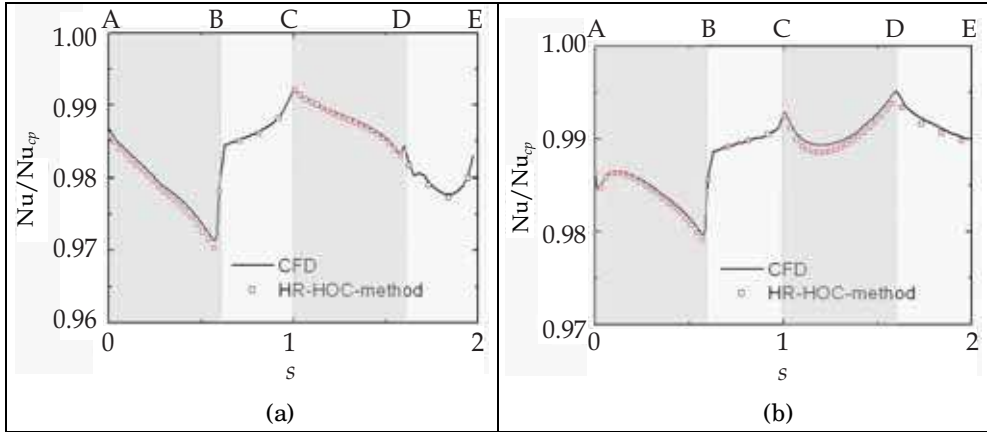


Fig. 2. Complex benchmark geometry, s. Fig. 1, ( $\dot{q}_w^* = 300 \text{ W/m}^2$ ,  $Re = 1.64 \times 10^6$ ) (a) heating element ① (b) heating element ②

**3.2 Locally constant property HOC-method (LCP-HOC)**

Without a real asymptotic justification all variable property effects outside the nondimensional groups are now neglected by setting all  $P_a = 0$  and  $\rho = 1$ . Equations (26)-(28) then are the equations for constant properties, but with locally variable properties in  $Re$  and  $Pr$ .

Solutions of these equations can be used as an easy to get answer with respect to the question how strong the influence of variable properties is and if it makes sense to apply the more sophisticated methods.

In this LCP-HOC approach  $Re$  and  $RePr$  in the reduced set of equations (26)-(28) are treated like variable properties, being called  $\alpha$  and  $\beta$ , i.e.

$$\alpha = Re = \frac{\rho^* u_R^* L^*}{\mu^*} \tag{35}$$

$$\beta = RePr = \frac{\rho^* c_p^* u_R^* L^*}{k^*} \tag{36}$$

Instead of equ (15) one now has for the Nusselt number with

$$K_{\alpha 1} = K_{\rho 1} - K_{\mu 1} \tag{37}$$

$$K_{\alpha 2} = K_{\rho 2} - K_{\mu 2} - 2K_{\rho 1}K_{\mu 1} + 2K_{\mu 1}^2$$

$$K_{\beta 1} = K_{\rho 1} + K_{c_p 1} - K_{k 1} \tag{38}$$

$$K_{\beta 2} = K_{c_p 2} + K_{\rho 2} - K_{k 2} + 2K_{c_p 1}K_{\rho 1} - 2K_{k 1}K_{c_p 1} - 2K_{\rho 1}K_{k 1} + 2K_{k 1}^2$$

the form:

$$\frac{Nu}{Nu_{cp}} = 1 + \varepsilon [K_{\alpha 1}A_{\alpha} + K_{\beta 1}A_{\beta}] + \varepsilon^2 [K_{\alpha 2}A_{\alpha 2} + K_{\beta 2}A_{\beta 2} + K_{\alpha 1}^2A_{\alpha\alpha} + K_{\beta 1}^2A_{\beta\beta} + K_{\alpha 1}K_{\beta 1}A_{\alpha\beta}] + O(\varepsilon^3) \tag{39}$$

The new  $h_{aj}$ -values (c.f. equ (17)) for the determination of the A-values are

$$h_{\alpha 1} = \varepsilon K_{\alpha 1}; \quad h_{\alpha 2} = \varepsilon^2 K_{\alpha 2} \tag{40}$$

$$h_{\beta 1} = \varepsilon K_{\beta 1}; \quad h_{\beta 2} = \varepsilon^2 K_{\beta 2} \tag{41}$$

Table 2 shows the reduction in terms and numerical solutions necessary for the determination of the A-values in equ (39) compared to those for equ (15).

Method	A-values		Numerical solutions	
	1. order	2. order	1. order	2. order
HOC	4	14	8	32
LCP-HOC	2	5	4	8

Table 2. Number of A-values and numerical solutions

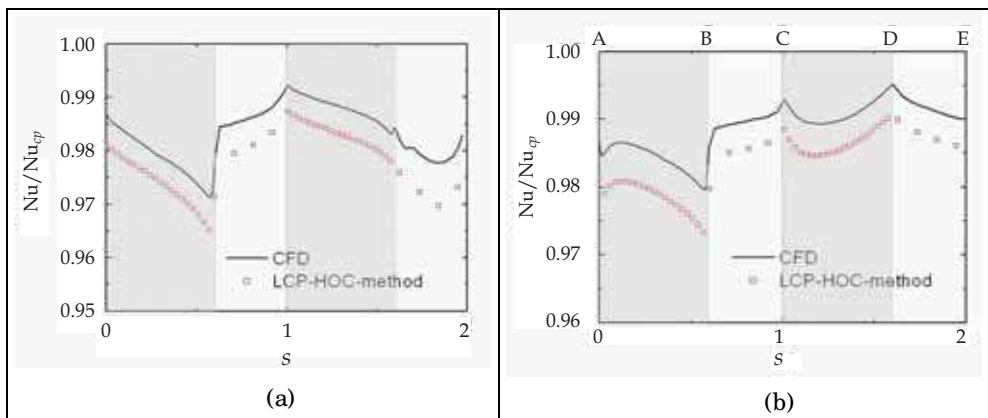


Fig. 3. Complex benchmark geometry, s. Fig. 1, ( $\dot{q}_w^* = 300 \text{ W/m}^2$ ,  $\text{Re} = 1.64 \times 10^6$ ) (a) heating element ① (b) heating element ②

Again, the complex benchmark geometry in Fig. 1 is chosen as a test case, now for the LCP-HOC approach. Fig. 3 shows that the results are of the same order as with the more comprehensive methods and thus can serve as a simple first approximation. So far, there is no explanation for the obviously constant offset in the results.

#### 4. Alternative determination HOC-method (AD-HOC)

So far the RANS-approach has been used to cope with turbulent flows. Computational times are moderate and the relatively large number of numerical solutions necessary for the various HOC-methods is available in acceptable time.

This is no longer true, however, when LES or DNS solutions are aimed at. Then, finding 32 solutions for the second order HOC method (c.f. table 1 or 2) is not a reasonable procedure. In table 3 CPU-times for RANS, LES and DNS solutions of a simple benchmark problem are compared to illustrate the discrepancy in typical CPU-times for the three methods. The

benchmark problem is that of a differentially heated cavity with adiabatic top and bottom walls shown in Fig. 4. In  $x_1$ -direction there is a periodic boundary condition  $L_1$  apart corresponding to an overall 2D geometry. As a result for constant properties (Boussinesq-approximation) the Nusselt number and shear stress distribution is shown for three different Rayleigh numbers at the hot wall in Fig. 5. Whereas DNS results from (Trias et al, 2010a; Trias et al, 2010b; Trias et al, 2007) and our own LES results are close to each other, RANS results show larger deviations. Some numerical details are given in table 3.

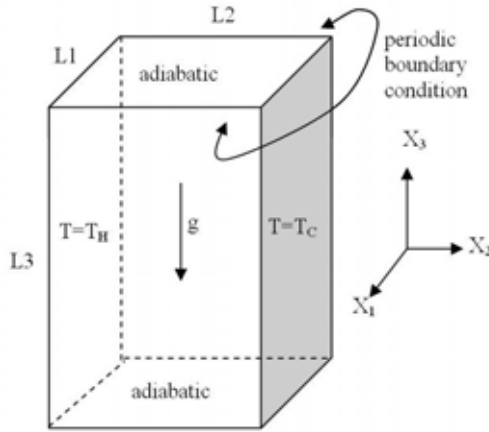


Fig. 4. Differentially heated cavity of aspect ratio  $L_3/L_1 = 4, L_2/L_1 = 1$

NUMERICAL METHOD	CPU-TIME in hours			NUMERICAL DETAILS
	$Ra = 2 \times 10^9$	$Ra = 4 \times 10^9$	$Ra = 10^{10}$	
RANS (own results)	1	1	1	$k - \omega$ SST model; mesh resolution: $100 \times 218$ ; steady simulation. Computer: Linux cluster (4x2.66GHz)
LES (own results)	330	330	330	$k$ -equation eddy-viscosity model; mesh resolution: $64 \times 100 \times 218$ ; time step: $2.4 \times 10^{-2}$ Computer: Linux cluster (4x2.66GHz)
DNS (Trias et al, 2010a; Trias et al, 2010b; Trias et al, 2007)	6200 (estimated)		26000 (estimated)	Mesh resolution $Ra = 2 \times 10^9$ : $64 \times 144 \times 318$ $Ra = 10^{10}$ : $128 \times 190 \times 462$ time step $Ra = 2 \times 10^9$ : $1.27 \times 10^{-3}$ $Ra = 10^{10}$ : $6.25 \times 10^{-4}$ Computer: Nec-Sx5 vectorial parallel machine.

CPU-TIME: sum of computational time of all processors in hours

Table 3. CPU-time and numerical details for different numerical methods applied to the benchmark problem (heated cavity), see Fig. 4

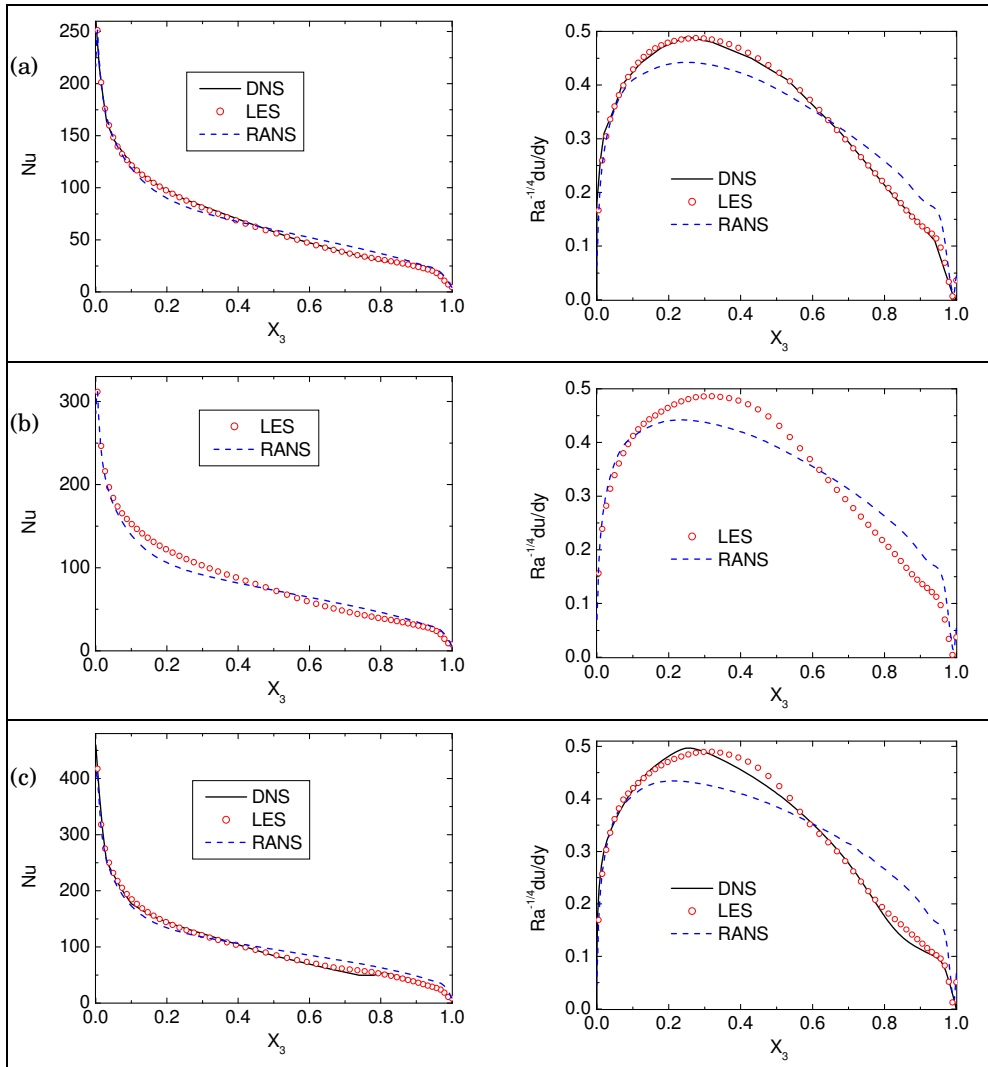


Fig. 5. Constant property results of the flow and heat transfer in a differentially heated cavity, see Fig. 4. (a)  $Ra = 2 \times 10^9$  (b)  $Ra = 4 \times 10^9$  (c)  $Ra = 1 \times 10^{10}$

If now variable property effects are of interest, the A-coefficients should be determined with a RANS-approach. It is assumed that the variable and the constant property RANS results show basically the same errors (compared to the “exact” DNS results) and thus have systematic errors that cancel when the difference is taken in the procedure to determine the A-values.

DNS and LES constant property results can thus be corrected with respect to variable property effects by “alternatively determined” (i.e. AD) A-values. This procedure is called AD-HOC-method, but is not really an ad-hoc-method!

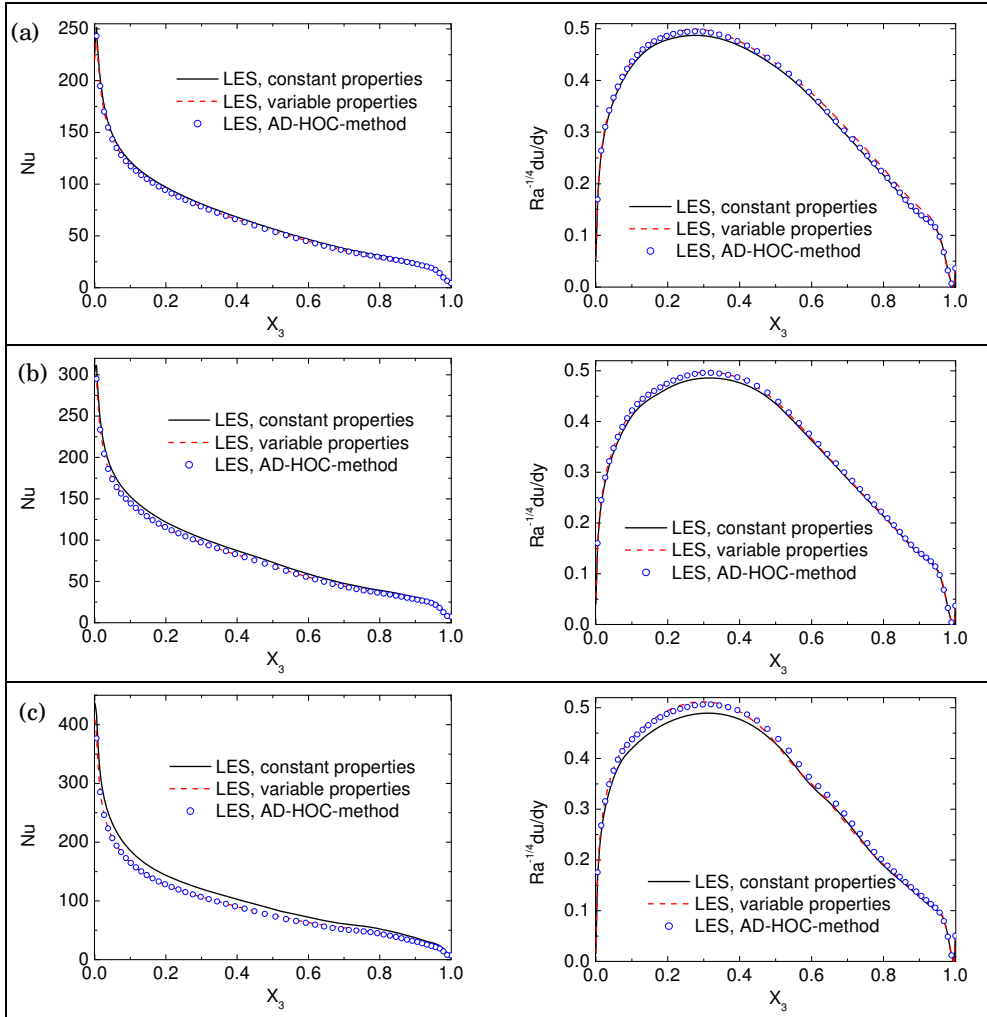


Fig. 6. Variable property results of the flow and heat transfer in a differentially heated cavity, see Fig. 4. (a)  $Ra = 2 \times 10^9$  (b)  $Ra = 4 \times 10^9$  (c)  $Ra = 1 \times 10^{10}$  Fluid: air with properties (42)-(45)

In Fig. 6 these AD-HOC-results that account for variable properties are compared to LES results that account for variable properties from the beginning. They are results for air with variable  $\rho^*$ ,  $\mu^*$ ,  $k^*$  and  $c_p^*$  asymptotically accounted for to the first order and for the LES calculations taken as

$$\rho^* = 1.188 \cdot (1 - \gamma^* (T^* - T_R^*)) \tag{42}$$

$$\mu^* = 1.18185 \times 10^{-5} \cdot \left( 1 + 0.775 \frac{T^* - T_R^*}{T_R^*} - 0.176 \left( \frac{T^* - T_R^*}{T_R^*} \right)^2 \right) \tag{43}$$

$$k^* = 0.025721 \cdot \left( 1 + 0.891 \frac{T^* - T_R^*}{T_R^*} - 0.1285 \left( \frac{T^* - T_R^*}{T_R^*} \right)^2 \right) \quad (44)$$

$$c_p^* = 1014 \cdot \left( 1 + 0.068 \frac{T^* - T_R^*}{T_R^*} - 0.038 \left( \frac{T^* - T_R^*}{T_R^*} \right)^2 \right) \quad (45)$$

where the reference temperature  $T_R^* = 293K$  and the expansion coefficient  $\gamma^* = 0.003413$ . The density variations are accounted for by the Boussinesq approximation like in (Trias et al, 2010a; Trias et al, 2010b; Trias et al, 2007). Fig. 6 shows that with the influence of variable properties determined by the AD-HOC-method there is a very good coincidence with LES results that account for variable property effects from the beginning. Thus the AD-HOC-method is an attractive approach when LES and DNS results have to be corrected with respect to the influence of variable properties.

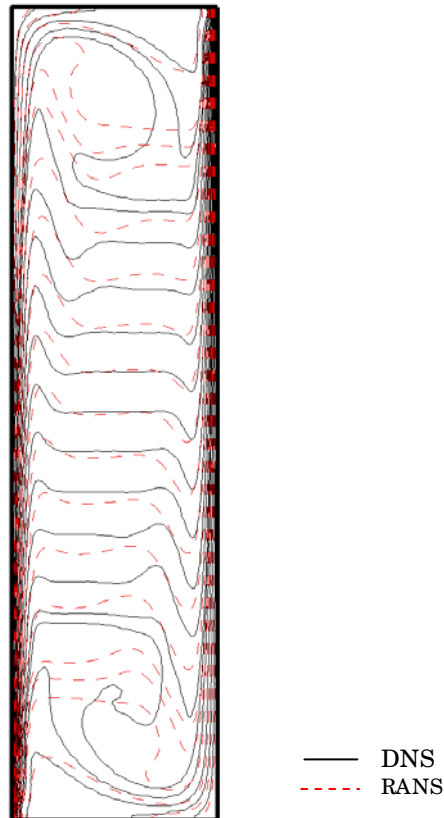


Fig. 7. Constant property results of the temperature field in a differentially heated cavity, see Fig. 4,  $Ra = 2 \times 10^8$

This is also true with respect to the field variables of these cases. For example, Fig. 7 shows that there is an appreciable difference when the temperature field is calculated by DNS compared to the RANS results.

However, as shown in Fig. 8, the iso-temperature lines for variable properties, calculated by DNS are well represented by iso-lines from the AD-HOC method, i.e. those lines from constant property DNS results corrected by A-values from RANS solutions for variable properties.

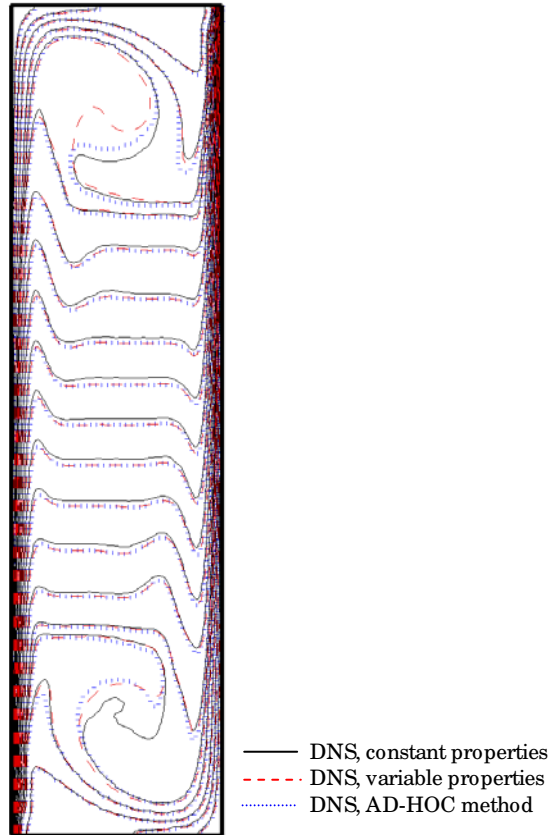


Fig. 8. Variable property results of the temperature field in a differentially heated cavity, see Fig. 4,  $Ra = 2 \times 10^8$

Distribution of the first order A-values  $A_\gamma$ ,  $A_\mu$ ,  $A_k$  and  $A_{c_p}$ , computed by RANS, are shown in Fig. 9. The variable properties behave differently in the core region, where a quasi-laminar flow prevails and in the large vortex region near the bottom and top walls. Also, signs within one region are different. For example  $A_\gamma$  and  $A_\mu$  are negative in the core

region, whereas  $A_k$  and  $A_{c_p}$  are positive in the same region. Altogether there is a non-negligible effect of variable properties on the temperature distribution.

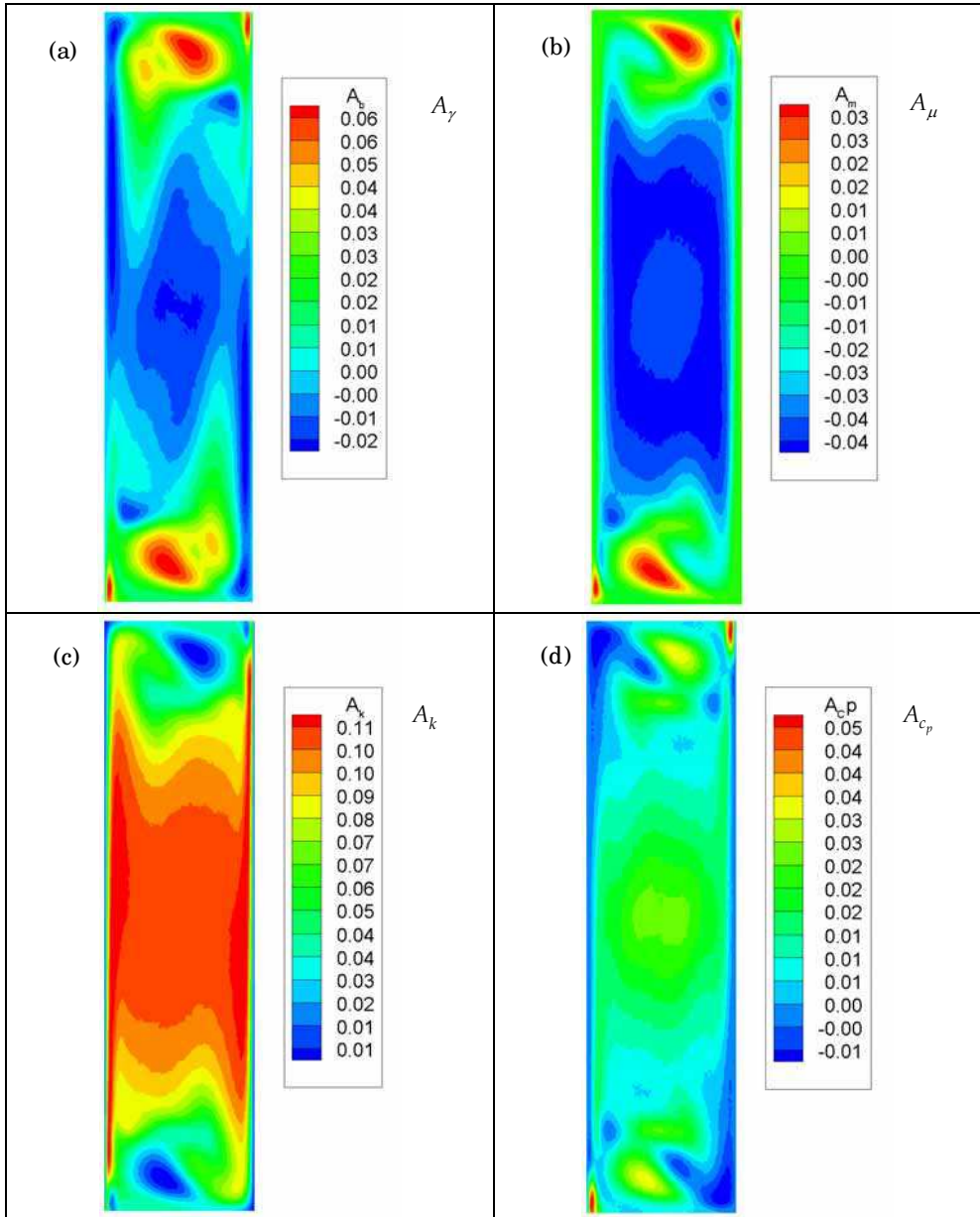


Fig. 9. A-values of temperature computed by RANS in a differentially heated cavity, see Fig. 4,  $Ra = 2 \times 10^8$  (a)  $A_\gamma$ ; (b)  $A_\mu$ ; (c)  $A_k$ ; (d)  $A_{c_p}$



## 5. Conclusions

Various methods to account for variable property effects in complex geometries and highly sophisticated numerical methods have been proposed. Due to the Taylor series expansions of all properties, which are the starting point for all methods, the influence of variable properties can be accounted for in a general manner, i.e. for all (small) heating rates and for all Newtonian fluids. For a special problem with A-values determined once, the effect of variable properties can be found in the final result by fixing  $\varepsilon$  (the heat transfer rate) and all K-values (from the fluid of interest).

This way of treating variable property effects is much closer to the physics than empirical methods like the property ratio and the reference temperature methods are.

## 6. Nomenclature

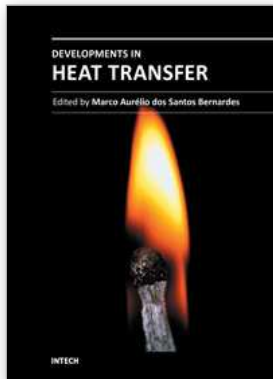
$A_a, A_{aa}, A_{ab}, A_{a2}$	A-values, $a, b \in \{\rho, \gamma, \mu, k, c_p\}, a \neq b$
$c_p$	specific heat capacity
$f$	friction factor
$\vec{g}$	gravity vector
$h_{aj}$	h-values, $\varepsilon^j K_{aj}$
$j$	empirical parameter
$K_a, K_{a2}, K_{an}$	K-values, $a = \rho, \gamma, \mu, k, c_p$
$k$	heat conductivity
$L^*$	characteristic length
$m_a, n_a$	empirical exponents
<b>Nu</b>	Nußelt number
$p$	pressure
<b>Pr</b>	Prandtl number
$P_a$	nondimensional properties about variable fluid property
$\dot{q}$	heat flux
<b>Re</b>	Reynolds number
$T$	temperature
$\vec{u}$	velocity vector
$V$	variables
<i>Greek symbols</i>	
$\alpha, \beta, \sigma, \delta$	variables composed of fluid properties
$\gamma$	expansion coefficient
$\Delta T^*$	temperature difference
$\varepsilon$	nondimensional temperature difference
$\mu$	dynamic viscosity
$\rho$	density
<i>subscripts</i>	
$cp$	constant properties
$R$	reference state
*	dimensional

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This book comprises heat transfer fundamental concepts and modes (specifically conduction, convection and radiation), bioheat, entransy theory development, micro heat transfer, high temperature applications, turbulent shear flows, mass transfer, heat pipes, design optimization, medical therapies, fiber-optics, heat transfer in surfactant solutions, landmine detection, heat exchangers, radiant floor, packed bed thermal storage systems, inverse space marching method, heat transfer in short slot ducts, freezing and drying mechanisms, variable property effects in heat transfer, heat transfer in electronics and process industries, fission-track thermochronology, combustion, heat transfer in liquid metal flows, human comfort in underground mining, heat transfer on electrical discharge machining and mixing convection. The experimental and theoretical investigations, assessment and enhancement techniques illustrated here aspire to be useful for many researchers, scientists, engineers and graduate students.

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