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Applications of Nash's Theorem to Cosmology

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1. Introduction

The understanding of gravitational phenomena has been considered a fundamental problem in modern Cosmology. Recent observations of the CMBR power spectrum in the 7-year data from WMAP (Komatsu et.al., 2011; Jarosik et.al., 2011) tell that the gravitational field perturbations amplify the higher acoustic modes due to the gravitational field of baryons and mainly on the influence of Dark matter. Dark matter has been regarded as to be responsible for inducing a strong gravitational effect on cosmological scale that would lead the young universe to form large scale structures. Such perturbations are also verified at the local scales of galaxies and clusters of galaxies. Moreover, the gravitational perturbations also play an important role in the acceleration of the universe. Due to the cosmological constant paradigm, modifications of gravity have been studied as a alternative route to obtain the require correction for Friedman's equations.

In this sense, Nash's theorem on gravitational perturbations along extra dimensions has been revealed to be an appropriated tool in a manner of dealing with such perturbations. In our present discussion, we seek such explanation within the foundations of geometry, notably using the notion of geometric or gravitational flow, determined by the extrinsic curvature. In order to understand the concept of geometric flow, we give a brief review of the problem of embedding space-times and of its compatibility with the observational aspects of physics.

We discuss the structure and concepts related to the embedding theory as the basis for a more general theory of gravitation. In this framework, for instance, the cosmological constant problem is seen as a symptom of the ambiguity of the Riemann curvature in general relativity. The solution of that ambiguity provided by Nash's theorem eliminates the direct comparison between the vacuum energy density and Einstein's cosmological constant, besides being compatible with the formation of structures and the accelerated expansion of the universe. Moreover, it is shown how space-times solutions of Einstein's equations can be smoothly deformed along the extra dimensions of an embedding space and how the deformation, described by the extrinsic curvature, produces an observable effect of topological character in the universe.

In the following section, we begin reviewing the brane-world program motivated by the problem of unification of the fundamental interactions. The third section is devoted to Nash's embedding theorem and its relation to the gravitational perturbations. The correct embedding structure of space-time is present here without using junction conditions. In the fourth section, we show some of the cosmological applications when considering a correct embedding structure of the space-time. Hence, final remarks are commented in the Conclusion section.

2. On the gravitational constant and Brane-world program

As well known, the gravitational constant in the Newton's Law given by

$$\vec{F} = m\vec{a} = G\frac{mm'}{r^2}\frac{\vec{r}}{r} \ , \tag{1}$$

was introduced to convert the physical dimensions $[M^2]/[L^2]$ to the dimensions of force $[M][L]/[T^2]$. It has the value $G = 6,67 \times 10^{-8}$ $\text{cm}^3/\text{g.sec}^2$, with the same value in a wide range of applications of (1). In 1914, Max Planck suggested a natural units system in which $G = c = \hbar = 1$ and everything else would be measured in centimeters. For that purpose it was assumed that Newton's equation (1) also holds at quantum level. Under this condition, comparing the gravitational energy for $m = m'$ with the quantum energy for a wavelength $\lambda \sim r$, it follows that

$$E = <\vec{F}.\vec{r}> = G\frac{m^2}{\lambda} = \frac{\hbar c}{\lambda} \ .$$

Together with Maxwell equations and the laws of thermodynamics, this leads to three quantities which characterize the so-called Planck regime:

$$m_{pl} = \sqrt{\frac{\hbar c}{G}} \sim 10^{19}\text{Gev}, \ \lambda_{pl} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33}\text{cm}, \ t_{pl} = \sqrt{\frac{\hbar G}{c^5}} \sim 10^{-44}\text{sec}. \tag{2}$$

Planck's conclusion established a landmark in the development of modern physics:

> "These quantities retain their natural significance as long as the law of gravitation and that of the propagation of light in a vacuum and the two principles of thermodynamics remain valid; they therefore must be found always the same, when measured by the most widely different intelligences according to the most different methods" (Planck, 1914)

Today, we can safely say that electrodynamics, actually all known gauge theories, and the laws of thermodynamics remain solid. However, the validity of Newton's law at 10^{-33}cm has not been experimentally confirmed. It has been recently shown to hold at 10^{-3}cm, but with strong hints that it breaks down at 10^{-4}cm (Decca et al., 2007). It should be noted also that the constant G is valid for the Newtonian space-time which has the product topology $\Sigma_3 \times \mathbb{R}$, where Σ_3 denotes the 3-dimensional simultaneity sections, implying that the gravitational constant has the physical dimensions $[G] = [L]^3/[M][T]^2$, appropriate for 3-dimensional manifolds only.

In 1916, Newton's gravitational law changed dramatically to General Relativity, including the principles of equivalence, the general covariance and Einstein's equations in a 4-dimensional space-time

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \ . \tag{3}$$

The Newtonian gravitational constant G, was retained in (3), to guarantee that the theory would reproduce the Newtonian theory in its weak field limit, without the need to change constants. However, the consequences of this are quite embarrassing: indeed, the maintenance of G in (3) originates the hierarchy problem of the fundamental interactions. While all relativistic gauge interactions are quantized at the Tev scales of energies, gravitation would be quantized only at $\sim 10^{19}$Gev, which, as we have seen, coincide with the level predicted by Planck for Newtonian quantum gravity which is the weak field limit of General

Relativity. Furthermore, the relativistic quantum gravitational theory compatible with the physical dimensions of G would be defined only in a 3-dimensional foliation of the space-time, as originally conceived by Dirac (Dirac, 1959), Arnowitt, Deser and Misner (Arnowitt et al., 1962). However, such foliation is not consistent with the diffeomorphism invariance of General Relativity (Kuchar, 1992).
The criticism on the validity of Planck's regime for quantum gravity is the basis of the brane-world program by Arkani-Hamed, G. Dvali and S. Dimopolous (ADD for short) (Arkani-Hamed et al., 1998) proposing a solution of the hierarchy problem of the two fundamental energy scales in nature, namely, the electroweak and Planck scales $[M_{Pl}/m_{EW} \sim 10^{16}]$ (Carter, 2001). It contains essentially three fundamental postulates:

1. the space-time or brane-world is an embedded differentiable sub manifold of another space (the bulk) whose geometry is defined by the Einstein-Hilbert action (therefore this should not be confused with the "brane" of string/M-theory);
2. all gauge interactions are confined to the four-dimensional brane-world (this is a consequence of the poincaré symmetry of the electromagnetic field and in general of the dualities of yang-mills fields, which are consistent in four-dimensional space-time only);
3. gravitation is defined by Einstein's equations for the bulk, propagating along the extra dimensions at Tev energy scale.

It follows from (2) that all ordinary matter fields interacting with gauge fields must also be confined to the same space-time; the original ADD paper refers to graviton probes to the extra dimensions, but classically it means that the bulk is locally foliated by a family brane-world sub-manifolds, whose metric depend on the extra-dimensional coordinates in the bulk.
The impact of such program in theoretical and observational cosmology has been discussed at length as, e.g., in Refs. (Randall, 1999, a;b; Dvali, 2000; Sahni, 2002; 2003; Shiromizu, 2000; Dick, 2001; Hogan, 2001; Deffayet, 2002; Alcaniz, 2002; Jain et al., 2002; Lue, 2006). For instance, concerning the dark matter problem, just like the gravitational field of ordinary matter, dark matter gravity could also propagate in the bulk and in principle should be derived from the same bulk gravitational equations. When considering the acceleration expansion problem, modifications of gravity at very large scales also have been regarded as an alternative route to deal with the accelerated expansion of the universe, often described by something called dark energy. That route in turn has been predominantly associated with the existence of extra-dimensions which a modified friedman's equation can be obtained and provide the correct acceleration expansion.
Some popular brane-world models use Strings/M-theory motivations and use additional postulates such as a z_2 symmetry across the brane-world (or d-brane-world) as in the Randall-Sundrum models (Randall, 1999, b). This symmetry was not considered here essentially because the z_2 symmetry breaks the regularity of the embedding, thus preventing the use of the perturbation mechanism which is the essential feature in our arguments.
To be free from these limitations we require a model independent formulation based on the perturbational theory of embedded submanifolds as stated in (Maia et al., 2005; 2007), rather than particular junction conditions that we discuss more details in the next section.

3. The embedding problem

The embedding of a manifold into another is a non-trivial problem and has its roots in the classic problem in differential geometry, originated in the early days of the Riemannian

geometry. The curvature tensor defined by Riemann can describe the local shape of a Riemannian manifold only up to the condition that it does not “stretch".
Reviewing the concept, given a basis $\{e_\mu\}$ the Riemann tensor describes the curvature of a manifold by displacing a vector field e_ρ along a closed parallelogram defined by e_μ and e_ν and comparing the result with the original vector obtaining:

$$R(e_\mu, e_\nu)e_\rho = R_{\mu\nu\rho\sigma}e^\sigma = [\nabla_\mu, \nabla_\rho]e^\sigma \ .$$

When the difference is zero, the manifold is said to be flat. Such Riemannian flat space is not necessarily equal to a flat space in Euclidean geometry. For instance, it could likewise be a cylinder or a helicoid. After Riemann conceptualized a manifold intrinsically, the question if the geometry of a Riemannian manifold has the same geometry of a manifold embedded in an Euclidean soon arose. Today we know that every Riemannian manifold defined intrinsically can be embedded isometrically, locally or globally, in a Euclidean space with *appropriate* dimensions (Odon, 2010).
Nonetheless, the existence of a background geometry is necessary to fix the ambiguity of the Riemann curvature of a given manifold, without a reference structure. General Relativity solves this ambiguity problem by specifying that the tangent Minkowski space is a flat plane, as decided by the Poincaré symmetry, and not by the Riemann geometry itself. The same space-time is chosen as the ground state for the gravitational field, where particles and quantum field are defined. This choice would be fine, were not for the experimental evidences of a small but non-zero cosmological constant. Since the presence of this constant is not compatible with the Minkowski space-time, we face a conflicting situation: Either we define particles, quantum fields and their vacua states in the Minkowski space-time using the Poincaré group, or else these properties should be defined in a De Sitter space-time using the De Sitter group (Maia et al., 2009). The cosmological constant and the vacuum energy density based on the Poincaré symmetry cannot be present simultaneously in Einstein’s equations, without bringing up the current cosmological constant issue.
The ambiguity of the curvature tensor was known by Riemann himself, when he acknowledged that his curvature tensor defines a class of objects and not just one (Riemann, 1854). This is explicit in Riemann’s words when he states “*by considering arbitrary bendings -without stretching*” of such surfaces which are equivalent to a plane due to the lines on the surfaces remain unaltered even when bending. It imposes a serious constraint on the dynamics of the geometry itself. This means that the Riemann curvature has a degree of ambiguity, characterizing classes of equivalence of manifolds which would otherwise have different shapes or topologies where it cannot evolve nor stretch. In particular, there are infinite many flat Riemannian manifolds, all with zero Riemann curvature, but with different shapes.
A solution of such ambiguity was conjectured by L. Schlaefli in 1871, proposing that *all Riemannian manifolds must be embedded in a larger space*, so that the components of the extrinsic curvature may decide the difference between two Riemann-flat geometries (Schlaefli, 1873). However, the embedding depend on the solution of the Gauss-Codazzi-Ricci equations, involving the metric, the extrinsic curvature and the third fundamental form as independent variables. They provide the necessary and sufficient conditions for the existence of the embedded manifold (Eisenhart, 1966). Until recently those equations could be solved only with the help of positive power series expansions of the embedding functions (that is, they must be analytic functions), and so each embedding had to be examined separately.

The proof that all differentiable Riemannian manifolds can be embedded in a space with sufficient number of dimensions using exclusively smooth functions was given by Nash (Nash, 1956) in 1956, when he introduced the notion of smoothing operators in Riemannian geometry, leading to the geometric flow condition

$$k_{\mu\nu} = -\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial y} \tag{4}$$

where $k_{\mu\nu}$ denotes the extrinsic curvature and y represents a coordinate on a direction orthogonal to the embedded geometry.
In the following we derive the condition (4) in the simple case of just one extra dimension. Higher dimensional cases were also implicit in Nash's paper and this was applied as a possible extension of the ADM quantization of the gravitational field (Maia et al., 2007).

4. Geometric flow

Consider a Riemannian manifold $\bar{V}_n$ with metric $\bar{g}_{\mu\nu}$, and its local isometric embedding in a D-dimensional Riemannian manifold V_D, $D = n+1$, given by a differentiable and regular map $\mathcal{X} : \bar{V}_n \to V_D$ satisfying the embedding

$$g_{\mu\nu} = \mathcal{G}_{AB}\mathcal{X}^A_{,\mu}\mathcal{X}^B_{,\nu};\ \mathcal{G}_{AB}\mathcal{X}^A_{,\mu}\eta^B_b = 0;\ \mathcal{G}_{AB}\eta^A_a\eta^B_b = g_{ab} = \pm\delta_{ab}\ . \tag{5}$$

where we have denoted by $\mathcal{G}_{AB}$ the metric components of V_D in arbitrary coordinates, and where $\bar{\eta}$ denotes the unit vector field orthogonal to $\bar{V}_n$. The extrinsic curvature of $\bar{V}_n$ is by definition the projection of the variation of η on the tangent plane (Eisenhart, 1966)

$$\bar{k}_{\mu\nu} = -\mathcal{X}^A_{,\mu}\bar{\eta}^B_{,\nu}\mathcal{G}_{AB} = \mathcal{X}^A_{,\mu\nu}\bar{\eta}^B\mathcal{G}_{AB}\ . \tag{6}$$

The integration of the system of equations gives the required embedding map $\mathcal{X}$.
In order to understand the meaning of the extrinsic curvature, construct the one-parameter group of diffeomorphisms defined by the map $h_y(p) : V_D \to V_D$, describing a continuous curve $\alpha(y) = h_y(p)$, passing through the point $p \in \bar{V}_n$, with unit normal vector $\alpha'(p) = \eta(p)$ (Crampin, 1986). The group is characterized by the composition $h_y \circ h_{\pm y'}(p) \overset{def}{=} h_{y\pm y'}(p)$, $h_0(p) \overset{def}{=} p$. Applying this diffeomorphisms to all points of a small neighborhood of p, we obtain a congruence of curves (or orbits) orthogonal to $\bar{V}_n$. It does not matter if the parameter y is time-like or not, nor if it is positive or negative.
Given a geometric object $\bar{\omega}$ in $\bar{V}_n$, its Lie transport along the flow for a small distance δy is given by $\Omega = \bar{\Omega} + \delta y \pounds_\eta \bar{\Omega}$, where $\pounds_\eta$ denotes the Lie derivative with respect to η Crampin (1986). In particular, the Lie transport of the Gaussian frame $\{\mathcal{X}^A_\mu, \bar{\eta}^A_a\}$, defined on $\bar{V}_n$ gives

$$\mathcal{Z}^A_{,\mu} = \mathcal{X}^A_{,\mu} + \delta y\ \pounds_\eta \mathcal{X}^A_{,\mu} = \mathcal{X}^A_{,\mu} + \delta y\ \eta^A_{,\mu} \tag{7}$$

$$\eta^A = \bar{\eta}^A + \delta y\ [\bar{\eta}, \bar{\eta}]^A \quad = \quad \bar{\eta}^A \tag{8}$$

However, from (6) we note that in general $\eta_{,\mu} \neq \bar{\eta}_{,\mu}$.
It is important to note that the set of coordinates $\mathcal{Z}^A$ obtained by integrating these equations *does not necessarily describe another manifold*. In order to be so, they need to satisfy embedding equations similar to (5):

$$\mathcal{Z}^A_{,\mu}\mathcal{Z}^B_{,\nu}\mathcal{G}_{AB} = g_{\mu\nu},\ \mathcal{Z}^A_{,\mu}\eta^B\mathcal{G}_{AB} = 0,\ \eta^A\eta^B\mathcal{G}_{AB} = 1\ . \tag{9}$$

Replacing (7) and (8) in (9) and using the definition (6) we obtain the metric and the extrinsic curvature of the new manifold

$$g_{\mu\nu} = \bar{g}_{\mu\nu} - 2y\bar{k}_{\mu\nu} + y^2\bar{g}^{\rho\sigma}\bar{k}_{\mu\rho}\bar{k}_{\nu\sigma} \quad (10)$$
$$k_{\mu\nu} = \bar{k}_{\mu\nu} - 2y\bar{g}^{\rho\sigma}\bar{k}_{\mu\rho}\bar{k}_{\nu\sigma} \; . \quad (11)$$

Taking the derivative of (10) with respect to y we obtain Nash's deformation condition (4). The analogy of geometry with fluid flows is similar but different from the Ricci flow proposed by R. Hamilton using the caloric fluid and Fourier's heat flux to obtain the expression

$$R_{\mu\nu} = -\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial y}$$

that resembles (4) (Hamilton, 1982). This result was subsequently applied with enormous success by G. Perelman to solve the Poincaré conjecture (Perelman, 2002). Unfortunately the Ricci-flow is not relativistic and it is not compatible with Einstein's equations or with relativistic cosmology.

The equations (9) need to be integrated so define a new manifold. The integrability conditions for these equations are given by the non-trivial components of the Riemann tensor of the embedding space[1], expressed in the frame $\{\mathcal{Z}^A_\mu, \eta^A\}$ as

$$^5\mathcal{R}_{ABCD}Z^A{}_{,\alpha}Z^B{}_{,\beta}Z^C{}_{,\gamma}Z^D{}_{,\delta} = R_{\alpha\beta\gamma\delta} + (k_{\alpha\gamma}k_{\beta\delta} - k_{\alpha\delta}k_{\beta\gamma}) \quad (12)$$
$$^5\mathcal{R}_{ABCD}Z^A{}_{,\alpha}Z^B{}_{,\beta}Z^C{}_{,\gamma}\eta^D = k_{\alpha[\beta;\gamma]} \quad (13)$$

These are the mentioned Gauss-Codazzi equations (the third equation -the Ricci equation- does not appear in the case of just one extra dimension) (Eisenhart, 1966). The first of these equation (Gauss) shows that the Riemann curvature of the embedding space acts as a reference for the Riemann curvature of the embedded space-time. Both Riemann curvatures are ambiguous in the sense described by Riemann, but Gauss' equation (12) shows that their difference is given by the extrinsic curvature, completing the proof of the Schlaefli embedding conjecture by use of Nash's deformation condition (4). The second equation (Codazzi) complements this interpretation, stating that the projection of the Riemann tensor of the embedding space along the normal direction is given by the tangent variation of the extrinsic curvature.

Equations (10) and (11) describe the metric and extrinsic curvature of the deformed geometry V_4. By varying y they describe a continuous sequence of deformations in the the embedding space. The existence of these deformations are given by the integrability conditions (12) and (13) which are therefore not dynamical equations.

As in Kaluza-Klein and in the brane-world theories, the embedding space V_5 has a metric geometry defined by the higher-dimensional Einstein's equations

$$^5\mathcal{R}_{AB} - \frac{1}{2}{}^5\mathcal{R}\mathcal{G}_{AB} = G_* T^*_{AB} \; . \quad (14)$$

where G_* is the new gravitational constant and where T^*_{AB} denotes the components of the energy-momentum tensor of the known gauge fields and material sources. From these

[1] To avoid confusion with the four dimensional Riemann tensor $R_{\alpha\beta\gamma\delta}$, the five-dimensional Riemann tensor is denoted by $^5\mathcal{R}_{ABCD}$. The extrinsic curvature terms in these equations follows from the five-dimensional Christoffel symbols together with the use of (4).

dynamical equations we may derive the gravitational field in the embedded space-times. Taking the tangent, vector and scalar components[2] of (14) and using the previous confinement conditions (19) one can obtain

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - Q_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{15}$$

$$k^{\rho}_{\mu;\rho} - h_{,\mu} = 0 \; , \tag{16}$$

where the term $Q_{\mu\nu}$ in the first equation results from the expression of R_{AB} in (14), involving the orthogonal and mixed components of the Christoffel symbols for the metric $\mathcal{G}_{AB}$. Explicitly this new term is

$$Q_{\mu\nu} = g^{\rho\sigma}k_{\mu\rho}k_{\nu\sigma} - k_{\mu\nu}h - \frac{1}{2}\left(K^2 - h^2\right)g_{\mu\nu} \; , \tag{17}$$

where $h^2 = g^{\mu\nu}k_{\mu\nu}$ is the squared mean curvature and $K^2 = k^{\mu\nu}k_{\mu\nu}$ is the squared Gauss curvature. This quantity is therefore entirely geometrical and it is conserved in the sense of

$$Q^{\mu\nu}{}_{;\nu} = 0 \; . \tag{18}$$

Therefore we may derive observable effects associated with the extrinsic curvature capable to be seen by four-dimensional observers in space-times.

With all these tools at hand, modern Cosmology has been investigated and represents an important source of data that can provide a deeper comprehension of the gravitational structure and evolution of the universe. Not only this, but it calls for new gravitational theories far beyond Einstein's approach. Even though we are long way from a concrete fully-developed theory, dark matter and dark energy play a major role on this quest, representing fundamental constraints to these new gravitational models. It is also important to make the following observations:

1) A cosmological constant was not included in the equation for the higher dimensional space V_5 in (14), so that the cosmological constant problem does not appear. With this choice we also ensure the existence of an embedded 4-dimensional Minkowski space-time (a cosmological constant was included in (Maia et al., 2005), but here we see no reason for it).

2) In contrast with the extra dimensional perturbative behaviour of the gravitational field, all gauge fields of the standard model remain confined to the four-dimensional space-time. This is a direct consequence of the gauge field structure. Just as a reminder, the Yang- Mills equations can be written as $D \wedge F = 0$, $D \wedge F^* = 4\pi J^*$, where $F = F_{\mu\nu}dx^\mu \wedge dx^\nu$, $F_{\rho\sigma} = [D_\rho, D_\sigma]$, $D_\mu = I\partial_\mu + A_\mu$, $F^* = F^*_{\mu\nu}dx^\mu \wedge dx^\nu$ and $F^*_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{*\,\rho\sigma}$. The duality operation $F \to F^*$ requires the existence of an isomorphism between 3-forms and 1-forms, which can only be realized in a four dimensional space-time manifold. Therefore, the confinement of gauge fields, matter and vacuum states is a property that is independent of the perturbation of the brane-world geometry.

There are two relevant consequences of the confinement. In the first place, it implies that all ordinary matter which interacts with the gauge fields, and also the vacuum states and its energy-momentum tensor associated with the confined fields also remain confined to the four-dimensional brane-world. Secondly, the diffeomorphism invariance of General

[2] The third gravitational equation was omitted here due to the fact that it vanishes in 5-D, but when the higher dimensional space-time is considered, one can obtain the equation $R - (K^2 - H^2) + \mathcal{R}(D-5) = 0$, sometimes called gravitational scalar equation.

Relativity cannot apply to the bulk manifold V_D, for it would imply in breaking the confinement. Of course, such limitation could be fixed by applying a coordinate gauge, but then we will be imposing a modification to Nash's theorem. Nash's theorem demands the embedding to be differentiable and regular, so that there is a 4×4 non-singular sub-matrix of the Jacobian determinant of the embedding map, thus guaranteeing the diffeomorphism invariance in the four-dimensional embedded submanifold only. Admitting that the original (on-embedded) space-time is a solution of Einstein's equations, the gauge fields, matter and its vacuum states keep a 1 : 1 correspondence with the source fields in the embedded space-time structure. Consequently, the confinement can be generally set as a condition on the embedding map such that

$$8\pi G T_{\mu\nu} = G_* Z^A_{,\mu} Z^B_{,\nu} T^*_{AB}, \quad Z^A_{,\mu} \eta^B T^*_{AB} = 0, \quad \text{and} \quad \eta^A \eta^B T^*_{AB} = 0 \tag{19}$$

3) Einstein's equations can be written as

$$^5\mathcal{R}_{AB} = G_* \left(T^*_{AB} - \frac{1}{3} T^* \mathcal{G}_{AB} \right)$$

The tensor $^5\mathcal{R}_{AB}$ may be evaluated in the embedded space-times by contracting it with $Z^A_{,\mu}$, $Z^B_{,\nu}$, $Z^A_{\mu}\eta^B$ and $\eta^A\eta^B$. Using (4), (9) and the confinement conditions (19), Einstein's equations become

$$^5\mathcal{R}_{\mu\nu} = R_{\mu\nu} + \frac{\partial k_{\mu\nu}}{\partial y} - 2k_{\mu\rho}k^{\rho}_{\nu} + hh_{\mu\nu} \tag{20}$$

$$^5\mathcal{R}_{\mu 5} = k^{\rho}_{\mu;\rho} + \frac{\partial \Gamma^{\rho}_{\mu 5}}{\partial y} \tag{21}$$

It follows that the Israel-Lanczos condition does not follow from Einstein's equations (3) by themselves. It becomes necessary that the embedded geometry does satisfy particular conditions such that the Ricci curvature of the embedding space coincide with the extrinsic curvature of the embedded space-time, that is $^5\mathcal{R}_{\mu\nu} = k_{\mu\nu}$, which is not generally true. One of these conditions is that the embedded space-time acts as a mirror boundary between two regions of the embedding space (see e.g. (Israel, 1966)). In this case we may evaluate the difference of $^5\mathcal{R}_{\mu\nu}$ from both sides of the space-times and the above mentioned boundary condition holds. However, in doing so the deformation given by (4) ceases to be. Therefore, to find the deformations caused by the extrinsic curvature, such special conditions are not applied and they are not needed. To make it clear how it works, one can first take (14) and contracting with the metric $\mathcal{G}^{AB}$ and using the confinement conditions in (19) and (14), one can find

$$\mathcal{R} = -\frac{2}{3}\alpha_* T^* , \tag{22}$$

and also

$$\mathcal{R}_{AB} = \alpha_* \left(T^*_{AB} - \frac{1}{3} T^* \mathcal{G}_{AB} \right) , \tag{23}$$

where the components can be obtained in the Gaussian frame $\{Z^A_{,\mu}, \eta^A\}$. Hence, we have

$$\mathcal{R}_{AB} Z^A_{,\mu} Z^B_{,\nu} = \alpha_* \left(T^*_{AB} - \frac{1}{3} T^* \mathcal{G}_{AB} \right) Z^A_{,\mu} Z^B_{,\nu} = \alpha_* \left(T^*_{\mu\nu} - \frac{1}{3} T^* g_{\mu\nu} \right) .$$

As we can see, the right side of the previous equation is the same expression as that verified in the IDL condition which must coincide with the extrinsic curvature in the brane-world. However, this is not true inasmuch as the left side of the equation is the contracted form of Gauss equations. We may check it writing the components in the Gaussian frame of (14) and obtain (15). As a consequence of Gauss-Codazzi-Ricci equations, in the higher dimensional space-time structure, the direct contraction of the Ricci equation gives

$$R = \mathcal{R} - (K^2 + H^2) + 2\frac{\partial h}{\partial y} , \tag{24}$$

where $\mathcal{R}_{AB}\eta^A\eta^B = \frac{\partial h}{\partial y} + K^2$.
Taking (22) and (24), and applying in (15), one can find

$$R_{\mu\nu} - \frac{\partial k_{\mu\nu}}{\partial y} - 2k^{\rho}_{\mu}k_{\rho\nu} + hk_{\mu\nu} = \alpha_* \left(T^*_{\mu\nu} - \frac{1}{3}T^* g_{\mu\nu} \right) . \tag{25}$$

In fact, it shows that the IDL condition only can be obtained by imposing some serious constraints on the embedding process. Still, if we want to insist on obtaining the IDL condition, we must assume some simplifying conditions. Let the brane-world has a boundary such that it separated into two sides labeled (+) and (-) regions. The difference calculated in each side of the brane-world is zero when $y \to 0$. In other words, we have the same equation obtained in (25) the more we approach $y = 0$ from each side inasmuch as there is not a effective distinction in the riemannian geometry when evaluated from each side to the other. This situation turns to be quite different when the Z_2 is considered. In this case, the extrinsic curvature (or any object that could access extra-dimensions) has its image mirrored in the brane-world (which acts as a mirror). For instance, if we have $k^+_{\mu\nu} = -k^-_{\mu\nu}$, the derivatives $\left[-\left(\frac{\partial k_{\mu\nu}}{\partial y} \right) \right] = \alpha_* \left(T^*_{\mu\nu} - \frac{1}{3}T^* g_{\mu\nu} \right)$ constantly change when they approach $y \to 0$. By using the mean value theorem in the interval $[-y, y]$, we can evaluate the difference between both sides and obtain

$$\left[-\left(\frac{\partial k_{\mu\nu}}{\partial y} \right) \right] = \frac{-k^+_{\mu\nu} + k^-_{\mu\nu}}{y}.$$

Denoting $[X] = X^+ - X^-$ and $X = \bar{X}(x)\delta(y)$, we have

$$y[X] = \int_{-y}^{y} \frac{d}{d\xi}(|\xi|X)d\xi = \int_{-y}^{y} \frac{\partial|\xi|}{\partial\xi} X d\xi + \int_{-y}^{y} |\xi| \frac{dX}{d\xi} d\xi$$

$$= \int_{-y}^{y} \frac{\partial|\xi|}{\partial\xi} \bar{X}\delta(\xi)d\xi + \int_{-y}^{y} |\xi| \frac{\partial\delta(\xi)}{\partial\xi} \bar{X} d\xi = 2\bar{X} .$$

In the case that $[X] = \alpha_* \left(T^*_{\mu\nu} - \frac{1}{3}T^* g_{\mu\nu} \right)$, we obtain Lanczos equation

$$k^+_{\mu\nu} - k^-_{\mu\nu} = -2\alpha_* \left(T^*_{\mu\nu} - \frac{1}{3}T^* g_{\mu\nu} \right) , \tag{26}$$

that describes the jump of the extrinsic curvature in the background separation point $y = 0$. Hence, the IDL condition is obtained when the Z_2 symmetry is applied to (26) obtaining

$$k_{\mu\nu} = \alpha_* \left(T^*_{\mu\nu} - \frac{1}{3}T^* g_{\mu\nu} \right) . \tag{27}$$

The use of Z_2 symmetry induces a serious constraint on the embedding differentiable structure. Once a perturbation occurs in a point of the background it is mirrored in the brane-world background and two tangent vectors on each side can be defined. The projections of these vectors point in opposite directions which means that the embedding differentiable functions cannot be properly defined (Maia, 2004).

In summary, the theoretical scheme presented here are consequence of a fundamental perturbational process stated by Nash's embedding theorem. Nash's perturbation method innovates in two basic aspects: first, there is no need to apply the restrictive convergent series power of analytical function hypothesis to make an embedding between two manifolds. Secondly, the perturbational nature of the process we can obtain dynamical equations as well as integrating them such as in Cauchy's problem in Mechanics and it also gives a prescription on how to construct geometrical structures by deforming simpler ones. It seems that this geometric perturbation process has to do with the formation of structures in the early universe. When Nash's theorem is applied to physics, it provides a general mathematical tool appropriated to the brane-world program. In the model independent covariant formulation the extrinsic curvature appears as an independent symmetric tensor field which evolves together with the brane-world dynamics. Interestingly, the presence of the independent symmetric rank-two tensor field has been considered long before the observation of the accelerated expansion of the universe under different motivations and circumstances as a possible repulsive gravitational field (Isham et al., 1971).

5. Cosmological applications

After all these geometrical considerations, in the following we summarize important ideas of works on the applications of Nash's theorem to Cosmology as seen in (Maia et al., 2009; 2005; Odon, 2010; Capistrano, 2010). The first step to do is to defined the background geometry. The standard Friedman-Lemaître-Robertson-Walker(FLRW) model is sufficiently simple to make it locally embedded in a 5-dimensional flat space, satisfying Nash's differentiable conditions. Therefore, it can be taken as a background cosmology, which can be deformed along the fifth-dimension. However, here the effects of the extrinsic geometry are shown in the FLWR background only (that is without perturbations).

5.1 The Cosmological Constant problem

The so-called *Cosmological Constant problem* had its first seeds planted in 1916, with the ideas of Nernst (Nernst, 1916). He studied the non-vanishing vacuum energy density that was fulfilled with radiation-only content, which was confirmed by the Casimir effect in 1948 (Casimir, 1948; Mostepanenko, 1997; Jaffe, 2005). In late 1920's, Pauli (Pauli, 1933; Straumann, 2002; Rugh, 2002) made studies about the gravitational influence of the vacuum energy density of the radiation field, suggesting a conflict between the vacuum energy density and gravitation. If vacuum energy density is considered, then gravity should be dispensed. Intriguingly, the conflicting Pauli's results passed unnoticed by scientific community. Only on subsequent decades, the observations of quasars in the mid-late of the 1960's suggested the reconsideration of Λ (Petrosian, 1974).

Here we refer to the cosmological constant problem described in (Weinberg, 1989). Using the semiclassical Einstein's equations in General Relativity the quantum vacuum can be described as a perfect fluid with state equation $p_v = -<\rho>_v=$ constant (Zel'dovich, 1967):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T^m_{\mu\nu} + 8\pi G <\rho>_v g_{\mu\nu} , \tag{28}$$

where $T^{m}_{\mu\nu}$ stands for the classical sources. Comparing the constant terms in both sides of this equation we obtain $\Lambda/8\pi G =< \rho >$, or as it is commonly stated, the cosmological constant is the vacuum energy density. However, current observations tell that $\Lambda/8\pi G \sim 10{-}47(\text{Gev})^4$ (here, $c = 1$). On the other hand, admitting that quantum field theory holds up to the Planck scale, the vacuum energy density would be $< \rho >_v \sim (10^{19}\text{Gev})^4 = 10^{76}(\text{Gev})^4$. This difference cannot be resolved by any known theoretical procedure in quantum field theory. Even supposing that quantum field theory holds to the Tev scale or less, the difference would be still too large to compensate. This difficulty has become to known as the cosmological constant problem.

In one proposal to solve this problem, a scalar field is added to the right hand side of Einstein's equations, such that it adjusts the difference between the two constants (Chen & Wu, 1990; Waga, 1993; Caldwell & Linder, 2005; Lima, 2004; Padmanabham, 2007). Of course, this scalar field must also agree the other cosmological conditions, such as the structure formation, the past and present inflationary periods, and the smooth transition to and from the standard cosmology period. The adjustments of this field to such conditions have proven to be not so simple. A more geometrical approach to the problem, the Einstein-Hilbert action principle has been tentatively modified, using for example higher derivative Lagrangians, or more generally a Lagrangean defined by an arbitrary function of the Ricci curvature, in the so called f(R) theories (Capozziello et.al., 1998). However, it becomes a necessity to give a meaning to the resulting action principle, which is after all a fundamental principle. In comparison, the Einstein-Hilbert principle has a specific meaning, stating that the geometry of the space-time must be as smooth as possible. Furthermore, it comes after Newton's gravitational law, when it is expressed geometrically, so that at the end, it is founded in experimental facts. In this respect, given the arbitrariness of f(R), it is not at all clear that the present astrophysical observations are sufficient to decide on such function (Sokolowski, 2007). Another fine-tuning approach suggests new two fundamental scales (Alfonso-Faus, 2009), the cosmological quantum black hole (CQBH) and the quantum black hole (QBH) in order to solve the ambiguity of Λ in the cosmological problem by using an appropriate choice of parameters, e.g $\hbar \sim 10^{-122}$ that lead from the Planck scale to the Cosmological scale without conflicting with $\Lambda\hbar \sim 1$, instead of using $G = c = \hbar = 1$.

As also suggest in (Alfonso-Faus, 2009), we must emphasize that the previous difference in the cosmological problem is not only numerical, but it is mainly conceptual, resulting from the superposition of two incompatible ground states for the gravitational field in General Relativity: The flat Minkowski ground state was chosen to be the reference of curvature, but the experimental evidences of $\Lambda/8\pi G \neq 0$ however small, point to a De Sitter ground state, which is conceptually incompatible with the Minkowski's choice. The implications being that particles and fields, their masses and spins defined by the Casimir operators of the De Sitter group are different from those defined by the Poincaré group, and they coincide only when Λ vanishes. The above numerical and conceptual conflicts can be resolved with the Schlaefli embedding conjecture as implemented by Nash, where the De Sitter and Minkowski space-times may coexist. Indeed, in (15), $\Lambda/8\pi G$ is a gravitational component resulting from the gravitational equations in the embedding space. However, the vacuum energy density $< \rho >_v$ is a confined quantity in the space-time, regardless of the perturbations of its metric. Finally, the presence of the extrinsic curvature $k_{\mu\nu}$ in the conserved quantity $Q_{\mu\nu}$ of (15), imply that those constants cannot be canceled without imposing a constraint on the extrinsic curvature, which is now part of the gravitational dynamics in the embedding space (Maia et al., 2009; Capistrano & Odon, 2010).

5.2 The accelerated expansion

A interesting situation occurs when Nash's theorem is applied to the Dark energy problem as proposed in (Maia et al., 2005). One of the most known brane-world models is the Randall-Sundrum type II (RSII) (Randall, 1999, b). When applied to Cosmology, the vacuum energy density in a 3-brane is still smaller than the one predicted by quantum field theory, which means that the cosmological constant problem persists, even though the fundamental Tev scale energy is preserved. A similar situation occurs when dealing with the Dark energy problem in which the RS model II provides the following modified Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3m_{pl}^2}\rho + \frac{16\pi^2}{9m_5^6}\rho^2 \ , \tag{29}$$

where m_5 is the 5-dimensional planck scale, m_{pl} is the 4-dimensional planck scale. The correction term corresponds to the square of the energy density ρ^2 of the confined matter (Tujikawa, 2004; Tujikawa et.al., 2004; Maia, 2004). As it is well known, this result is not compatible with recent observational data (Komatsu et.al., 2011; Jarosik et.al., 2011) since the additional term on Friedmann's equation, i.e, the energy density ρ^2, provides a deceleration scenario of the universe, besides affecting the nucleosynthesis of large structures. To remedy this situation, other attempts have been studied, such as particular classes of bulk and brane scalar potentials (Langlois, 2001), notwithstanding they lead to a fine-tuning mechanisms.
In (Maia et al., 2005), the Friedmann-Lemaître-Robertson-Walker (FLRW) line element was embedded in a 5-dimensional space with constant curvature bulk space whose geometry satisfy Einstein's equations with a cosmological constant given by (14). When the equations are written in the Gaussian frame defined by the embedded space-time, we obtain a larger set of gravitational field equations. The general solution of (16) for the FLRW geometry was found to be

$$k_{ij} = \frac{b}{a^2}g_{ij}, \quad i,j = 1,2,3, \quad k_{44} = \frac{-1}{\dot{a}}\frac{d}{dt}\frac{b}{a} \ , \tag{30}$$

where we notice that the function $b(t) = k_{11}$ remains an arbitrary function of time. As a direct consequence of the confinement of the gauge fields, equation (16) is homogeneous, meaning that one component $k_{11} = b(t)$ remains arbitrary. Denoting the Hubble and the extrinsic parameters by $H = \dot{a}/a$ and $B = \dot{b}/b$, respectively, we may write all components of the extrinsic geometry in terms of B/H as follows

$$k_{44} = -\frac{b}{a^2}(\frac{B}{H} - 1)g_{44}, \tag{31}$$

$$K^2 = \frac{b^2}{a^4}\left(\frac{B^2}{H^2} - 2\frac{B}{H} + 4\right), \quad h = \frac{b}{a^2}(\frac{B}{H} + 2) \tag{32}$$

$$Q_{ij} = \frac{b^2}{a^4}\left(2\frac{B}{H} - 1\right)g_{ij}, \qquad Q_{44} = -\frac{3b^2}{a^4}, \tag{33}$$

$$Q = -(K^2 - h^2) = \frac{6b^2}{a^4}\frac{B}{H} \ , \tag{34}$$

Next, by replacing the above results in (15) and applying the conservation laws, we obtain the Friedmann equation modified by the presence of the extrinsic curvature, i.e.,

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{4}{3}\pi G\rho + \frac{\Lambda_*}{3} + \frac{b^2}{a^4} \ . \tag{35}$$

When compared with the phenomenological quintessence phenomenology with constant EoS we have found a very close match with the golden set of cosmological data on the accelerated expansion of the universe.

Notice that we have not used the Israel-Lanczos condition (27) as used in (Randall, 1999, b). If we do so, in the case of the usual perfect fluid matter, then we obtain in (35) a term proportional to ρ^2. It is possible to argue that the above energy-momentum tensor $T_{\mu\nu}$ also include a dark energy component in the energy density ρ. However, in this case we gain nothing because we will be still in darkness concerning the nature of this energy. Finally, as it was shown in the previous section, the Israel-Lanczos condition requires that the four-dimensional space-time behaves like a boundary brane-world, with a mirror symmetry on it, which is not compatible with the regularity condition for local and differentiable embedding.

Therefore, the conclusion from (Maia et al., 2005) is that the extrinsic curvature is a good candidate for the universe accelerator. In the next section we start anew, with a mathematical explanation on why only gravitation access the extra dimensions using the mentioned theorem of Nash on local embeddings, and the geometric properties of spin-2 fields defined on space-times.

5.3 The dynamics of extrinsic curvature

Hitherto, we did not have at the time any previous information on the dynamics of the extrinsic curvature. The only widely accepted relation of that curvature with matter sources is the Israel-Lanczos boundary condition, as applied to the Randall-Sundrum brane-world cosmology. However, this condition fixes once for all the extrinsic curvature, so that it does not follow the dynamics of the brane-world. Thus, a more fundamental explanation for the dynamics of the extrinsic curvature is required. In the purpose of complementing the study shown in (Maia et al., 2005) is to show that the extrinsic curvature behaves as an independent spin-2 field whose effect on the gravitational field is precisely the observed accelerated expansion.

From the theoretical point of view, it would be a satisfactory solution for the dark energy problem if the $b(t)$ (35) function was a unique solution, but, in fact, it depends on a choice of a family of solutions for the extrinsic curvature induced by the homogeneity of the Codazzi equation (16) which is well-known equation in differential geometry. Thus, to be free from these pathologies a proper mechanism or an additional dynamical equation for extrinsic curvature should be implemented. In spite of Brane-world models get some attention on recent years due to several options for dark energy, their mechanisms are still not completely understood or justified. These are mostly based on specific models using special conditions. For such large scale phenomenology as the expansion of the universe, a general theory based on fundamental principles and on solid mathematical foundations is still lacking.

Another aspect of Nash's theorem is that the extrinsic curvature are the generator of the perturbations of the gravitational field along the extra dimensions. The symmetric rank-2 tensor structure of the extrinsic curvature lends the physical interpretation of an independent spin-2 field on the embedded space-time. The study of linear massless spin-2 fields in Minkowski space-time dates back to late 1930s (Pauli, 1939). Some years later, Gupta (Gupta, 1954) noted that the Fierz-Pauli equation has a remarkable resemblance with the linear approximation of Einstein's equations for the gravitational field, suggesting that such equation could be just the linear approximation of a more general, non-linear equation for massless spin-2 fields. In reality, he also found that any spin-2 field in Minkowski space-time

must satisfy an equation that has the same formal structure as Einstein's equations. This amounts to saying that, in the same way as Einstein's equations can be obtained by an infinite sequence of infinitesimal perturbations of the linear gravitational equation, it is possible to obtain a non-linear equation for any spin-2 field by applying an infinite sequence of infinitesimal perturbations to the Fierz-Pauli equations. The result is an Einstein-like system of equations, the Gupta equations (Gupta, 1954; Fronsdal, 1978).
In order to write the Gupta equations for the extrinsic curvature $k_{\mu\nu}$ of an embedded Riemannian geometry with metric $g_{\mu\nu}$, we may use an analogy with the derivation of the Riemann tensor, defining the "connection" associated with $k_{\mu\nu}$ and then the corresponding Riemann tensor, but keeping in mind that the geometry of the embedded space-time is already defined by the metric tensor $g_{\mu\nu}$. Let us define the tensor

$$f_{\mu\nu} = \frac{2}{K}k_{\mu\nu}, \text{ and } f^{\mu\nu} = \frac{2}{K}k^{\mu\nu} , \tag{36}$$

so that $f^{\mu\rho}f_{\rho\nu} = \delta^{\mu}_{\nu}$. Subsequently, we construct the "Levi-Civita connection" associated with $f_{\mu\nu}$, based on the analogy with the "metricity condition". Let us denote by $||$ the covariant derivative with respect to $f_{\mu\nu}$ (while keeping the usual $(;)$ notation for the covariant derivative with respect to $g_{\mu\nu}$), so that $f_{\mu\nu||\rho} = 0$. With this condition we obtain the "f-connection"

$$\Upsilon_{\mu\nu\sigma} = \frac{1}{2}\left(\partial_\mu f_{\sigma\nu} + \partial_\nu f_{\sigma\mu} - \partial_\sigma f_{\mu\nu}\right)$$

and

$$\Upsilon_{\mu\nu}{}^{\lambda} = f^{\lambda\sigma}\,\Upsilon_{\mu\nu\sigma}$$

The "f-Riemann tensor" associated with this f-connection is

$$\mathcal{F}_{\nu\alpha\lambda\mu} = \partial_\alpha \Upsilon_{\mu\lambda\nu} - \partial_\lambda \Upsilon_{\mu\alpha\nu} + \Upsilon_{\alpha\sigma\mu}\Upsilon^{\sigma}_{\lambda\nu} - \Upsilon_{\lambda\sigma\mu}\Upsilon^{\sigma}_{\alpha\nu}$$

and the "f-Ricci tensor" and the "f-Ricci scalar", defined with $f_{\mu\nu}$ are, respectively,

$$\mathcal{F}_{\mu\nu} = f^{\alpha\lambda}\mathcal{F}_{\nu\alpha\lambda\mu} \text{ and } \mathcal{F} = f^{\mu\nu}\mathcal{F}_{\mu\nu}$$

Finally, write the Gupta equations for the $f_{\mu\nu}$ field

$$\mathcal{F}_{\mu\nu} - \frac{1}{2}\mathcal{F}f_{\mu\nu} = \alpha_f \tau_{\mu\nu} \tag{37}$$

where $\tau_{\mu\nu}$ stands for the source of the f-field, with coupling constant α_f. Note that the above equation can be derived from the action

$$\delta \int \mathcal{F}\sqrt{|f|}dv$$

Note also that, unlike the case of Einstein's equations, here we have not the equivalent to the Newtonian weak field limit, so that we cannot tell about the nature of the source term $\tau_{\mu\nu}$. For this reason, we start with the simplest Ricci-flat-like equation for $f_{\mu\nu}$, i.e.,

$$\mathcal{F}_{\mu\nu} = 0 \,. \tag{38}$$

For simplicity, the equations were written in 5-d but it remains valid for a higher dimensional bulk. With this new set of equations, in principle the homogeneity of Codazzi equations can be lift. The work on Gupta's theorem is currently on progress and applications to the Dark energy problem have been recently investigated. A more detailed discussion can be found in (Maia et.al., 2011; Capistrano, 2010)

5.4 Local gravity and structure formation

Current local dark matter observations based on gravitational micro-lensing, optical and x-ray astronomical observations tell that the local dark matter phenomenology is different from that in cosmology. In fact, there is no evidence that the same structure formation caused by geometric perturbations similar to the cosmological situation is still present around the already formed structures, at least at the same rate. Gravitational lensing evidences a gravitational field with a certain metric symmetry. In some cases the dark matter gravitational field is anchored to an observed structure (spiral galaxies, gravitational halos in clusters etc.) and its metric symmetry is the same as that of the observed structure. Until very recently these observations indicated that the source of the local dark matter gravitation (that is, the dark matter itself) was usually attached to galaxies and clusters. In other cases, as in the example of the Abell 520 cluster (MS0451+02), the dark matter gravitational field seems to be away from any baryon substructures. Another recent evidence of the local dark matter gravity is observed through x-ray astronomy in near colliding clusters (exemplified by the bullet cluster 1E0657-558). The observed effect is the formation of a sonic bullet-like substructure moving through the intercluster plasma, long before the cluster themselves collide. This is attributed to the collision of the real dark matter halos assumed to be around the colliding clusters. Admitting Newtonian gravity, the center of mass of the moving object coincide with the Newtonian halos. Such wide range of experimental evidences from cosmology to local gravity suggests the necessity of a comprehensive analysis of the dark matter gravitational field *per se*, regardless of any other attributes that dark matter may eventually possess. Therefore, it is possible that the theoretical power spectrum obtained from (35) coincide with the observed one. In a preliminary analysis, we obtained a power spectrum which is similar to the power spectrum from the cosmic microwave background radiation obtained from the WMAP experiment. On the other hand, Nash's geometric perturbations may be present as a local

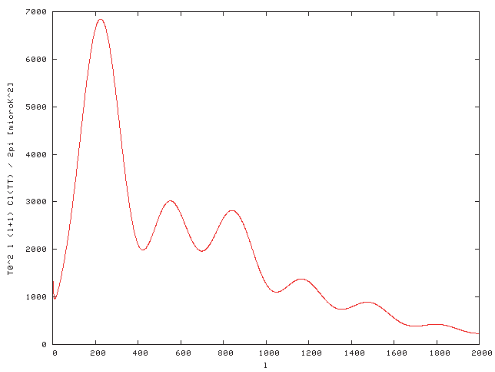

Fig. 1. The theoretical power spectrum calculated with the CAMB for $-1 \leq \omega_0 \leq -1/3$, Massive Neutrinos=1, massless neutrinos =3.04.

process, as for example in young galaxies and in cluster collisions. However, in most other cases there are not sufficient experimental evidences that it is still going on. The formation of large structures in the early universe has been mostly attributed to gravitational perturbations produced by other than baryons sources, generally referred to as the dark matter component of the universe. In the present case, the extrinsic curvature solution of (37) should have an observable effect in space-time, independently of the perturbations.

6. Conclusions

The fundamental problems of Modern cosmology are three-fold: the Λ paradigm, dark energy and dark matter. With the high developing of the observational methods and devices, these problems have demanded a series of theoretical needs also stimulating the development of theories beyond Einstein's. Our approach here was to stress the study of the embedding process between manifolds and its necessity for the contemporary physics. By its own nature, the embedding between manifolds is a perturbational process of geometry and the recent fundamental problems on Cosmology seem to point to the same question: what is gravity and how it can be perturbed? The studies on the extrinsic curvature have been made at length in the literature but with no the required accuracy by using junction conditions that induce the extrinsic geometry to be minimized to gauge fields and matter. Since we understand the embedding conditions, the using of any junction condition can be dispensed and the geometrical limitation for the embedding can be lifted.

In the early days of Riemannian geometry, the embedding between two Riemannian geometries was such a problem due to the fact the need of a relative geometric reference was missing. The existence of a background geometry is necessary to fix the ambiguity of the Riemann curvature of a given manifold, without a reference structure. General relativity solves this ambiguity problem by specifying that the tangent Minkowski space is a flat plane, as decided by the Poincaré symmetry, and not by the Riemann geometry itself. Such difficulty was known by Riemann himself, when he acknowledged that his curvature tensor defines a class of objects and not just one (Riemann, 1854). Unlike the case of string theory the bulk geometry is a solution of Einstein's equations, acting as a dynamic reference of shape for all embedded Riemann geometries. This generality follows from the remarkable accomplishment of Nash's theorem on embedded geometries. Nash showed that any Riemannian geometry can be generated by continuous sequence of infinitesimal perturbations defined by the extrinsic curvature. It seems natural that this result provides the required geometrical structure to describe a dynamically changing universe. This plays an essential feature for a new gravitational theory.

The four-dimensionality of the embedded space-times is determined by the dualities of the gauge fields, which corresponds to the equivalent concept of confinement gauge fields and ordinary matter in the brane-world program. However, this confinement implies that the extrinsic curvature cannot be completely determined, simply because Codazzi's equations becomes homogeneous. Incidently, the Randall-Sundrum model avoids this problem by imposing the Israel-Lanczos condition on a fixed boundary-like brane-world. Since the extrinsic curvature assumes a fundamental role in Nash's theorem, an additional equation is required. Recently, works on the subject noted that the extrinsic curvature is an independent rank-2 symmetric tensor, which corresponds to a spin-2 field defined on the embedded space-time. However, as it was demonstrated by Gupta, any spin-2 field satisfy an Einstein-like equation. After the due adaption to an embedded space-time, the analysis of Gupta's equations for the extrinsic curvature of the FLWR geometry and the study of the behavior of the extrinsic curvature at the various stages of the evolution of the universe is still an open question and the works on the subject are currently on progress.

The embedding of a space-time manifold into another defined by the Einstein-Hilbert principle may lead to an interesting gravitational theory, not only because its mathematical consistency provided by the Schlaefli conjecture as resolved by Nash's theorem, but mainly because it can meet the demands of modern cosmology, with the minimum of additional assumptions which can be fundamental for the development of a soft-after gravitational quantum field theory.

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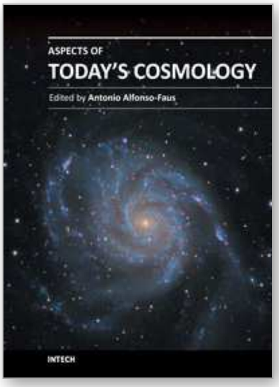

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This book presents some aspects of the cosmological scientific odyssey that started last century. The chapters vary with different particular works, giving a versatile picture. It is the result of the work of many scientists in the field of cosmology, in accordance with their expertise and particular interests. Is a collection of different research papers produced by important scientists in the field of cosmology. A sample of the great deal of efforts made by the scientific community, trying to understand our universe. And it has many challenging subjects, like the possible doomsday to be confirmed by the next decade of experimentation. May be we are now half way in the life of the universe. Many more challenging subjects are not present here: they will be the result of further future work. Among them, we have the possibility of cyclic universes, and the evidence for the existence of a previous universe.