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Decentralized Adaptive Control of Discrete-Time Multi-Agent Systems

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1. Introduction

In this chapter, we report some work on decentralized adaptive control of discrete-time multi-agent systems. Multi-agent systems, one important class of models of the so-called complex systems, have received great attention since 1980s in many areas such as physics, biology, bionics, engineering, artificial intelligence, and so on. With the development of technologies, more and more complex control systems demand new theories to deal with challenging problems which do not exist in traditional single-plant control systems.
The new challenges may be classified but not necessarily restricted in the following aspects:

- The increasing number of connected plants (or subsystems) adds more complexity to the control of whole system. Generally speaking, it is very difficult or even impossible to control the whole system in the same way as controlling one single plant.
- The couplings between plants interfere the evolution of states and outputs of each plant. That is to say, it is not possible to completely analyze each plant independently without considering other related plants.
- The connected plants need to exchange information among one another, which may bring extra communication constraints and costs. Generally speaking, the information exchange only occurs among coupled plants, and each plant may only have local connections with other plants.
- There may exist various uncertainties in the connected plants. The uncertainties may include unknown parameters, unknown couplings, unmodeled dynamics, and so on.

To resolve the above issues, multi-agent system control has been investigated by many researchers. Applications of multi-agent system control include scheduling of automated highway systems, formation control of satellite clusters, and distributed optimization of multiple mobile robotic systems, etc. Several examples can be found in Burns (2000); Swaroop & Hedrick (1999).
Various control strategies developed for multi-agent systems can be roughly assorted into two architectures: centralized and decentralized. In the decentralized control, local control for each agent is designed only using locally available information so it requires less

computational effort and is relatively more scalable with respect to the swarm size. In recent years, especially since the so-called Vicsek model was reported in Vicsek et al. (1995), decentralized control of multi-agent system has received much attention in the research community (e.g. Jadbabaie et al. (2003a); Moreau (2005)). In the (discrete-time) Vicsek model, there are *n* agents and all the agents move in the plane with the same speed but with different headings, which are updated by averaging the heading angles of neighor agents. By exploring matrix and graph properties, a theoretical explanation for the consensus behavior of the Vicsek model has been provided in Jadbabaie et al. (2003a). In Tanner & Christodoulakis (2005), a discrete-time multi-agent system model has been studied with fixed undirected topology and all the agents are assumed to transmit their state information in turn. In Xiao & Wang (2006), some sufficient conditions for the solvability of consensus problems for discrete-time multi-agent systems with switching topology and time-varying delays have been presented by using matrix theories. In Moreau (2005), a discrete-time network model of agents interacting via time-dependent communication links has been investigated. The result in Moreau (2005) has been extended to the case with time-varying delays by set-value Lyapunov theory in Angeli & Bliman (2006). Despite the fact that many researchers have focused on problems like consensus, synchronization, etc., we shall notice that the involved underlying dynamics in most existing models are essentially evolving with time in an invariant way determined by fixed parameters and system structure. This motivates us to consider decentralized adaptive control problems which essentially involve distributed agents with ability of adaptation and learning. Up to now, there are limited work on decentralized adaptive control for discrete-time multi-agent systems.

The theoretical work in this chapter has the following motivations:

1. The research on the capability and limitation of the feedback mechanism (e.g. Ma (2008a;b); Xie & Guo (2000)) in recent years focuses on investigating how to identify the maximum capability of feedback mechanism in dealing with *internal uncertainties of one single system*.
2. The decades of studies on traditional adaptive control (e.g. Aström & Wittenmark (1989); Chen & Guo (1991); Goodwin & Sin (1984); Ioannou & Sun (1996)) focus on investigating how to identify *the unknown parameters of a single plant*, especially a linear system or linear-in-parameter system.
3. The extensive studies on complex systems, especially the so-called *complex adaptive systems* theory Holland (1996), mainly focus on *agent-based modeling and simulations* rather than rigorous mathematical analysis.

Motivated by the above issues, to investigate how to deal with *coupling uncertainties* as well as internal uncertainties, we try to consider decentralized adaptive control of multi-agent systems, which exhibit complexity characteristics such as parametric internal uncertainties, parametric coupling uncertainties, unmodeled dynamics, random noise, and communication limits. To facilitate mathematical study on adaptive control problems of complex systems, the following simple yet nontrivial theoretical framework is adopted in our theoretical study:

1. The whole system consists of many dynamical agents, and evolution of each agent can be described by a state equation with optional output equation. Different agents may have different structures or parameters.
2. The evolution of each agent may be interacted by other agents, which means that the dynamic equations of agents are coupled in general. Such interactions among agents are usually restricted in local range, and the extent or intensity of reaction can be parameterized.

3. There exist information limits for all of the agents: (a) Each agent does *not* have access to internal structure or parameters of other agents while it may have complete or limited knowledge to its own internal structure and values of internal parameters. (b) Each agent does *not* know the intensity of influence from others. (c) However, each agent can observe the states of neighbor agents besides its own state.
4. Under the information limits above, each agent may utilize all of the information in hand to estimate the intensity of influence and to design local control so as to change the state of itself, consequently to influence neighbor agents. In other words, each agent is selfish and it aims to maximize its local benefits via minimizing the local tracking error.

Within the above framework, we are to explore the answers to the following basic problem: *Is it possible for all of the agents to achieve a global goal based on the local information and local control?* Here the global goal may refer to global stability, synchronization, consensus, or formation, etc. We shall start from a general model of discrete-time multi-agent system and discuss adaptive control design for several typical cases of this model. The ideas in this chapter can be also applied in more general or complex models, which may be considered in our future work and may involve more difficulties in the design and theoretical analysis of decentralized adaptive controller.

The remainder of this chapter is organized as follows: first, problem formulation will be given in Section 2 with the description of the general discrete-time multi-agent system model and several cases of local tracking goals; then, for these various local tracking tasks, decentralized adaptive control problem for a stochastic synchronization problem is discussed in Section 3 based on the recursive least-squares estimation algorithm; in Section 4, decentralized adaptive control for a special deterministic tracking problem, whereas the system has uncertain parameters, is given based on least-squares estimation algorithm; and Section 5 studies decentralized adaptive control for the special case of a hidden leader tracking problem, based on the normalized gradient estimation algorithm; finally, we give some concluding remarks in the last section.

2. Problem formulation

In this section, we will first describe the network of dynamic systems and then formulate the problems to be studied. We shall study a simple discrete-time dynamic network. In this model, there are N subsystems (plants), and each subsystem represents evolution of one agent. We denote the state of Agent i at time t by $x_i(t)$, and, for simplicity, we assume that linear influences among agents exist in this model. For convenience, we define the concepts of "neighbor" and "neighborhood" as follows: Agent j is a *neighbor* of Agent i if Agent j has influence on Agent i. Let $\mathcal{N}_i$ denote the set of all neighbors of Agent i and Agent i itself. Obviously *neighborhood* $\mathcal{N}_i$ of Agent i is a concept describing the communication limits between Agent i and others.

2.1 System model

The general model of each agent has the following state equation ($i = 1, 2, \ldots, N$) :

$$x_i(t+1) = f_i(z_i(t)) + u_i(t) + \gamma_i \bar{x}_i(t) + w_i(t+1) \tag{2.1}$$

with $z_i(t) = [\underline{x}_i(t), \underline{u}_i(t)]^T$, $\underline{x}_i(t) = [x_i(t), x_i(t-1), \ldots, x_i(t-n_i+1)]^T$ and $\underline{u}_i(t) = [u_i(t), u_i(t-1), \ldots, u_i(t-m_i+1)]^T$, where $f_i(\cdot)$ represents the internal structure of Agent i, $u_i(t)$ is the local control of Agent i, $w_i(t)$ is the unobservable random noise sequence, and

$\gamma_i \bar{x}_i(t)$ reflects the influence of the other agents towards Agent i. Hereinafter, $\bar{x}_i(t)$ is the weighted average of states of agents in the neighborhood of Agent i, i.e.,

$$\bar{x}_i(t) = \sum_{j \in \mathcal{N}_i} g_{ij} x_j(t) \tag{2.2}$$

where the nonnegative constants $\{g_{ij}\}$ satisfy $\sum\limits_{j \in \mathcal{N}_i} g_{ij} = 1$ and γ_i denotes the *intensity of influence*, which is unknown to Agent i. From graph theory, the network can be represented by a directed graph with each node representing an agent and the neighborhood of Node i consists of all the nodes that are connected to Node i with an edge directing to Node i. This graph can be further represented by an adjacent matrix

$$G = (g_{ij}), g_{ij} = 0 \text{ if } j \notin \mathcal{N}_i. \tag{2.3}$$

***Remark* 2.1.** Although model (2.1) is simple enough, it can capture all essential features that we want, and the simple model can be viewed as a prototype or approximation of more complex models. Model (2.1) highlights the difficulties in dealing with coupling uncertainties as well as other uncertainties by feedback control.

2.2 Local tracking goals

Due to the limitation in the communication among the agents, generally speaking, agents can only try to achieve local goals. We assume that the *local tracking goal* for Agent i is to follow a reference signal x_i^{ref}, which can be a known sequence or a sequence relating to other agents as discussed below:

Case I (deterministic tracking). In this case, $x_i^{\text{ref}}(t)$ is a sequence of deterministic signals (bounded or even unbounded) which satisfies $|x_i^{\text{ref}}(t)| = O(t^\delta)$.

Case II (center-oriented tracking). In this case, $x_i^{\text{ref}}(t) = \bar{x}(t) \stackrel{\Delta}{=} \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ is the center state of all agents, i.e., average of states of all agents.

Case III (loose tracking). In this case, $x_i^{\text{ref}}(t) = \lambda \bar{x}_i(t)$, where constant $|\lambda| < 1$. This case means that the tracking signal $x_i^{\text{ref}}(t)$ is close to the (weighted) average of states of neighbor agents of Agent i, and factor λ describes how close it is.

Case IV (tight tracking). In this case, $x_i^{\text{ref}}(t) = \bar{x}_i(t)$. This case means that the tracking signal $x_i^{\text{ref}}(t)$ is exactly the (weighted) average of states of agents in the neighborhood of Agent i.

In the first two cases, all agents track a common signal sequence, and the only differences are as follows: In Case I the common sequence has nothing with every agent's state; however, in Case II the common sequence is the center state of all of the agents. The first two cases mean that a common "leader" of all of agents exists, who can communicate with and send commands to all agents; however, the agents can only communicate with one another under certain *information limits*. In Cases III and IV, no common "leader" exists and all agents attempt to track the average state $\bar{x}_i(t)$ of its neighbors, and the difference between them is just the factor of tracking tightness.

2.3 Decentralized adaptive control problem

In the framework above, Agent i does not know the intensity of influence γ_i; however, it can use the historical information

$$\{x_i(t), \bar{x}_i(t), u_i(t-1), x_i(t-1), \bar{x}_i(t-1), u_i(t-2), \dots, x_i(1), \bar{x}_i(1), u_i(0)\} \tag{2.4}$$

to estimate γ_i and can further try to design its local control $u_i(t)$ to achieve its local goal. Such a problem is called a *decentralized adaptive control problem* since the agents must be smart enough so as to design a stabilizing adaptive control law, rather than to simply follow a common rule with fixed parameters such as the so-called *consensus protocol*, in a coupling network. Note that in the above problem formulation, besides the uncertain parameters γ_i, other uncertainties and constraints are also allowed to exist in the model, which may add the difficulty of decentralized adaptive control problem. In this chapter, we will discuss several concrete examples of designing decentralized adaptive control laws, in which coupling uncertainties, external noise disturbance, internal parametric uncertainties, and even functional structure uncertainties may exist and be dealt with by the decentralized adaptive controllers.

3. Decentralized synchronization with adaptive control

Synchronization is a simple global behavior of agents, and it means that all agents tend to behave in the same way as time goes by. For example, two fine-tuned coupled oscillators may gradually follow almost the same pace and pattern. As a kind of common and important phenomenon in nature, synchronization has been extensively investigated or discussed in the literature (e.g., Time et al. (2004); Wu & Chua (1995); Zhan et al. (2003)) due to its usefulness (e.g. secure communication with chaos synchronization) or harm (e.g. passing a bridge resonantly). Lots of existing work on synchronization are conducted on chaos (e.g.Gade & Hu (2000)), coupled maps (e.g.Jalan & Amritkar (2003)), scale-free or small-world networks (e.g.Barahona & Pecora (2002)), and complex dynamical networks (e.g.Li & Chen (2003)), etc. In recent years, several synchronization-related topics (*coordination*, *rendezvous*, *consensus*, *formation*, etc.) have also become active in the research community (e.g.Cao et al. (2008); Jadbabaie et al. (2003b); Olfati-Saber et al. (2007)). As for adaptive synchronization, it has received the attention of a few researchers in recent years (e.g.Yao et al. (2006); Zhou et al. (2006)), and the existing work mainly focused on deterministic continuous-time systems, especially chaotic systems, by constructing certain update laws to deal with parametric uncertainties and applying classical Lyapunov stability theory to analyze corresponding closed-loop systems.

In this section, we are to investigate a synchronization problem of a stochastic dynamic network. Due to the presence of random noise and unknown parametric coupling, unlike most existing work on synchronization, we need to introduce new concepts of synchronization and the decentralized learning (estimation) algorithm for studying the problem of decentralized adaptive synchronization.

3.1 System model

In this section, for simplicity, we assume that the internal function $f_i(\cdot)$ is known to each agent and the agents are in a common noisy environment, i.e. the random noise $\{w(t), \mathcal{F}_t\}$ are commonly present for all agents. Hence, the dynamics of Agent i ($i = 1, 2, \ldots, N$) has the following state equation:

$$x_i(t+1) = f_i(z_i(t)) + u_i(t) + \gamma_i \bar{x}_i(t) + w(t+1). \tag{3.1}$$

In this model, we emphasize that coupling uncertainty γ_i is the main source to prevent the agents from achieving synchronization with ease. And the random noise makes that traditional analysis techniques for investigating synchronization of deterministic systems cannot be applied here because it is impossible to determine a fixed common orbit for all agents to track asymptotically. These difficulties make the rather simple model here non-trivial for studying the synchronization property of the whole system, and we will find that proper estimation algorithms, which can be somewhat regarded as learning algorithms and make the agents smarter than those machinelike agents with fixed dynamics in previous studies, is critical for each agent to deal with these uncertainties.

3.2 Local controller design

As the intensity of influence γ_i is unknown, Agent i is supposed to estimate it on-line via commonly-used *recursive least-squares* (RLS) algorithm and design its local control based on the intensity estimate $\hat{\gamma}_i(t)$ via the certainty equivalence principle as follows:

$$u_i(t) = -f_i(z_i(t)) - \hat{\gamma}_i(t)\bar{x}_i(t) + x_i^{\mathrm{ref}}(t) \tag{3.2}$$

where $\hat{\gamma}_i(t)$ is updated on-line by the following recursive LS algorithm

$$\begin{aligned} \hat{\gamma}_i(t+1) &= \hat{\gamma}_i(t) + \bar{\sigma}_i(t)\bar{p}_i(t)\bar{x}_i(t)[y_i(t+1) - \hat{\gamma}_i(t)\bar{x}_i(t)] \\ \bar{p}_i(t+1) &= \bar{p}_i(t) - \bar{\sigma}_i(t)[\bar{p}_i(t)\bar{x}_i(t)]^2 \end{aligned} \tag{3.3}$$

with $y_i(t) = x_i(t) - f_i(z_i(t-1)) - u_i(t-1)$ and

$$\bar{\sigma}_i(t) \stackrel{\Delta}{=} \left[1 + \bar{p}_i(t)\bar{x}_i^2(t)\right]^{-1}, \bar{p}_i(t) \stackrel{\Delta}{=} \left[\sum_{k=0}^{t-1} \bar{x}_i^2(k)\right]^{-1} \tag{3.4}$$

Let $e_{ij}(t) \stackrel{\Delta}{=} x_i(t) - x_j(t)$, and suppose that $x_i^{\mathrm{ref}}(t) = x^*(t)$ for $i = 1, 2, \ldots, N$ in Case I. And suppose also matrix G is an irreducible primitive matrix in Case IV, which means that all of the agents should be connected and matrix G is cyclic (or periodic from the point of view of Markov chain).

Then we can establish almost surely convergence of the decentralized LS estimator and the global synchronization in Cases I—IV.

3.3 Assumptions

We need the following assumptions in our analysis:

Assumption 3.1. *The noise sequence $\{w(t), \mathcal{F}_t\}$ is a martingale difference sequence (with $\{\mathcal{F}_t\}$ being a sequence of nondecreasing σ-algebras) such that*

$$\sup_t E\left[|w(t+1)|^\beta | \mathcal{F}_t\right] < \infty \quad a.s. \tag{3.5}$$

for a constant $\beta > 2$.

Assumption 3.2. *Matrix $G = (g_{ij})$ is an irreducible primitive matrix.*

3.4 Main result

Theorem 3.1. *For system (3.1), suppose that Assumption 3.1 holds in Cases* I—IV *and Assumption 3.2 holds also in Case* IV*. Then the decentralized LS-based adaptive controller has the following closed-loop properties:*

(1) *All of the agents can asymptotically correctly estimate the intensity of influence from others, i.e.,*

$$\lim_{t\to\infty} \hat{\gamma}_i(t) = \gamma_i. \tag{3.6}$$

(2) *The system can achieve synchronization in sense of mean, i.e.,*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}|e_{ij}(t)|=0,\quad \forall i\neq j. \tag{3.7}$$

(3) *The system can achieve synchronization in sense of mean squares, i.e.,*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}|e_{ij}(t)|^2=0,\quad \forall i\neq j. \tag{3.8}$$

3.5 Lemmas

Lemma 3.1. *Suppose that Assumption 3.1 holds in Cases* I, II, III, *and* IV. *Then, in either case, for* $i=1,2,\ldots,N$ *and* $m\geq 1$, $0\leq d<m$, *we have*

$$\begin{aligned}\sum_{k=1}^{t}|\tilde{\gamma}_i(mk-d)\bar{x}_i(mk-d)|^2&=o(t)\ a.s.,\\ \sum_{k=1}^{t}|\tilde{\gamma}_i(mk-d)\bar{x}_i(mk-d)|&=o(t)\ a.s.\end{aligned} \tag{3.9}$$

Proof. See Ma (2009). ■

Lemma 3.2. *Consider the following iterative system:*

$$X_{t+1}=A_tX_t+W_t, \tag{3.10}$$

where $A_t\to A$ *as* $t\to\infty$ *and* $\{W_t\}$ *satisfies*

$$\sum_{k=1}^{t}\|W_k\|^2=o(t). \tag{3.11}$$

If the spectral radius $\rho(A)<1$, *then*

$$\sum_{k=1}^{t}\|X_k\|=o(t),\ \ \sum_{k=1}^{t}\|X_k\|^2=o(t). \tag{3.12}$$

Proof. See Ma (2009). ■

Lemma 3.3. *The estimation* $\hat{\gamma}_i(t)$ *of* γ_i *converges to the true value* γ_i *almost surely with the convergence rate*

$$|\tilde{\gamma}_i(t)|=O\left(\sqrt{\frac{\log\bar{r}_i(t)}{\bar{r}_i(t)}}\right). \tag{3.13}$$

where $r_i(t)$ *and* $\bar{r}_i(t)$ *are defined as follows*

$$\begin{aligned}r_i(t)&\triangleq 1+\textstyle\sum_{k=0}^{t-1}x_i^2(k)\\ \bar{r}_i(t)&\triangleq 1+\textstyle\sum_{k=0}^{t-1}\bar{x}_i^2(k)\end{aligned} \tag{3.14}$$

Proof. This lemma is just the special one-dimensional case of (Guo, 1993, Theorem 6.3.1). ■

3.6 Proof of theorem 3.1

Putting (3.2) into (3.1), we have

$$\begin{aligned} x_i(t+1) &= -\hat{\gamma}_i(t)\bar{x}_i(t) + x_i^{\text{ref}}(t) + \gamma_i\bar{x}_i(t) + w(t+1) \\ &= x_i^{\text{ref}}(t) + \tilde{\gamma}_i(t)\bar{x}_i(t) + w(t+1). \end{aligned} \tag{3.15}$$

Denote

$$\begin{aligned} X(t) &= (x_1(t), x_2(t), \ldots, x_N(t))^T, \\ Z(t) &= (x_1^{\text{ref}}(t), x_2^{\text{ref}}(t), \ldots, x_N^{\text{ref}}(t))^T, \\ \bar{X}(t) &= (\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_N(t))^T, \\ W(t+1) &= w(t+1)\mathbf{1} = (w(t+1), w(t+1), \ldots, w(t+1))^T, \\ \tilde{\Gamma}(t) &= \text{diag}(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t), \ldots, \tilde{\gamma}_N(t)), \\ 1 &= [1, \ldots, 1]^T. \end{aligned} \tag{3.16}$$

Then we get

$$X(t+1) = Z(t) + \tilde{\Gamma}(t)\bar{X}(t) + W(t+1). \tag{3.17}$$

According to (2.2), we have

$$\bar{X}(t) = GX(t), \tag{3.18}$$

where the matrix $G = (g_{ij})$. Furthermore, we have

$$\bar{X}(t+1) = GX(t+1) = GZ(t) + G\tilde{\Gamma}(t)\bar{X}(t) + W(t+1). \tag{3.19}$$

By Lemma 3.3, we have $\tilde{\gamma}(t) \to 0$ as $t \to \infty$. Thus, $\tilde{\Gamma}(t) \to 0$.
By (3.15), we have

$$x_i(t+1) - x_i^{\text{ref}}(t) - w(t+1) = \tilde{\gamma}_i(t)\bar{x}_i(t). \tag{3.20}$$

Let $e_{ij}(t) \triangleq x_i(t) - x_j(t)$, $\eta_i(t) = \tilde{\gamma}_i(t)\bar{x}_i(t)$. Then

$$e_{ij}(t+1) = [\eta_i(t) - \eta_j(t)] + [x_i^{\text{ref}}(t) - x_j^{\text{ref}}(t)]. \tag{3.21}$$

For convenience of later discussion, we introduce the following notations:

$$\begin{aligned} G^T &= (\zeta_1, \zeta_2, \ldots, \zeta_N), \\ E(t) &= (e_{1N}(t), e_{2N}(t), \ldots, e_{N-1,N}(t), 0)^T, \\ \eta(t) &= (\eta_1(t), \eta_2(t), \ldots, \eta_N(t))^T. \end{aligned} \tag{3.22}$$

Case I. In this case, $x_i^{\text{ref}}(t) = x^*(t)$, where $x^*(t)$ is a bounded deterministic signal. Hence,

$$e_{ij}(t+1) = \eta_i(t) - \eta_j(t). \tag{3.23}$$

Consequently, by Lemma 3.1, we obtain that $(i \neq j)$

$$\sum_{k=1}^{t} |e_{ij}(k+1)|^2 = O\left(\sum_{k=1}^{t} \eta_i^2(t)\right) + O\left(\sum_{k=1}^{t} \eta_j^2(t)\right) = o(t), \tag{3.24}$$

and similarly $\sum_{k=1}^{t} |e_{ij}(k+1)| = o(t)$ also holds.

Case II. In this case, $x_i^{\text{ref}}(t) = \bar{x}(t) = \frac{1}{N}\sum\limits_{i=1}^{N} x_i(t)$. The proof is similar to Case I.

Case III. Here $x_i^{\text{ref}}(t) = \lambda\bar{x}_i(t) = \lambda\zeta_i^T X(t)$. Noting that $\zeta_i^T\mathbf{1} = 1$ for any i, we have

$$\zeta_i^T X(t) - \zeta_j^T X(t) = \zeta_i^T [X(t) - x_N(t)\mathbf{1}] - \zeta_j^T [X(t) - x_N(t)\mathbf{1}] = \zeta_i^T E(t) - \zeta_j^T E(t), \tag{3.25}$$

and, thus,

$$\begin{aligned} e_{ij}(t+1) &= [\eta_i(t) - \eta_j(t)] + \lambda[\bar{x}_i(t) - \bar{x}_j(t)] \\ &= [\eta_i(t) - \eta_j(t)] + \lambda[\zeta_i^T X(t) - \zeta_j^T X(t)] \\ &= [\eta_i(t) - \eta_j(t)] + \lambda[\zeta_i^T E(t) - \zeta_j^T E(t)]. \end{aligned} \tag{3.26}$$

Taking $j = N$ and $i = 1, 2, \ldots, N$, we can rewrite (3.26) into matrix form as

$$E(t+1) = [\eta(t) - \eta_N(t)\mathbf{1}] + \lambda[G - \mathbf{1}\zeta_N^T]E(t) = \lambda H E(t) + \xi(t), \tag{3.27}$$

where

$$H = G - G_N = G - \mathbf{1}\zeta_N^T, \;\; \xi(t) = \eta(t) - \eta_N(t). \tag{3.28}$$

By Lemma 3.1, we have

$$\sum_{k=1}^{t} \|\eta(k)\|^2 = o(t). \tag{3.29}$$

Therefore,

$$\sum_{k=1}^{t} \|\xi(k)\|^2 = o(t). \tag{3.30}$$

Now we prove that $\rho(H) \leq 1$. In fact, for any vector v such that $v^T v = 1$, we have

$$\begin{aligned} |v^T H v| &= |v^T G v - v^T G_N v| \\ &\leq \max\left(\lambda_{\max}(G)\|v\|^2 - \lambda_{\min}(G_N)\|v\|^2,\right. \\ &\quad \left.\lambda_{\max}(G_N)\|v\|^2 - \lambda_{\min}(G)\|v\|^2\right) \\ &\leq \max\left(\|v\|^2, \lambda_{\max}(G_N)\|v\|^2\right) \\ &= 1 \end{aligned} \tag{3.31}$$

which implies that $\rho(H) \leq 1$.
Finally, by (3.27), together with Lemma 3.2, we can immediately obtain

$$\sum_{k=1}^{t} \|E(k)\| = o(t), \quad \sum_{k=1}^{t} \|E(k)\|^2 = o(t). \tag{3.32}$$

Thus, for $i = 1, 2, \ldots, N-1$, as $t \to \infty$, we have proved

$$\frac{1}{t}\sum_{k=1}^{t} |e_{iN}(k)| \to 0, \quad \frac{1}{t}\sum_{k=1}^{t} [e_{iN}(k)]^2 \to 0. \tag{3.33}$$

Case IV. The proof is similar to that for Case III. We need only prove that the spectral radius $\rho(H)$ of H is less than 1, i.e., $\rho(H) < 1$; then we can apply Lemma 3.2 like in Case III. Consider the following linear system:

$$z(t+1) = Gz(t). \tag{3.34}$$

Noting that G is a stochastic matrix, then, by Assumption 3.2 and knowledge of the Markov chain, we have

$$\lim_{t \to \infty} G^t = \mathbf{1}\pi^T, \tag{3.35}$$

where π is the unique stationary probability distribution of the finite-state Markov chain with transmission probability matrix G. Therefore,

$$z(t) = G^t z(0) \to \mathbf{1}\pi^T x_0^{\text{ref}} = (\pi^T x_0^{\text{ref}})\mathbf{1} \tag{3.36}$$

which means that all elements of $z(t)$ converge to a same constant $\pi^T x_0^{\text{ref}}$. Furthermore, let $z(t) = (x_1^{\text{ref}}(t), x_2^{\text{ref}}(t), \ldots, x_N^{\text{ref}}(t))^T$ and $\nu(t) = (\nu_1(t), \nu_2(t), \ldots, \nu_{N-1}(t), 0)^T$, where $\nu_i(t) = x_i^{\text{ref}}(t) - x_N^{\text{ref}}(t)$ for $i = 1, 2, \ldots, N$. Then we can see that

$$\nu(t+1) = (G - G_N)\nu(t) = H\nu(t) \tag{3.37}$$

and $\lim_{t\to\infty} \nu(t) = 0$ for any initial values $\nu_i(0) \in \mathcal{R}$, $i = 1, 2, \ldots, N-1$. Obviously $\nu(t) = H^t\nu(0)$, and each entry in the Nth row of H^t is zero since each entry in the Nth row of H is zero. Thus, denote

$$H^t \triangleq \begin{bmatrix} H_0(t) & * \\ 0 & 0 \end{bmatrix}, \tag{3.38}$$

where $H_0(t)$ is an $(N-1) \times (N-1)$ matrix. Then, for $i = 1, 2, \ldots, N-1$, taking $\nu(0) = \mathbf{e}_i$, respectively, by $\lim\limits_{t\to\infty} \nu(t) = 0$ we easily know that the ith column of $H_0(t)$ tends to zero vector as $t \to \infty$. Consequently, we have

$$\lim_{t\to\infty} H_0(t) = 0, \tag{3.39}$$

which implies that each eigenvalue of $H_0(t)$ tends to zero too. By (3.38), eigenvalues of H^t are identical with those of $H_0(t)$ except for zero, and, thus, we obtain that

$$\lim_{t\to\infty} \rho\left(H^t\right) = 0 \tag{3.40}$$

which implies that

$$\rho(H) < 1. \tag{3.41}$$

This completes the proof of Theorem 3.1. ■

4. Decentralized tracking with adaptive control

Decentralized tracking problem is critical to understand the fundamental relationship between (local) stability of individual agents and the global stability of the whole system, and tracking problem is the basis for investigating more general or complex problems such as formation control. In this section, besides the parametric coupling uncertainties and external random noise, parametric internal uncertainties are also present for each agent, which require each agent to do more estimation work so as to deal with all these uncertainties. If each agent needs to deal with both parametric and non-parametric uncertainties, the agents should adopt more complex and smart leaning algorithms, whose ideas may be partially borrowed from Ma & Lum (2008); Ma et al. (2007a); Yang et al. (2009) and the references therein.

4.1 System model

In this section, we study the case where the internal dynamics function $f_i(\cdot)$ is not completely known but can be expressed into a linear combination with unknown coefficients, such that (2.1) can be expressed as follows:

$$x_i(t+1) + \sum_{k=1}^{n_i} a_{ik} x_i(t-k+1) = \sum_{k=1}^{m_i} b_{ik} u_i(t-k+1) + \gamma_i \sum_{j\in\mathcal{N}_i} g_{ij} x_j(t) + w_i(t+1) \tag{4.1}$$

which can be rewritten into the well-known ARMAX model with additional coupling item $\sum_{j\in\mathcal{N}_i}\bar{g}_{ij}x_j(t)$ (letting $\bar{g}_{ij}\stackrel{\Delta}{=}\gamma_i g_{ij}$) as follows:

$$A_i(q^{-1})x_i(t+1) = B_i(q^{-1})u_i(t) + w_i(t+1) + \sum_{j\in\mathcal{N}_i}\bar{g}_{ij}x_j(t) \tag{4.2}$$

with $A_i(q^{-1}) = 1 + \sum_{j=1}^{n_i} a_{ij}q^{-j}$, $B_i(q^{-1}) = b_{i1} + \sum_{j=2}^{m_i} b_{ij}q^{-j+1}$ and back shifter q^{-1}.

4.2 Local controller design

For Agent i, we can rewrite its dynamic model as the following regression model

$$x_i(t+1) = \boldsymbol{\theta}_i^T\boldsymbol{\phi}_i(t) + w_i(t+1) \tag{4.3}$$

where $\boldsymbol{\theta}_i$ holds all unknown parameters and $\boldsymbol{\phi}_i(t)$ is the corresponding regressor vector. Then, by the following LS algorithm

$$\begin{array}{lll} \hat{\boldsymbol{\theta}}_i(t+1) &=& \hat{\boldsymbol{\theta}}_i(t) + \sigma_i(t)P_i(t)\boldsymbol{\phi}_i(t)[x_i(t+1) - \boldsymbol{\phi}_i^T(t)\hat{\boldsymbol{\theta}}_i(t)] \\ P_i(t+1) &=& P_i(t) - \sigma_i(t)P_i(t)\boldsymbol{\phi}_i(t)\boldsymbol{\phi}_i^T(t)P_i(t) \\ \sigma_i(t) &=& [1 + \boldsymbol{\phi}_i^T(t)P_i(t)\boldsymbol{\phi}_i(t)]^{-1} \end{array} \tag{4.4}$$

we can obtain the estimated values $\hat{\boldsymbol{\theta}}_i(t)$ of $\boldsymbol{\theta}_i$ at time t. For Agent i, to track a given local reference signal $x_i^{ref}(t) \triangleq x_i^*(t)$, with the parameter estimate $\hat{\boldsymbol{\theta}}_i(t)$ given by the above LS algorithm, it can then design its adaptive control law $u_i(t)$ by the "certainty equivalence" principle, that is to say, it can choose $u_i(t)$ such that

$$\hat{\boldsymbol{\theta}}_i^T(t)\boldsymbol{\phi}_i(t) = x_i^*(t+1) \tag{4.5}$$

where $x_i^*(t)$ is the bounded desired reference signal of Agent i, i.e. Agent i is to track the deterministic given signal $x_i^*(t)$.
Consequently we obtain

$$\begin{array}{rl} u_i(t) = & \frac{1}{\hat{b}_{i1}(t)}\{x_i^*(t+1) \\ & +[\hat{a}_{i1}(t)x_i(t) + \cdots + \hat{a}_{i,p_i}(t)x_i(t-p_i+1)] \\ & -[\hat{b}_{i2}(t)u_i(t-1) + \cdots + \hat{b}_{i,q_i}(t)u_i(t-q_i+1)] \\ & -\hat{\bar{\boldsymbol{g}}}_i^T(t)\bar{\boldsymbol{X}}_i(t)\} \end{array} \tag{4.6}$$

where $\hat{\bar{\boldsymbol{g}}}_i(t)$ is a vector holding the estimates $\hat{\bar{g}}_{ij}(t)$ of g_{ij} ($j \in \mathcal{N}_i$) and $\bar{\boldsymbol{X}}_i(t)$ is a vector holding the states $x_{ij}(t)$ ($j \in \mathcal{N}_i$).
In particular, when the high-frequency gain b_{i1} is known *a priori*, let $\bar{\boldsymbol{\theta}}_i$ denote the parameter vector $\boldsymbol{\theta}_i$ without component b_{i1}, $\bar{\boldsymbol{\phi}}_i(t)$ denote the regression vector $\boldsymbol{\phi}_i(t)$ without component $u_i(t)$, and similarly we introduce notations $\bar{a}_i(t), \bar{P}_i(t)$ corresponding to $a_i(t)$ and $P_i(t)$, respectively. Then, the estimate $\bar{\boldsymbol{\theta}}_i(t)$ at time t of $\bar{\boldsymbol{\theta}}_i$ can be updated by the following algorithm:

$$\begin{array}{lll} \bar{\boldsymbol{\theta}}_i(t+1) &=& \bar{\boldsymbol{\theta}}_i(t) + \bar{\sigma}_i(t)\bar{P}_i(t)\bar{\boldsymbol{\phi}}_i(t) \\ & & \times[x_i(t+1) - b_{i1}u_i(t) - \bar{\boldsymbol{\phi}}_i^T(t)\bar{\boldsymbol{\theta}}_i(t)] \\ \bar{P}_i(t+1) &=& \bar{P}_i(t) - \bar{\sigma}_i(t)\bar{P}_i(t)\bar{\boldsymbol{\phi}}_i(t)\bar{\boldsymbol{\phi}}_i^T(t)\bar{P}_i(t) \\ \bar{\sigma}_i(t) &=& [1 + \bar{\boldsymbol{\phi}}_i^T(t)\bar{P}_i(t)\bar{\boldsymbol{\phi}}_i(t)]^{-1} \end{array} \tag{4.7}$$

When the high-frequency gain b_{i1} is unknown *a priori*, to avoid the so-called singularity problem of $\hat{b}_{i1}(t)$ being or approaching zero, we need to use the following modified $\hat{b}_{i1}(t)$, denoted by $\hat{\hat{b}}_{i1}(t)$, instead of original $\hat{b}_{i1}(t)$:

$$\hat{\hat{b}}_{i1}(t) = \begin{cases} \hat{b}_{i1}(t) & \text{if } |\hat{b}_{i1}(t)| \geq \frac{1}{\sqrt{\log r_i(t)}} \\ \hat{b}_{i1}(t) + \frac{\text{sgn}(\hat{b}_{i1}(t))}{\sqrt{\log r_i(t)}} & \text{if } |\hat{b}_{i1}(t)| < \frac{1}{\sqrt{\log r_i(t)}} \end{cases} \tag{4.8}$$

and consequently the local controller of Agent i is given by

$$\begin{aligned} u_i(t) = \frac{1}{\hat{\hat{b}}_{i1}(t)} \{ & x_i^*(t+1) \\ & + [\hat{a}_{i1}(t)x_i(t) + \cdots + \hat{a}_{i,p_i}(t)x_i(t-p_i+1)] \\ & - [\hat{b}_{i2}(t)u_i(t-1) + \cdots + \hat{b}_{i,q_i}(t)u_i(t-q_i+1)] \\ & - \hat{\boldsymbol{g}}_i^T(t)\bar{\boldsymbol{X}}_i(t)\}. \end{aligned} \tag{4.9}$$

4.3 Assumptions

Assumption 4.1. (noise condition) $\{w_i(t), \mathcal{F}_t\}$ *is a martingale difference sequence, with* $\{\mathcal{F}_t\}$ *being a sequence of nondecreasing* σ*-algebras, such that*

$$\sup_{t\geq 0} E[|w_i(t+1)|^\beta | \mathcal{F}_t] < \infty, a.s.$$

for some $\beta > 2$ *and*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} |w_i(k)|^2 = R_i > 0, a.s.$$

Assumption 4.2. (minimum phase condition) $B_i(z) \neq 0, \forall z \in \mathcal{C} : |z| \leq 1.$

Assumption 4.3. (reference signal) $\{x_i^*(t)\}$ *is a bounded deterministic signal.*

4.4 Main result

Theorem 4.1. *Suppose that Assumptions 4.1—4.3 hold for system (4.1). Then the closed-loop system is stable and optimal, that is to say, for* $i = 1, 2, \ldots, N$*, we have*

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} [|x_i(k)|^2 + |u_i(k-1)|^2] < \infty, \quad a.s.$$

and

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} |x_i(k) - x_i^*(k)|^2 = R_i, \quad a.s.$$

Although each agent only aims to track a local reference signal by local adaptive controller based on recursive LS algorithm, the whole system achieves global stability. The optimality can also be understood intuitively because in the presence of noise, even when all the parameters are known, the limit of

$$J_i(t) \stackrel{\Delta}{=} \lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} |x_i(k+1) - x_i^*(k+1)|^2$$

cannot be smaller than R_i.

4.5 Lemmas

Lemma 4.1. *Under Assumption 4.1, we have* $|w_i(t)| = O(d_i(t))$*, where* $\{d_i(t)\}$ *is an increasing sequence and can be taken as* t^δ *(*δ *can be any positive number).*

Proof. In fact, by using Markov inequality, we obtain that

$$\sum_{t=1}^{\infty} P(|w_i(t+1)|^2 \geq t^{2\delta}|\mathcal{F}_t) \leq \sum_{t=1}^{\infty} \frac{E[|w_i(t+1)|^\beta|\mathcal{F}_t]}{t^{\beta\delta}} < \infty$$

holds almost surely. By applying the Borel-Cantelli-Levy lemma, immediately we have $|w_i(t+1)| = O(t^\delta)$, *a.s.* ■

Lemma 4.2. *If* $\xi(t+1) = B(z)u(t), \forall t > 0$*, where polynomial* $(q \geq 1)$

$$B(z) = b_1 + b_2 z + \cdots + b_q z^{q-1}$$

satisfies

$$B(z) \neq 0, \forall z : |z| \leq 1, \tag{4.10}$$

then there exists a constant $\lambda \in (0,1)$ *such that*

$$|u(t)|^2 = O(\sum_{k=0}^{t+1} \lambda^{t+1-k}|\xi(k)|^2). \tag{4.11}$$

Proof. See Ma et al. (2007b). ■

Lemma 4.3. *Under Assumption 4.1, for* $i = 1, 2, \ldots, N$*, the LS algorithm has the following properties almost surely:*

(a)

$$\tilde{\boldsymbol{\theta}}_i^T(t+1)P_i^{-1}(t+1)\tilde{\boldsymbol{\theta}}_i(t+1) = O(\log r_i(t))$$

(b)

$$\sum_{k=1}^{t} \alpha_i(k) = O(\log r_i(t))$$

where

$$\begin{aligned}
\delta_i(t) &\stackrel{\Delta}{=} \mathrm{tr}(P_i(t) - P_i(t+1)) \\
\sigma_i(k) &\stackrel{\Delta}{=} [1 + \boldsymbol{\phi}_i^T(k)P_i(k)\boldsymbol{\phi}_i(k)]^{-1} \\
\alpha_i(k) &\stackrel{\Delta}{=} \sigma_i(k)|\tilde{\boldsymbol{\theta}}_i^T(k)\boldsymbol{\phi}_t(k)|^2 \\
r_i(t) &\stackrel{\Delta}{=} 1 + \sum_{k=1}^{t} \boldsymbol{\phi}_i^T(k)\boldsymbol{\phi}_i(k)
\end{aligned} \tag{4.12}$$

Proof. This is a special case of (Guo, 1994, Lemma 2.5). ■

Lemma 4.4. *Under Assumption 4.1, for* $i = 1, 2, \ldots, N$*, we have*

$$\sum_{k=1}^{t} |x_i(k)|^2 \to \infty, \ \liminf_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} |x_i(k)|^2 \geq R_i > 0, \ a.s. \tag{4.13}$$

Proof. This lemma can be obtained by estimating lower bound of $\sum_{k=1}^{t} [x_i(k+1)]^2$ with the help of Assumption 4.1 and the martingale estimation theorem. Similar proof can be found in Chen & Guo (1991). ■

4.6 Proof of Theorem 4.1

To prove Theorem 4.1, we shall apply the main idea, utilized in Chen & Guo (1991) and Guo (1993), to estimate the bounds of signals by analyzing some linear inequalities. However, there are some difficulties in analyzing the closed-loop system of decentralized adaptive control law. Noting that each agent only uses local estimate algorithm and control law, but the agents are coupled, therefore for a fixed Agent i, we cannot estimate the bounds of state $x_i(t)$ and control $u_i(t)$ without knowing the corresponding bounds for its neighborhood agents. This is the main difficulty of this problem. To resolve this problem, we first analyze every agent, and then consider their relationship globally, finally the estimation of state bounds for each agent can be obtained through both the local and global analysis.

In the following analysis, $\delta_i(t)$, $\sigma_i(k)$, $\alpha_i(k)$ and $r_i(t)$ are defined as in Eq. (4.12).

Step 1: In this step, we analyze dynamics of each agent. We consider Agent i for $i = 1, 2, \ldots, N$. By putting the control law (4.9) into (4.3), noting that (4.5), we have

$$\begin{aligned} x_i(t+1) &= \boldsymbol{\theta}_i^T \boldsymbol{\phi}_i(t) + w_i(t+1) \\ &= x_i^*(t+1) - \hat{\boldsymbol{\theta}}_i^T(t)\boldsymbol{\phi}_i(t) + \boldsymbol{\theta}_i^T \boldsymbol{\phi}_i(t) + w_i(t+1) \\ &= x_i^*(t+1) + \tilde{\boldsymbol{\theta}}_i^T(t)\boldsymbol{\phi}_i(t) + w_i(t+1) \end{aligned}$$

By Lemma 4.1, we have $|w_i(t)|^2 = O(d_i(t))$. Noticing also

$$\begin{aligned} |\tilde{\boldsymbol{\theta}}_i(t)\boldsymbol{\phi}_i(t)|^2 &= \alpha_i(t)[1 + \boldsymbol{\phi}_i^T(t)P_i(t)\boldsymbol{\phi}_i(t)] \\ &= \alpha_i(t)[1 + \boldsymbol{\phi}_i^T(t)P_i(t+1)\boldsymbol{\phi}_i(t)] \\ &\quad + \alpha_i(t)\boldsymbol{\phi}_i^T(t)[P_i(t) - P_i(t+1)]\boldsymbol{\phi}_i(t) \\ &\le \alpha_i(t)[2 + \delta_i(t)\|\boldsymbol{\phi}_i(t)\|^2] \end{aligned}$$

and the boundedness of $x_i^*(t+1)$, we can obtain that

$$|x_i(t+1)|^2 \le 2\alpha_i(t)\delta_i(t)\|\boldsymbol{\phi}_i(t)\|^2 + O(d_i(t)) + O(\log r_i(t)). \tag{4.14}$$

Now let us estimate $\|\boldsymbol{\phi}_i(t)\|^2$. By Lemma 4.2, there exists $\lambda_i \in (0, 1)$ such that

$$|u_i(t)|^2 = O(\sum_{k=0}^{t+1} \lambda_i^{t+1-k}(|x_i(k)|^2 + \|\bar{\boldsymbol{X}}_i(k)\|^2 + |w_i(k+1)|^2)).$$

It holds for all $i = 1, 2, \ldots, N$, but we cannot estimate $|u_i(t)|^2$ directly because it involves $\{x_j(k), j \in \mathcal{N}_i\}$ in $\bar{\boldsymbol{X}}_i(k)$.

Let

$$\begin{aligned} \rho &= \max(\lambda_1, \cdots, \lambda_N) \in (0, 1) \\ \boldsymbol{X}(k) &= [x_1(k), \cdots, x_N(k)]^T \\ \bar{d}(k) &= \max(d_1(k), \cdots, d_N(k)). \end{aligned}$$

Obviously we have

$$|x_i(k)|^2 = O(\|\boldsymbol{X}(k)\|^2), \|\bar{\boldsymbol{X}}_i(k)\|^2 = O(\|\boldsymbol{X}(k)\|^2).$$

Now define

$$L_t \triangleq \sum_{k=0}^{t} \rho^{t-k}\|\boldsymbol{X}(k)\|^2.$$

Then, for $i = 1, 2, \ldots, N$, we have

$$\begin{aligned} |u_i(t)|^2 &= O(L_{t+1}) + O(\sum_{k=0}^{t+1} \rho^{t+1-k}\bar{d}(k)) \\ &= O(L_{t+1}) + O(\bar{d}(t+1)). \end{aligned}$$

Since

$$\phi_i(t) = [x_i(t), \cdots, x_i(t-p_i+1), \quad u_i(t-1), \cdots, u_i(t-q_i+1), \bar{\boldsymbol{X}}_i^T(t)]^T$$

we can obtain that

$$\begin{aligned} \|\phi_i(t)\|^2 &= O(\|\boldsymbol{X}(t)\|^2) + O(L_t) + O(\bar{d}(t)) \\ &\quad + O(\log r_i(t) + d_i(t)) \\ &= O(L_t + \log \bar{r}(t) + \bar{d}(t)) \end{aligned}$$

where

$$\bar{r}(t) \stackrel{\Delta}{=} \max(r_1(t), r_2(t), \cdots, r_N(t)).$$

Hence by (4.14), for Agent i, there exists $C_i > 0$ such that

$$\begin{aligned} |x_i(t+1)|^2 \le\ & C_i \alpha_i(t)\delta_i(t) L_t \\ & + O(\alpha_i(t)\delta_i(t)[\log \bar{r}(t) + \bar{d}(t)]) \\ & + O(d_i(t) + \log r_i(t)). \end{aligned}$$

Then noticing

$$\alpha_i(t)\delta_i(t) = O(\log r_i(t))$$

we obtain that

$$|x_i(t+1)|^2 \le C_i \alpha_i(t)\delta_i(t) L_t + O(\log r_i(t)[\log \bar{r}(t) + \bar{d}(t)]). \tag{4.15}$$

Step 2: Because (4.15) holds for $i = 1, 2, \ldots, N$, we have

$$\begin{aligned} \|\boldsymbol{X}(t+1)\|^2 &= \sum_{i=1}^{N} |x_i(t+1)|^2 \\ &\le [\sum_{i=1}^{N} C_i \alpha_i(t)\delta_i(t)] L_t \\ &\quad + O(N\bar{d}(t)\log \bar{r}(t)) + O(N \log^2 \bar{r}(t)). \end{aligned}$$

Thus by the definition of L_t, we have

$$\begin{aligned} L_{t+1} &= \rho L_t + \|\boldsymbol{X}(t+1)\|^2 \\ &\le [\rho + C \sum_{i=1}^{N} \alpha_i(t)\delta_i(t)] L_t \\ &\quad + O(N\bar{d}(t)\log \bar{r}(t)) + O(N \log^2 \bar{r}(t)) \end{aligned}$$

where

$$C = \max(C_1, C_2, \cdots, C_N).$$

Let $\eta(t) = \sum_{i=1}^{N} \alpha_i(t)\delta_i(t)$, then

$$\begin{aligned} L_{t+1} &= O(N\bar{d}(t)\log r(t) + N \log^2 \bar{r}(t)) \\ &\quad + O(N \sum_{k=0}^{t-1} \prod_{l=k+1}^{t} [\rho + C\eta(l)] \\ &\quad \times [\bar{d}(k)\log \bar{r}(k) + \log^2 \bar{r}(k)]). \end{aligned} \tag{4.16}$$

Since

$$\sum_{k=0}^{\infty}\delta_i(k) = \sum_{k=0}^{\infty}[\operatorname{tr} P_i(k) - \operatorname{tr} P_i(k+1)] \le \operatorname{tr} P_i(0) < \infty,$$

we have $\delta_i(k) \to 0$ as $k \to \infty$. By Lemma 4.3,

$$\sum_{k=0}^{\infty}\alpha_i(k) = O(\log r_i(k)) = O(\log \bar{r}(k)).$$

Then, for $i = 1, 2, \ldots, N$ and arbitrary $\epsilon > 0$, there exists $k_0 > 0$ such that

$$\rho^{-1}C\sum_{k=t_0}^{t}\alpha_i(k)\delta_i(k) \le \frac{1}{N}\epsilon \log \bar{r}(t)$$

for all $t \ge t_0 \ge k_0$. Therefore

$$\rho^{-1}C\sum_{k=t_0}^{t}\eta(k) \le \epsilon \log \bar{r}(t).$$

Then, by the inequality $1 + x \le e^x, \forall x \ge 0$ we have

$$\begin{aligned}\prod_{k=t_0}^{t}[1+\rho^{-1}C\eta(k)] &\le \exp\{\rho^{-1}C\sum_{k=t_0}^{t}\eta(k)\}\\ &\le \exp\{\epsilon \log \bar{r}(t)\} = \bar{r}^{\epsilon}(t).\end{aligned}$$

Putting this into (4.16), we can obtain

$$L_{t+1} = O(\log r_i(t)[\log \bar{r}(t) + \bar{d}(t)]\bar{r}^{\epsilon}(t)).$$

Then, by the arbitrariness of ϵ, we have

$$L_{t+1} = O(\bar{d}(t)\bar{r}^{\epsilon}(t)), \forall \epsilon > 0.$$

Consequently, for $i = 1, 2, \ldots, N$, we obtain that

$$\begin{aligned}&\|\boldsymbol{X}(t+1)\|^2 \le L_{t+1} = O(\bar{d}(t)\bar{r}^{\epsilon}(t))\\ &|u_i(t)|^2 = O(L_{t+1} + \bar{d}(t+1)) = O(\bar{d}(t)\bar{r}^{\epsilon}(t))\\ &\|\phi_i(t)\|^2 = O(L_t + \log \bar{r}(t) + \bar{d}(t)) = O(\bar{d}(t)\bar{r}^{\epsilon}(t)).\end{aligned} \tag{4.17}$$

Step 3: By Lemma 4.4, we have

$$\liminf_{t\to\infty}\frac{r_i(t)}{t} \ge R_i > 0, \quad a.s.$$

Thus $t = O(r_i(t)) = O(\bar{r}(t))$, together with $\bar{d}(t) = O(t^{\delta}), \forall \delta \in (\frac{2}{\beta}, 1)$, then we conclude that $\bar{d}(t) = O(\bar{r}^{\epsilon}(t))$. Putting this into (4.17), and by the arbitrariness of ϵ, we obtain that

$$\|\phi_i(t)\|^2 = O(\bar{r}^{\delta}(t)), \forall \delta \in (\tfrac{2}{\beta}, 1).$$

Therefore

$$\begin{aligned}
&\sum_{k=0}^{t} |\tilde{\boldsymbol{\theta}}_i^T(k)\boldsymbol{\phi}_i(k)|^2 \\
&\quad = \sum_{k=0}^{t} \alpha_i(k)[1 + \boldsymbol{\phi}_i^T(k)P_i(k)\boldsymbol{\phi}_i(k)] \\
&\quad = O(\log r_i(t)) + O(\sum_{k=0}^{t} \alpha_i(k)\|\boldsymbol{\phi}_i(k)\|^2) \\
&\quad = O(\log \bar{r}(t)) + O(\bar{r}^\delta(t) \sum_{k=0}^{t} \alpha_i(k)) \\
&\quad = O(\bar{r}^\delta(t) \log \bar{r}(t)), \forall \delta \in (\tfrac{2}{\beta}, 1).
\end{aligned}$$

Then, by the arbitrariness of δ, we have

$$\sum_{k=0}^{t} |\tilde{\boldsymbol{\theta}}_i^T(k)\boldsymbol{\phi}_i(k)|^2 = O(\bar{r}^\delta(t)), \forall \delta \in (\tfrac{2}{\beta}, 1). \tag{4.18}$$

Since

$$x_i(t+1) = \tilde{\boldsymbol{\theta}}_i^T(t)\boldsymbol{\phi}_i(t) + x_i^*(t+1) + w_i(t+1)$$

we have

$$\begin{aligned}
\sum_{k=0}^{t} |x_i(k+1)|^2 &= O(\bar{r}^\delta(t)) + O(t) + O(\log \bar{r}(t)) \\
&= O(\bar{r}^\delta(t)) + O(t) \\
\sum_{k=0}^{t} |u_i(k-1)|^2 &= O(\bar{r}^\delta(t)) + O(t)
\end{aligned}$$

From the above, we know that for $i = 1, 2, \ldots, N$,

$$\begin{aligned}
r_i(t) &= 1 + \sum_{k=0}^{t} \|\boldsymbol{\phi}_i(k)\|^2 = O(\bar{r}^\delta(t)) + O(t) \\
&\quad \forall \delta \in (\tfrac{2}{\beta}, 1).
\end{aligned}$$

Hence

$$\begin{aligned}
\bar{r}(t) &= \max\{r_i(t), 1 \le i \le N\} \\
&= O(\bar{r}^\delta(t)) + O(t), \forall \delta \in (\tfrac{2}{\beta}, 1).
\end{aligned}$$

Furthermore, we can obtain

$$\bar{r}(t) = O(t)$$

which means that the closed-loop system is stable.

Step 4: Now we give the proof of the optimality.

$$\begin{aligned}
&\sum_{k=0}^{t} |x_i(k+1) - x_i^*(k+1)|^2 \\
&\quad = \sum_{k=0}^{t} [w_i(k+1)]^2 + \sum_{k=0}^{t} [\psi_i(k)]^2 + 2\sum_{k=0}^{t} \psi_i(k)w_i(k+1)
\end{aligned} \tag{4.19}$$

where

$$\psi_i(k) \stackrel{\Delta}{=} \tilde{\boldsymbol{\theta}}_i^T(k)\boldsymbol{\phi}_i(k).$$

By (4.18) and the martingale estimate theorem, we can obtain that the orders of last two items in (4.19) are both $O(\bar{r}^{\delta}(t)), \forall\delta \in (\frac{2}{\beta}, 1)$. Then we can obtain

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t} |x_i(k+1) - x_i^*(k+1)|^2 = R_i, \quad a.s.$$

Furthermore

$$\begin{aligned}\sum_{k=0}^{t} |x_i(k) - x_i^*(k) - w_i(k)|^2 &= \sum_{k=0}^{t} \|\psi_i(k)\|^2 \\ &= O(\bar{r}^{\delta}(t)) = o(t), \quad a.s.\end{aligned}$$

This completes the proof of the optimality of the decentralized adaptive controller. ■

5. Hidden leader following with adaptive control

In this section, we consider a hidden leader following problem, in which the leader agent knows the target trajectory to follow but the leadership of itself is unknown to all the others, and the leader can only affect its neighbors who can sense its outputs. In fact, this sort of problems may be found in many real applications. For example, a capper in the casino lures the players to follow his action but at the same time he has to keep not recognized. For another example, the plainclothes policeman can handle the crowd guide work very well in a crowd of people although he may only affect people around him. The objective of hidden leader following problem for the multi-agent system is to make each agent eventually follow the hidden leader such that the whole system is in order. It is obvious that the hidden leader following problem is more complicated than the conventional leader following problem and investigations of this problem are of significance in both theory and practice.

5.1 System model

For simplicity, we do not consider random noise in this section. The dynamics of the multi-agent system under study is in the following manner:

$$A_i(q^{-1})x_i(t+1) = B_i(q^{-1})u_i(t) + \gamma_i \bar{x}_i(t) \tag{5.1}$$

with $A_i(q^{-1}) = 1 + \sum_{j=1}^{n_i} a_{ij}q^{-j}$, $B_i(q^{-1}) = b_{i1} + \sum_{j=2}^{m_i} b_{ij}q^{-j+1}$ and back shifter q^{-1}, where $u_i(t)$ and $x_i(t)$, $i = 1, 2, \ldots, N$, are input and output of Agent i, respectively. Here $\bar{x}_i(t)$ is the average of the outputs from the neighbors of Agent i:

$$\bar{x}_i(t) \stackrel{\Delta}{=} \frac{1}{N_i} \sum_{j\in\mathcal{N}_i} x_j(t) \tag{5.2}$$

where

$$\mathcal{N}_i = \{s_{i,1}, s_{i,2}, \ldots, s_{i,N_i}\} \tag{5.3}$$

denotes the indices of Agent i's neighbors (excluding Agent i itself) and N_i is the number of Agent i's neighbors. In this model, we suppose that the parameters a_{ij} ($j = 1, 2, \ldots, m_j$), b_{ij} ($j = 1, 2, \ldots, n_j$) and γ_i are all *a priori* unknown to Agent i.

Remark 5.1. *From (5.1) we can find that there is no information to indicate which agent is the leader in the system representation.*

5.2 Local Controller design

Dynamics equation (5.1) for Agent i can be rewritten into the following regression form

$$x_i(t+1) = \boldsymbol{\theta}_i^T \boldsymbol{\phi}_i(t)$$

where $\boldsymbol{\theta}_i$ holds all unknown parameters and $\boldsymbol{\phi}_i(t)$ is the corresponding regressor vector.
We assume that the bounded desired reference $x^*(k)$ is only available to the hidden leader and satisfies $x^*(k+1) - x^*(k) = o(1)$. Without loss of generality, we suppose that the first agent is the hidden leader, so the control $u_1(t)$ for the first agent can be directly designed by using the certainty equivalence principle to track $x_1^{\text{ref}}(k) \stackrel{\Delta}{=} x^*(k)$:

$$\hat{\boldsymbol{\theta}}_1^T(t)\boldsymbol{\phi}_1(t) = x^*(t+1) \tag{5.4}$$

which leads to

$$\begin{aligned} u_1(t) = \; & \tfrac{1}{\hat{b}_{11}(t)}\{x^*(t+1) + [\hat{a}_{11}(t)x_1(t) + \cdots + \hat{a}_{1,n_1}(t)x_1(t-n_1+1)] \\ & -[\hat{b}_{12}(t)u_1(t-1) + \cdots + \hat{b}_{1,m_1}(t)u_1(t-m_1+1)] \\ & -\hat{\gamma}_1(t)\bar{x}_1(t)\}. \end{aligned} \tag{5.5}$$

As for the other agents, they are unaware of either the reference trajectory or the existence of the leader and the outputs of their neighbors are the only external information available for them, consequently, the jth ($j = 2, 3, \cdots, N$) agent should design its control $u_j(t)$ to track corresponding local center $x_j^{\text{ref}}(t) \stackrel{\Delta}{=} \bar{x}_j(t)$ such that

$$\hat{\boldsymbol{\theta}}_j^T(t)\boldsymbol{\phi}_j(t) = \bar{x}_j(t) \tag{5.6}$$

from which we can obtain the following local adaptive controller for Agent j:

$$\begin{aligned} u_j(t) = \; & \tfrac{1}{\hat{b}_{j1}(t)}\{\bar{x}_j(t) + [\hat{a}_{j1}(t)x_j(t) + \cdots + \hat{a}_{j,n_1}(t)x_j(t-n_j+1)] \\ & -[\hat{b}_{j2}(t)u_j(t-1) + \cdots + \hat{b}_{j,m_j}(t)u_j(t-m_j+1)] \\ & -\hat{\gamma}_j(t)\bar{x}_j(t)\}. \end{aligned} \tag{5.7}$$

Define

$$\tilde{y}_1(t) = x_1(t) - x^*(t) \tag{5.8}$$

and

$$\tilde{y}_j(t) = x_j(t) - \bar{x}_j(t-1), \quad j = 2, 3, \cdots, N. \tag{5.9}$$

The update law for the estimated parameters in the adaptive control laws (5.5) and (5.7) is given below ($j = 1, 2, \ldots, N$):

$$\begin{aligned} \hat{\theta}_j(t) &= \hat{\theta}_j(t-1) + \tfrac{\mu_j \tilde{y}_j(t)\phi_j(t-1)}{D_j(t-1)} \\ D_j(k) &= 1 + \|\phi_j(k)\|^2 \end{aligned} \tag{5.10}$$

where $0 < \mu_j < 2$ is a tunable parameter for tuning the convergence rate. Note that the above update law may not guarantee that $\hat{b}_{j1}(t) \geq \underline{b}_{j1}$, hence when the original $\hat{b}_{j1}(t)$ given by (5.10), denoted by $\hat{b}'_{j1}(t)$ hereinafter, is smaller than $\underline{b}_{j1}$, we need to make minor modification to $\hat{b}_{j1}(t)$ as follows:

$$\hat{b}_{j1}(t) = \underline{b}_{j1} \quad \text{if } \hat{b}'_{j1}(t) < \underline{b}_{j1}. \tag{5.11}$$

In other words, $\hat{b}_{j1}(t) = \max(\hat{b}'_{j1}(t), \underline{b}_{j1})$ in all cases.

5.3 Assumptions

Assumption 5.1. *The desired reference $x^*(k)$ for the multi-agent system is a bounded sequence and satisfies $x^*(k+1) - x^*(k) = o(1)$.*

Assumption 5.2. *The graph of the multi-agent system under study is strongly connected such that its adjacent matrix G_A is irreducible.*

Assumption 5.3. *Without loss of generality, it is assumed that the first agent is a hidden leader who knows the desired reference $x^*(k)$ while other agents are unaware of either the desired reference or which agent is the leader.*

Assumption 5.4. *The sign of control gain b_{j1}, $1 \leq j \leq n$, is known and satisfies $|b_{j1}| \geq \underline{b}_{j1} > 0$. Without loss of generality, it is assumed that b_{j1} is positive.*

5.4 Main result

Under the proposed decentralized adaptive control, the control performance for the multi-agent system is summarized as the following theorem.

Theorem 5.1. *Considering the closed-loop multi-agent system consisting of open loop system in (5.1) under Assumptions 5.1-5.4, adaptive control inputs defined in (5.5) and (5.7), parameter estimates update law in (5.10), the system can achieve synchronization and every agent can asymptotically track the reference $x^*(t)$, i.e.,*

$$\lim_{t\to\infty} e_j(t) = 0, \ j = 1, 2, \ldots, N \tag{5.12}$$

where $e_j(k) = x_j(k) - x^*(k)$.

Corollary 5.1. *Under conditions of Theorem 5.1, the system can achieve synchronization in sense of mean and every agent can successfully track the reference $x^*(t)$ in sense of mean, i.e.,*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} |e_j(k)| = 0, \ j = 1, 2, \ldots, N \tag{5.13}$$

5.5 Notations and lemmas

Define

$$\begin{aligned} X(k) &= [x_1(k), x_2(k), \ldots, x_N(k)]^T && (5.14)\\ \tilde{Y}(k) &= [\tilde{y}_1(k), \tilde{y}_2(k), \ldots, \tilde{y}_n(k)]^T && (5.15)\\ H &= [1, 0, \ldots, 0]^T \in \mathcal{R}^N && (5.16) \end{aligned}$$

From (5.2) and (5.14), we have

$$[0, \bar{x}_2(k), \ldots, \bar{x}_n(k)] = \Lambda G_A X(k) \tag{5.17}$$

where

$$\Lambda = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{N_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{N_N} \end{bmatrix}. \tag{5.18}$$

and G_A is an adjacent matrix of the multi-agent system (5.1), whose (i,j)th entry is 1 if $j \in \mathcal{N}_i$ or 0 if $j \notin \mathcal{N}_i$. Consequently, the closed-loop multi-agent system can be written in the following compact form by using equality

$$X(k+1) = \Lambda G_A X(k) + Hx^*(k+1) + \tilde{Y}(k+1) \tag{5.19}$$

Definition 5.1. *A sub-stochastic matrix is a square matrix each of whose rows consists of nonnegative real numbers, with at least one row summing strictly less than 1 and other rows summing to 1.*

Lemma 5.1. *According to Assumption 5.2, the product matrix ΛG_A is a substochastic matrix (refer to Definition 5.1) such that $\rho(\Lambda G_A) < 1$ (Dong et al. (2008)), where $\rho(A)$ stands for the spectral radius of a matrix A.*

Definition 5.2. *(Chen & Narendra, 2001) Let $x_1(k)$ and $x_2(k)$ be two discrete-time scalar or vector signals, $\forall k \in Z_t^+$, for any t.*

- *We denote $x_1(k) = O[x_2(k)]$, if there exist positive constants m_1, m_2 and k_0 such that $\|x_1(k)\| \leq m_1 \max_{k' \leq k} \|x_2(k')\| + m_2, \ \forall k > k_0$.*
- *We denote $x_1(k) = o[x_2(k)]$, if there exists a discrete-time function $\alpha(k)$ satisfying $\lim_{k\to\infty} \alpha(k) = 0$ and a constant k_0 such that $\|x_1(k)\| \leq \alpha(k) \max_{k' \leq k} \|x_2(k')\|$, $\forall k > k_0$.*
- *We denote $x_1(k) \sim x_2(k)$ if they satisfy $x_1(k) = O[x_2(k)]$ and $x_2(k) = O[x_1(k)]$.*

For convenience, in the followings we use $O[1]$ and $o[1]$ to denote bounded sequences and sequences converging to zero, respectively. In addition, if sequence $y(k)$ satisfies $y(k) = O[x(k)]$ or $y(k) = o[x(k)]$, then we may directly use $O[x(k)]$ or $o[x(k)]$ to denote sequence $y(k)$ for convenience.
According to Definition 5.2, we have the following lemma

Lemma 5.2. *According to the definition on signal orders in Definition 5.2, we have following properties:*

(i) $O[x_1(k+\tau)] + O[x_1(k)] \sim O[x_1(k+\tau)], \forall \tau \geq 0$.

(ii) $x_1(k+\tau) + o[x_1(k)] \sim x_1(k+\tau), \forall \tau \geq 0$.

(iii) $o[x_1(k+\tau)] + o[x_1(k)] \sim o[x_1(k+\tau)], \forall \tau \geq 0$.

(iv) $o[x_1(k)] + o[x_2(k)] \sim o[|x_1(k)| + |x_2(k)|]$.

(v) $o[O[x_1(k)]] \sim o[x_1(k)] + O[1]$.

(vi) If $x_1(k) \sim x_2(k)$ *and* $\lim_{k\to\infty} \|x_2(k)\| = 0$, *then* $\lim_{k\to\infty} \|x_1(k)\| = 0$.

(vii) If $x_1(k) = o[x_1(k)] + o[1]$, *then* $\lim_{k\to\infty} \|x_1(k)\| = 0$.

(viii) Let $x_2(k) = x_1(k) + o[x_1(k)]$. *If* $x_2(k) = o[1]$, *then* $\lim_{k\to\infty} \|x_1(k)\| = 0$.

The following lemma is a special case of Lemma 4.4 in Ma (2009).

Lemma 5.3. *Consider the following iterative system*

$$X(k+1) = A(k)X(k) + W(k) \tag{5.20}$$

where $\|W(k)\| = O[1]$, and $A(k) \to A$ as $k \to \infty$. Assume that $\rho(A)$ is the spectral radius of A, i.e. $\rho(A) = \max\{|\lambda(A)|\}$ and $\rho(A) < 1$, then we can obtain

$$X(k+1) = O[1]. \tag{5.21}$$

5.6 Proof of Theorem 5.1

In the following, the proof of mathematic rigor is presented in two steps. In the first step, we prove that $\tilde{x}_j(k) \to 0$ for all $j = 1, 2, \ldots, N$, which leads to $x_1(k) - x^*(k) \to 0$ such that the hidden leader follows the reference trajectory. In the second step, we further prove that the output of each agent can track the output of the hidden leader such that the control objective is achieved.

Step 1: Denote $\tilde{\theta}_j(k) \stackrel{\Delta}{=} \hat{\theta}_j(k) - \boldsymbol{\theta}_j(k)$, especially $\tilde{b}_{j1}(k) \stackrel{\Delta}{=} \hat{b}_{j1}(k) - b_{j1}$. For convenience, let $\tilde{b}'_{j1} \stackrel{\Delta}{=} \hat{b}'_{j1} - b_{j1}$, where $\hat{b}'_{j1}$ denotes the original estimate of b_{j1} without further modification. From the definitions of $\hat{b}'_{j1}$ and $\hat{b}_{j1}$, since $\hat{b}_{j1}(t) = \max(\hat{b}'_{j1}(t), \underline{b}_{j1})$ and $b_{j1} \geq \underline{b}_{j1}$, obviously we have

$$\tilde{b}^2_{j1}(k) \leq \tilde{b}'^2_{j1}(k). \tag{5.22}$$

Consider a Lyapunov candidate

$$V_j(k) = \|\tilde{\theta}_j(k)\|^2 \tag{5.23}$$

and we are to show that $V_j(k)$ is non-increasing for each $j = 1, 2, \ldots, N$, i.e. $V_j(k) \leq V_j(k-1)$. Noticing the fact given in (5.22), we can see that the minor modification given in (5.11) will not increase the value of $V_j(k)$ when $\hat{b}'_{j1}(k) < \underline{b}_{j1}$, therefore, in the sequel, we need only consider the original estimates without modification. Noting that

$$\|\hat{\theta}_j(k) - \hat{\theta}_j(k-1)\| = \|\tilde{\theta}_j(k) - \tilde{\theta}_j(k-1)\| \tag{5.24}$$

the difference of Lyapunov function $V_j(k)$ can be written as

$$\begin{aligned} \Delta V_j(k) &= V_j(k) - V_j(k-1) \\ &= \|\tilde{\theta}_j(k)\|^2 - \|\tilde{\theta}_j(k-1)\|^2 \\ &= \|\hat{\theta}_j(k) - \hat{\theta}_j(k-1)\|^2 + 2\tilde{\theta}_j^{\tau}(k-1)[\hat{\theta}_j(k) - \hat{\theta}_j(k-1)]. \end{aligned} \tag{5.25}$$

Then, according to the update law (5.10), the error dynamics (5.8) and (5.9), we have

$$\begin{aligned} &\|\hat{\theta}_j(k) - \hat{\theta}_j(k-1)\|^2 + 2\tilde{\theta}_j^{\tau}(k-1)[\hat{\theta}_j(k) - \hat{\theta}_j(k-1)] \\ &\leq \frac{\mu_j^2 \tilde{y}_j^2(k)}{D_j(k-1)} - \frac{2\mu_j \tilde{y}_j^2(k)}{D_j(k-1)} = -\frac{\mu_j(2-\mu_j)\tilde{y}_j^2(k)}{D_j(k-1)}. \end{aligned}$$

Noting $0 < \mu_j < 2$, we see that $\Delta V_j(k)$ is guaranteed to be non-positive such that the boundedness of $V_j(k)$ is obvious, and immediately the boundedness of $\hat{\theta}_j(k)$ and $\hat{b}_{j1}(k)$ is guaranteed. Taking summation on both sides of the above equation, we obtain

$$\sum_{k=0}^{\infty} \mu_j(2-\mu_j) \frac{\tilde{y}_j^2(k)}{D_j(k-1)} \leq V_j(0) \tag{5.26}$$

which implies

$$\lim_{k\to\infty} \frac{\tilde{y}_j^2(k)}{D_j(k-1)} = 0, \text{ or } \tilde{y}_j(k) = \alpha_j(k) D_j^{\frac{1}{2}}(k-1) \tag{5.27}$$

with $\alpha_j(k) \in L^2[0, \infty)$.
Define

$$\bar{Y}_j(k) = [x_j(k), Y_j^T(k)]^T \tag{5.28}$$

where $Y_j(k)$ is a vector holding states, at time k, of the jth agent's neighbors. By (5.5) and (5.7), we have

$$\begin{aligned} u_j(k) &= O[\bar{Y}_j(k+1)] \\ \phi_j(k) &= O[\bar{Y}_j(k)] \end{aligned} \tag{5.29}$$

then it is obvious that

$$\begin{aligned} D_j^{\frac{1}{2}}(k-1) &\leq 1 + \|\phi_j(k-1)\| + |u_j(k-1)| \\ &= 1 + O[\bar{Y}_j(k)]. \end{aligned} \tag{5.30}$$

From (5.27) we obtain that

$$\tilde{y}_j(k) = o[1] + o[\bar{Y}_j(k)], \ j = 1, 2, \ldots, N \tag{5.31}$$

Using $o[X(k)] \sim o[x_1(k)] + o[x_2(k)] + \ldots + o[x_n(k)]$, we may rewrite the above equation as

$$\begin{aligned} \tilde{Y}(k) \sim\ & \text{diag}(o[1], \ldots, o[1])(G_A + I)X(k) \\ & + [o[1], \ldots, o[1]]^T \end{aligned} \tag{5.32}$$

where I is the $n \times n$ identity matrix. Substituting the above equation into equation (5.19), we obtain

$$\begin{aligned} X(k+1) = & (\Lambda G_A + \text{diag}(o[1], \ldots, o[1])(G_A + I))X(k) \\ & + [x^*(k+1) + o[1], o[1], \ldots, o[1]]^T. \end{aligned}$$

Since

$$(\Lambda G_A + \text{diag}(o[1], \ldots, o[1])(G_A + I))Y(k) \to \Lambda G_A \tag{5.33}$$

as $k \to \infty$, noting $\rho(\Lambda G_A) < 1$, according to Lemma 5.1 and

$$[x^*(k+1) + o[1], o[1], \ldots, o[1]]^T = O[1] \tag{5.34}$$

from Lemma 5.3, we have

$$X(k+1) = O[1]. \tag{5.35}$$

Then, together with equation (5.32), we have $\tilde{Y}(k) = [o[1], \ldots, o[1]]^T$, which implies

$$\tilde{y}_j(k) \to 0 \text{ as } k \to \infty, j = 1, 2, \ldots, N \tag{5.36}$$

which leads to $x_1(k) - x^*(k) \to 0$.
Step 2: Next, we define a vector of the errors between each agent's output and the hidden leader's output as follows

$$E(k) = X(k) - [1,\ 1, \ldots,\ 1]^T x_1(k) = [e_{11}(k), e_{21}(k), \ldots, e_{n1}(k)]^T \tag{5.37}$$

where $e_{j1}(k)$ satisfies

$$\begin{aligned} e_{11}(k+1) &= x_1(k+1) - x_1(k+1) = 0, \\ e_{j1}(k+1) &= x_j(k+1) - x_1(k+1) = \bar{x}_j(k+1) - x_1(k+1) + \tilde{x}_j(k+1), \\ &\quad j = 2, 3, \ldots, N. \end{aligned} \tag{5.38}$$

Noting that except the first row, the summations of the other rows in the sub-stochastic matrix ΛG_A are 1, we have

$$[0, 1, \ldots, 1]^T = \Lambda G_A [0, 1, \ldots, 1]^T$$

such that equations in (5.38) can be written as

$$\begin{aligned} E(k+1) = \Lambda G X(k) - \Lambda G_A [0, 1, \ldots, 1]^T x_1(k+1) \\ + \operatorname{diag}(0, 1, \ldots, 1) \tilde{Y}(k). \end{aligned} \tag{5.39}$$

According to Assumption 5.1, we obtain

$$\begin{aligned} E(k+1) &= \Lambda G_A (X(k) - [0, 1, \ldots, 1]^T x_1(k)) \\ &\quad + [0, 1, \ldots, 1]^T (x_1(k) - x_1(k+1)) \\ &\quad + [o[1], \ldots, o[1]]^T \\ &= \Lambda G E(k) + [o[1], \ldots, o[1]]^T. \end{aligned} \tag{5.40}$$

Assume that ρ' is the spectral radius of ΛG_A, then there exists a matrix norm, which is denoted as $\| \cdot \|_p$, such that

$$\|E(k+1)\|_p \leq \rho' \|E(k)\|_p + o[1] \tag{5.41}$$

where $\rho' < 1$. Then, it is straightforward to show that

$$\|E(k+1)\|_p \to 0 \tag{5.42}$$

as $k \to \infty$. This completes the proof. ■

6. Summary

The decentralized adaptive control problems have wide backgrounds and applications in practice. Such problems are very challenging because various uncertainties, including coupling uncertainties, parametric plant uncertainties, nonparametric modeling errors, random noise, communication limits, time delay, and so on, may exist in multi-agent systems. Especially, the decentralized adaptive control problems for the discrete-time multi-agent systems may involve more technical difficulties due to the nature of discrete-time systems and lack of mathematical tools for analyzing stability of discrete-time nonlinear systems.

In this chapter, within a unified framework of multi-agent decentralized adaptive control, for a typical general model with coupling uncertainties and other uncertainties, we have investigated several decentralized adaptive control problems, designed efficient local adaptive controllers according to local goals of agents, and mathematically established the global properties (synchronization, stability and optimality) of the whole system, which in turn reveal the fundamental relationship between local agents and the global system.

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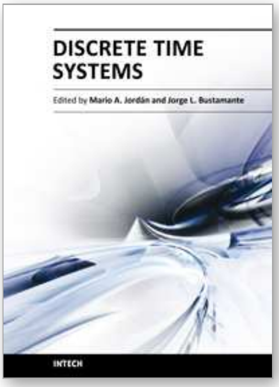

Discrete Time Systems

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Discrete-Time Systems comprehend an important and broad research field. The consolidation of digital-based computational means in the present, pushes a technological tool into the field with a tremendous impact in areas like Control, Signal Processing, Communications, System Modelling and related Applications. This book attempts to give a scope in the wide area of Discrete-Time Systems. Their contents are grouped conveniently in sections according to significant areas, namely Filtering, Fixed and Adaptive Control Systems, Stability Problems and Miscellaneous Applications. We think that the contribution of the book enlarges the field of the Discrete-Time Systems with signification in the present state-of-the-art. Despite the vertiginous advance in the field, we also believe that the topics described here allow us also to look through some main tendencies in the next years in the research area.

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