

Two Dimensional Sliding Mode Control

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1. Introduction

In nature, there are many processes, which their dynamics depend on more than one independent variable (e.g. thermal processes and long transmission lines (Kaczorek, 1985)). These processes are called multi-dimensional systems. Two Dimensional (2-D) systems are mostly investigated in the literature as a multi-dimensional system. 2-D systems are often applied to theoretical aspects like filter design, image processing, and recently, Iterative Learning Control methods (see for example Roesser, 1975; Hinamoto, 1993; Whalley, 1990; Al-Towaim, 2004; Hladowski et al., 2008). Over the past two decades, the stability of multi-dimensional systems in various models has been a point of high interest among researchers (Anderson et al., 1986; Kar, 2008; Singh, 2008; Bose, 1994; Kar & Singh, 1997; Lu, 1994). Some new results on the stability of 2-D systems have been presented – specifically with regard to the Lyapunov stability condition which has been developed for RM (Lu, 1994). Then, robust stability problem (Wang & Liu, 2003) and optimal guaranteed cost control of the uncertain 2-D systems (Guan et al., 2001; Du & Xie, 2001; Du et al., 2000) came to be the area of interest. In addition, an adaptive control method for SISO 2-D systems has been presented (Fan & Wen, 2003). However, in many physical systems, the goal of control design is not only to satisfy the stability conditions but also to have a system that takes its trajectory in the predetermined hyperplane. An interesting approach to stabilize the systems and keep their states on the predetermined desired trajectory is the sliding mode control method. Generally speaking, SMC is a robust control design, which yields substantial results in invariant control systems (Hung et al., 1993). The term invariant means that the system is robust against model uncertainties and exogenous disturbances. The behaviour of the underlying SMC of systems is indeed divided into two parts. In the first part, which is called reaching mode, system states are driven to a predetermined stable switching surface. And in the second part, the system states move across or intersect the switching surface while always staying there. The latter is called sliding mode. At a glance in the literature, it is understood that there are many works in the field of SMC for 1-D continuous and discrete time systems. (see Utkin, 1977; Asada & Slotine, 1986; Hung et al., 1993; DeCarlo et al., 1988; Wu and Gao, 2008; Furuta, 1990; Gao et al., 1995; Wu & Juang, 2008; Lai et al., 2006; Young et al., 1999; Furuta & Pan, 2000; Proca et al., 2003; Choa et al., 2007; Li & Wikander, 2004; Hsiao et al., 2008; Salarieh & Alasty, 2008) Furthermore SMC has been contributed to various control methods (see for example Hsiao et al., 2008; Salarieh & Alasty, 2008) and several experimental works (Proca et al., 2003). Recently, a SMC design for a 2-D system in RM

model has been presented (Wu & Gao, 2008) in which the idea of a 1-D quasi-sliding mode (Gao et al., 1995) has been extended for the 2-D system. Though the sliding surfaces design problem and the conditions for the existence of an ideal quasi-sliding mode has been solved in terms of LMI.

In this Chapter, using a 2-D Lyapunov function, the conditions ensuring the rest of horizontal and vertical system states on the switching surface and also the reaching condition for designing the control law are investigated. This function can also help us design the proper switching surface. Moreover, it is shown that the designed control law can be applied to some classes of 2-D uncertain systems. Simulation results show the efficiency of the proposed SMC design. The rest of the Chapter is organized as follows. In Section two, Two Dimensional (2-D) systems are described. Section three discusses the design of switching surface and the switching control law. In Section four, the proposed control design for two numerical examples in the form of 2-D uncertain systems is investigated. Conclusions and suggestions are finally presented in Section five.

2. Two dimensional systems

As the name suggests, two-dimensional systems represent behaviour of some processes which their variables depend on two independent varying parameters. For example, transmission lines are the 2-D systems where whose currents and voltages are changed as the space and time are varying. Also, dynamic equations governed to the motion of waves and temperatures of the heat exchangers are other examples of 2-D systems. It is interesting to note that some theoretical issues such as image processing, digital filter design and iterative processes control can be also used the 2-D systems properties.

2.1 Representation of 2-D systems

Especially, a well-known 2-D discrete systems called Linear Shift-Invariant systems has been presented which is described by the following input-output relation

$$\sum_m \sum_n b_{m,n} y_{i-m, j-n} = \sum_m \sum_n a_{m,n} u_{i-m, j-n} \quad (1)$$

Also, this input-output relation can be transformed into frequency-domain using 2-D Z transformation.

$$H(z_1, z_2) = \frac{Z\{y(i, j)\}}{Z\{u(i, j)\}} = \frac{P(z_1, z_2)}{Q(z_1, z_2)} \quad (2)$$

Similar to the one-dimensional systems, the 2-D systems are commonly represented in the state space model but what is makes different is being two independent variables in the 2-D systems so that this resulted in several state space models.

A well-known 2-D state space model was introduced by Roesser, 1975 which is called Roesser Model (RM or GR) and described by the following equations

$$\begin{aligned} \begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} &= \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} u^h(i, j) \\ u^v(i, j) \end{bmatrix} \\ y(i, j) &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} \end{aligned} \quad (3)$$

where $x^h(i, j) \in R^n$ and $x^v(i, j) \in R^m$ are the so called *horizontal* and *vertical* state variables respectively. Also $u(i, j) \in R^p$ is an input and $y(i, j) \in R^q$ is an output variable. Moreover, i and j represent two independent variables. $A_1, A_2, A_3, A_4, B_1, B_2, C_1$ and C_2 are constant matrices with proper dimensions. To familiar with other 2-D state space models (Kaczorek, 1985).

2.2 Stability of Rosser Model

One of the important topics in the 2-D systems is stability problem. Similar to 1-D systems, the stability of 2-D systems can be represent in two kinds, BIBO and Internally stability. First, a BIBO stability condition for RM is stated.

Theorem 1: A zero inputs 2-D system in RM (3) is BIBO stable if and only if one of the following conditions is satisfied

1. I. A_1 is stable, II. $A_4 + A_3 [I_n z_1 - A_1]^{-1} A_2$ is stable for $|z_1| = 1$.
2. I. A_2 is stable, II. $A_1 + A_2 [I_m z_2 - A_4]^{-1} A_3$ is stable for $|z_2| = 1$.

Note that, in the discrete systems, a matrix is stable if all whose eigenvalues are in the unite circle. Thus, from Theorem 1, it can be easily shown that a 2-D system in RM is unstable if A_1 or A_2 is not stable.

Similar to 1-D case, the Lyapunov stability for 2-D systems has been developed such that we represented in the following theorem.

Theorem 2: Zero inputs 2-D system (3) is asymptotically stable if there exist two positive definite matrices $P_1 \in R^n$ and $P_2 \in R^m$ such that

$$A^T P A - P = -Q \quad (4)$$

where Q is a positive matrix and

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \quad (5)$$

Remark 1: Note that the 2D system (3) is asymptotically stable if the state vector norms $x^h(i, j)$ and $x^v(i, j)$ converge to zero when $i+j \rightarrow \infty$.

Remark 2: The equality (3) is commonly called 2-D Lyapunov equation. As stated in the theorem 2, the condition for stability of 2-D systems in RM model is only sufficient not necessary and the Lyapunov matrix, P , is a block diagonal while in the 1-D case, the stability conditions is necessary and sufficient and the Lyapunov matrix is a full matrix.

However, it is worthy to know that the Lyapunov equation (3) can be used to define the 2-D Lyapunov function as shown below.

$$V_{00}(i, j) = X^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} X \quad (6)$$

where $X = [x^h(i, j) \quad x^v(i, j)]^T$. Regarding (5), define delayed 2-D Lyapunov function as follows

$$V_{11}(i, j) = X_{11}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} X_{11} \quad (7)$$

where $X = [x^h(i+1, j) \quad x^v(i, j+1)]^T$. Now, we can state following fact.

Theorem 3: 2-D system (3) is asymptotically stable if there are the Lyapunov function, (6) and the delayed function (7) such that

$$\Delta V(i, j) = V_{11}(i, j) - V_{00}(i, j) < 0 \tag{8}$$

As a result, the Theorem 3 can be used to design a 2-D control system.

3. Sliding mode control of 2-D systems

In this section, we review some prominence of the 1-D sliding mode control and then present the 2-D sliding mode control for RM.

3.1 One dimensional (1-D) Sliding Mode Control

Generally speaking, Sliding Mode Control (SMC) method is a robust control policy in which the control input is designed based on the reaching and remaining on the predetermined state trajectory. This state trajectory is commonly called switching surface (or manifold). Usually, first the switching surface is determined as a function of the state and/or time, and

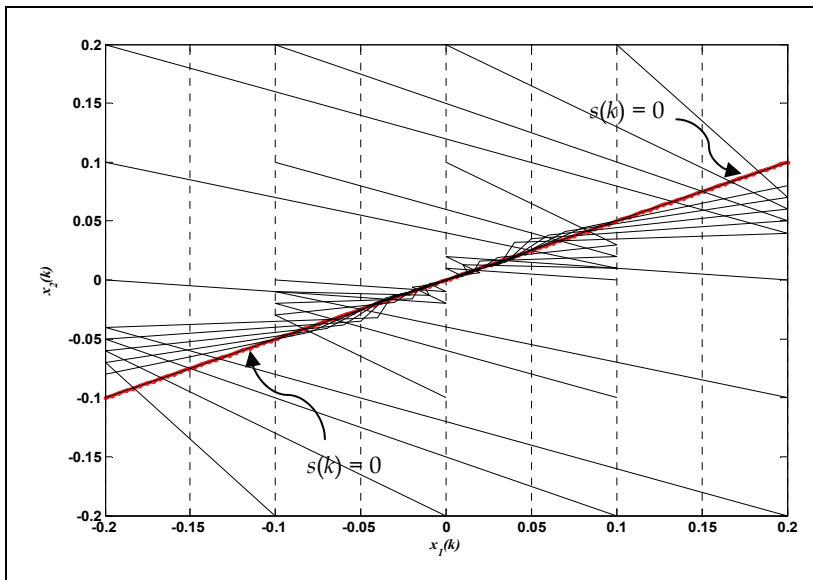


Fig. 1. State trajectory for some different initial conditions

then the control action is designed to reach and remain the state trajectory on the switching surface and move to the origin. Therefore, it can be appreciated that the switching surface should be contained the origin and designed such that the system is stable when remaining on it. Three main advantages of the SMC method are low sensitivity to the uncertainty (high robustness), dividing the system trajectory in two sections with low degree and also easily in implementation and applicability to various systems.

To make easier understanding the 1-D SMC, let consider a simple example in which a discrete time system is given as follows

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_1(k+1) = x_1(k) + x_2(k) + u(k) \end{cases} \quad (9)$$

Consider that the switching surface is

$$s(k) = x_1(k) - 2x_2(k) \quad (10)$$

also let us the control input is given as

$$u(k) = u_e(x, k) + 0.6s(k) \quad (11)$$

where $u_e(x, k) = -x_1(k) - 0.5x_2(k) - 0.5s(k)$. It is clear that the control input is $u_e(x, k)$ when the system remain on the surface (in other word when $s(k) = 0$). Fig. 1 illustrates the state trajectories of the system for some different initial conditions such that they converge to the surface and move to the origin in the vicinity of it. As it is shown in Fig. 1, the state trajectories switch around the surface when they reach the vicinity of it. The main reason of this phenomenon comes from the fact that the system dynamic equation is not exactly matched to the switching surface (Gao, 1995). In fact, the control policy in the SMC method is to reduce the error of the state trajectory to the switching surface using the switching surface feedback control. It is worthy to note that in the SMC method, the system trajectory is divided to two sections that are called reaching phase and sliding phase. Thus, the control input design is commonly performed in two steps, which named equivalent control law and switching are control law design. We want to use this strategy to present 2-D SMC design.

3.2 Two dimensional (2-D) sliding mode control

Consider the 2-D system in RM model as stated in (3).

In this chapter it is assumed that the 2-D system (3) starts from the boundary conditions that are satisfied following condition

$$\sum_{k=0}^{\infty} |x^h(0, k)|^2 + |x^v(k, 0)|^2 < \infty \quad (12)$$

where $x^h(0, k)$ and $x^v(k, 0)$ are horizontal and vertical boundary conditions. Before introducing 2-D SMC method, some definitions are represented.

Definition 1: The horizontal and vertical linear switching surfaces denoted by $s^h(i, j)$ and $s^v(i, j)$, are defined as the linear combination of the horizontal and vertical state of the 2-D system respectively as shown below

$$\begin{aligned} s^h(i, j) &= C^h x^h(i, j) \\ s^v(i, j) &= C^v x^v(i, j) \end{aligned} \quad (13)$$

where C^h and C^v are the proper constant matrices with proper dimensions.

Definition 2: Consider the 2-D system (1) starts from $(i, j) = (i_0, j_0)$. The boundary conditions can be out of or on the switching surface. So, if the system state trajectory moves toward the switching surface (11), this case is called reaching phase (or mode). After this, if it intersects switching surface at $(i, j) = (i_1, j_1)$ and remains there for all $(i, j) > (i_1, j_1)$ then this is called sliding motion or sliding phase for 2-D systems in RM.

As it is mentioned previously, a common approach to design SMC method contains two steps. First step is determination of the proper switching surface and second step is to design a control action to reach the state trajectory the surface and after it move toward the origin.

3.3 Two dimensional switching surface design

In order to design the 2-D switching surface, we want to extend a well-known method in 1-D case to 2-D case that is equivalent control approach. The equivalent control approach is based on the fact that the system state equation should be stable when it stays on the surface. In this method two points have to be considered, one is to find condition that assures staying on the surface and other is related to the stability of the system when is laid on the surface. It can be shown that two problems can be solved by 2-D Lyapunov stability presented in the theorem 3. For this purpose, let us define following Lyapunov functions

$$\begin{aligned} V_{00}(i, j) &= \frac{1}{2}[s^h(i, j)]^2 + \frac{1}{2}[s^v(i, j)]^2 \\ V_{11}(i, j) &= \frac{1}{2}[s^h(i+1, j)]^2 + \frac{1}{2}[s^v(i, j+1)]^2 \end{aligned} \quad (14)$$

According to theorem 3, the stability condition in the sense of reaching to the switching surface is occurred when the difference of two functions V_{11} and V_{00} is negative. Consequently, the condition that presents staying on the switching surface can be

$$V_{11}(i, j) - V_{00}(i, j) = 0 \quad (15)$$

Therefore, it can be concluded that

$$\begin{aligned} s^h(i+1, j) &= s^h(i, j) \\ s^v(i, j+1) &= s^v(i, j) \end{aligned} \quad (16)$$

Let define following functions as

$$S_{00}(i, j) = \begin{bmatrix} s^h(i, j) \\ s^v(i, j) \end{bmatrix}, \quad S_{11}(i, j) = \begin{bmatrix} s^h(i+1, j) \\ s^v(i, j+1) \end{bmatrix} \quad (17)$$

Thus, equality (17) can be written as

$$S_{11}(i, j) = S_{00}(i, j) \quad (18)$$

From the equations (18) we can derive the control input equivalent to the case that 2-D system (3) stay on the switching surfaces as shown below

$$\begin{aligned}
S_{11}(i, j) &= \begin{bmatrix} C^h & 0 \\ 0 & C^v \end{bmatrix} \begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} \\
&= \begin{bmatrix} C^h & 0 \\ 0 & C^v \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} C^h & 0 \\ 0 & C^v \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} u_{eq}^h(i, j) \\ u_{eq}^v(i, j) \end{bmatrix} \\
&= S_{00}(i, j)
\end{aligned} \tag{19}$$

So we have

$$\begin{bmatrix} u_{eq}^h(i, j) \\ u_{eq}^v(i, j) \end{bmatrix} = -F(C^h, C^v) \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} \tag{20}$$

Where

$$F(C^h, C^v) = \begin{bmatrix} C^h B_1 & C^h B_2 \\ C^v B_3 & C^v B_4 \end{bmatrix}^{-1} \begin{bmatrix} C^h(A_1 - I) & C^h A_2 \\ C^v A_3 & C^v(A_4 - I) \end{bmatrix} \tag{21}$$

and it is assumed that $\begin{bmatrix} C^h B_1 & C^h B_2 \\ C^v B_3 & C^v B_4 \end{bmatrix}$ is invertible. The control input (20) is called equivalent control law. Now, we should also guarantee the stability of the system when is laid on the surfaces. To perform this, it is sufficient that the following augmented system is stable.

$$\begin{aligned}
\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} &= \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} u_{eq}^h(i, j) \\ u_{eq}^v(i, j) \end{bmatrix} \\
\begin{bmatrix} s^h(i, j) \\ s^v(i, j) \end{bmatrix} &= 0
\end{aligned} \tag{22}$$

Aforementioned state updating equations (22) represents the 2-D system in the case that it is laid on the surface. By replacing the equivalent control law (20) we have

$$\begin{aligned}
\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} &= \left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} - \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} F(C^h, C^v) \right) \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} \\
\begin{bmatrix} s^h(i, j) \\ s^v(i, j) \end{bmatrix} &= 0
\end{aligned} \tag{23}$$

ith respect to the stability of the system (8), the switching surfaces can be designed.

3.4 Two dimensional control law design

After designing the proper horizontal and vertical switching surfaces, it has to be shown that the 2-D system in RM (3) with any boundary conditions, will move toward the surfaces and reach and also sliding on them toward the origin. This purpose can be interpreted as a regulating and/or tracking control strategies. To perform this purpose, consider that the control inputs are assigned as follows

$$\begin{bmatrix} u^h(i, j) \\ u^v(i, j) \end{bmatrix} = \begin{bmatrix} u_s^h(i, j) \\ u_s^v(i, j) \end{bmatrix} + \begin{bmatrix} u_{eq}^h(i, j) \\ u_{eq}^v(i, j) \end{bmatrix} \tag{24}$$

where $u_{eq}^h(i, j)$ and $u_{eq}^v(i, j)$ were designed as (20). $u_s^h(i, j)$ and $u_s^v(i, j)$ which are called switching control laws, has to be designed such that the control inputs ensure the reaching condition. In this method, it is shown that the duties of the switching control laws are to move the state trajectories toward the surfaces. Therefore, we will first determine the condition that guarantees the reaching phase. It is interesting to note that the reaching condition is also obtained in the sense of 2-D Lyapunov functions (6) and (7) using theorem 3 such that if we have

$$S_{11}^2(i, j) < S_{00}^2(i, j) \tag{25}$$

Then the state trajectories move to the surfaces. Now let us define $\Delta S = S_{11}(i, j) - S_{00}(i, j)$ and applying the equivalent control laws (20) we have

$$\begin{aligned} \Delta S = & \left(\begin{bmatrix} C^h A_1 & C^h A_2 \\ C^v A_3 & C^v A_4 \end{bmatrix} - \begin{bmatrix} C^h(A_1 - I) & C^h A_2 \\ C^v A_3 & C^v(A_4 - I) \end{bmatrix} \right) \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} \\ & + \begin{bmatrix} C^h B_1 & C^h B_2 \\ C^v B_3 & C^v B_4 \end{bmatrix} \begin{bmatrix} u_s^h(i, j) \\ u_s^v(i, j) \end{bmatrix} - \begin{bmatrix} s^h(i, j) \\ s^v(i, j) \end{bmatrix} \end{aligned} \tag{26}$$

So, this results in

$$\Delta S = \begin{bmatrix} C^h B_1 & C^h B_2 \\ C^v B_3 & C^v B_4 \end{bmatrix} \begin{bmatrix} u_s^h(i, j) \\ u_s^v(i, j) \end{bmatrix} \tag{27}$$

Theorem 4: For the 2-D system in RM (3) if the switching control law is designed as

$$\begin{bmatrix} u_s^h(i, j) \\ u_s^v(i, j) \end{bmatrix} = \begin{bmatrix} k^h s^h(i, j) \\ k^v s^v(i, j) \end{bmatrix} \tag{28}$$

where k^h and k^v are the positive constant numbers and also

$$2 \begin{bmatrix} C^h B_1 & C^h B_2 \\ C^v B_3 & C^v B_4 \end{bmatrix} + \begin{bmatrix} k^h C^h B_1 & C^h B_2 \\ C^v B_3 & k^v C^v B_4 \end{bmatrix}^T \begin{bmatrix} k^h C^h B_1 & C^h B_2 \\ C^v B_3 & k^v C^v B_4 \end{bmatrix} < 0 \tag{29}$$

then the reaching condition (25) is satisfied.

Proof:

As it is mentioned, to ensure the reaching phase it is sufficient that the (25) is satisfied. It is well-known we can write (25) as below

$$\frac{1}{2}(S_{00} + \Delta S)^2 < \frac{1}{2}S_{00}^2 \tag{30}$$

By replacing (20) into (30) we have

$$\begin{aligned}
& \frac{1}{2} \left(S_{00} + \begin{bmatrix} C^h B_1 & C^h B_2 \\ C^v B_3 & C^v B_4 \end{bmatrix} \begin{bmatrix} u_s^h(i, j) \\ u_s^v(i, j) \end{bmatrix} \right)^2 \\
&= \frac{1}{2} S_{00}^2 + S_{00}^T \begin{bmatrix} C^h B_1 & C^h B_2 \\ C^v B_3 & C^v B_4 \end{bmatrix} \begin{bmatrix} u_s^h(i, j) \\ u_s^v(i, j) \end{bmatrix} \\
&+ \frac{1}{2} \left(\begin{bmatrix} C^h B_1 & C^h B_2 \\ C^v B_3 & C^v B_4 \end{bmatrix} \begin{bmatrix} u_s^h(i, j) \\ u_s^v(i, j) \end{bmatrix} \right)^2 < \frac{1}{2} S_{00}^2
\end{aligned} \tag{31}$$

Therefore,

$$S_{00}^T \begin{bmatrix} C^h B_1 & C^h B_2 \\ C^v B_3 & C^v B_4 \end{bmatrix} \begin{bmatrix} u_s^h(i, j) \\ u_s^v(i, j) \end{bmatrix} < -\frac{1}{2} \left(\begin{bmatrix} C^h B_1 & C^h B_2 \\ C^v B_3 & C^v B_4 \end{bmatrix} \begin{bmatrix} u_s^h(i, j) \\ u_s^v(i, j) \end{bmatrix} \right)^2 \tag{32}$$

Now with respect to the switching control laws (28) we can write

$$S_{00}^T \begin{bmatrix} k^h C^h B_1 & C^h B_2 \\ C^v B_3 & k^v C^v B_4 \end{bmatrix} S_{00} + \frac{1}{2} S_{00}^T \begin{bmatrix} k^h C^h B_1 & C^h B_2 \\ C^v B_3 & k^v C^v B_4 \end{bmatrix}^T \begin{bmatrix} k^h C^h B_1 & C^h B_2 \\ C^v B_3 & k^v C^v B_4 \end{bmatrix} S_{00} < 0 \tag{33}$$

This completes the proof.

3.5 Robust control design

In this section, assume that the 2-D system in RM (3) is not given exactly and we have

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} = (A + \Delta A) \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + (B + \Delta B) \begin{bmatrix} u^h(i, j) \\ u^v(i, j) \end{bmatrix} \tag{34}$$

where ΔA and ΔB are denoted as the uncertainties in the system. Assume that

$$\begin{aligned}
\Delta A &= pA \\
\Delta B &= pB
\end{aligned} \tag{35}$$

where p is an unknown constant number and there exists a known positive real number, α , such that

$$|p| < \alpha \tag{36}$$

In this case we present following theorem.

Theorem 5: The state trajectories of the uncertain 2-D system (34) is converged to the switching surfaces (13) if

$$\alpha \begin{bmatrix} C^h B_1 & C^h B_2 \\ C^v B_3 & C^v B_4 \end{bmatrix} + \alpha^2 \begin{bmatrix} k^h C^h B_1 & C^h B_2 \\ C^v B_3 & k^v C^v B_4 \end{bmatrix}^T \begin{bmatrix} k^h C^h B_1 & C^h B_2 \\ C^v B_3 & k^v C^v B_4 \end{bmatrix} < 0 \tag{37}$$

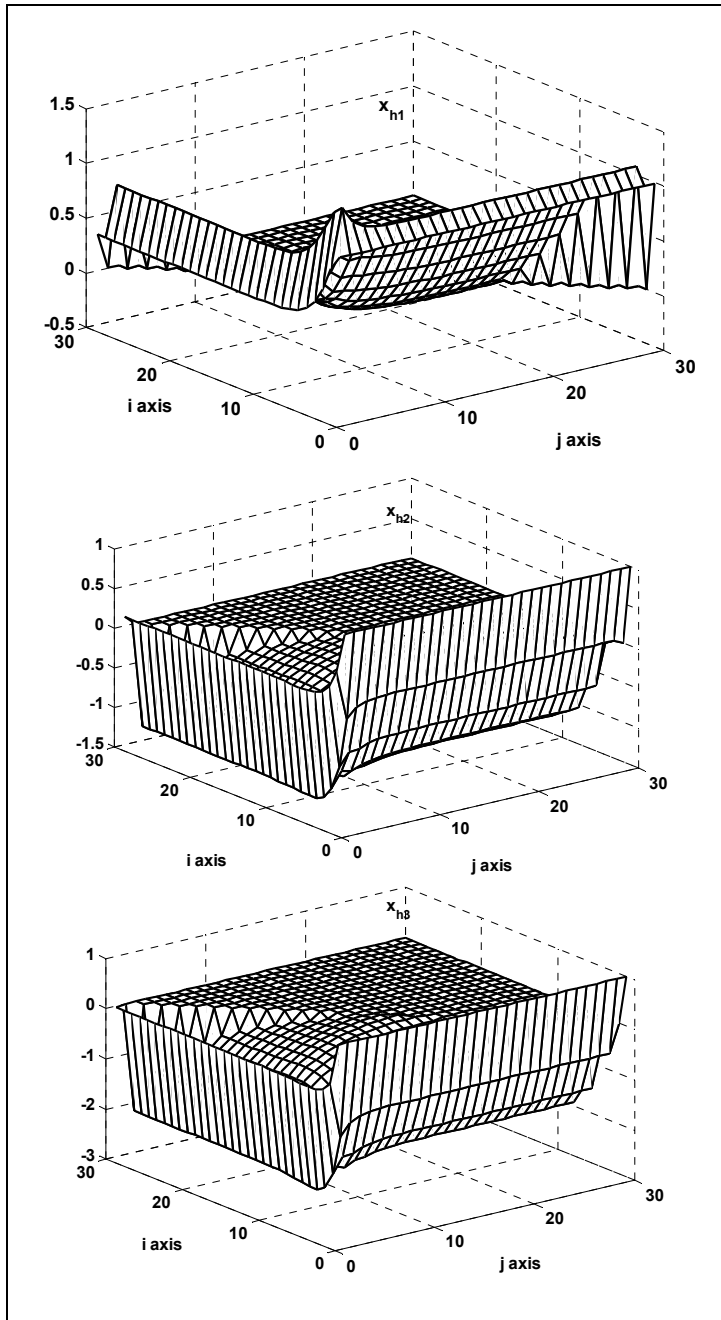


Fig. 2. The horizontal states of the system

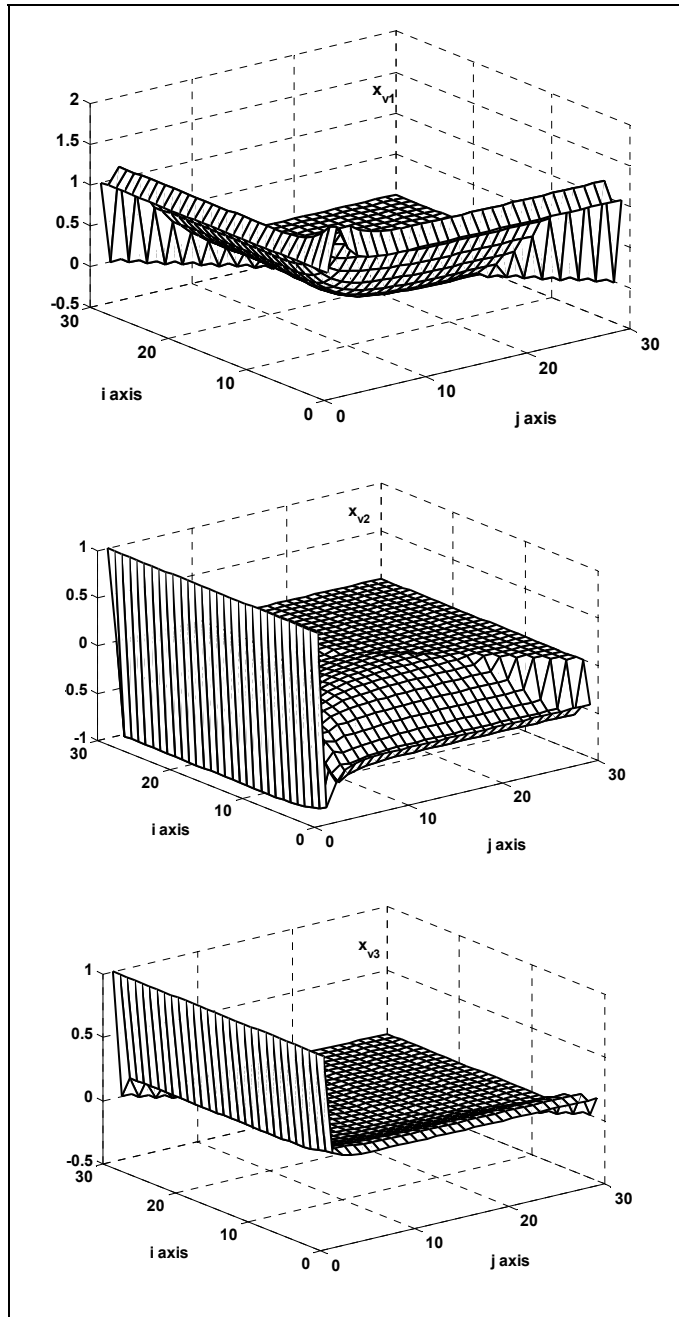


Fig. 3. The vertical states of the system

4. Numerical examples

4.1 As a first numerical example, consider a discretization of the partial differential equation of darbox equation as a 2-D system in RM (Wu & Gao, 2008) that is

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} u^h(i, j) \\ u^v(i, j) \end{bmatrix} \quad (38)$$

Where $u^v(i, j) \in R$, $u^h(i, j) \in R$, $x^h(i, j) \in R^3$, $x^v(i, j) \in R^3$ and

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.65 & -0.25 & 0.32 \\ -0.20 & 0.75 & -0.15 \\ 0.26 & 0.34 & 0.80 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0.25 & -0.30 & 0.20 \\ -0.30 & 0.15 & 0.24 \\ 0.15 & 0.36 & -0.48 \end{bmatrix} \\ A_3 &= \begin{bmatrix} 0.45 & 0.20 & -0.15 \\ 0.25 & -0.30 & 0.20 \\ -0.20 & 0.65 & 0.25 \end{bmatrix}, & A_4 &= \begin{bmatrix} 0.60 & 0.25 & 0.18 \\ -0.75 & -0.40 & 0.14 \\ 0.20 & 0.15 & -0.37 \end{bmatrix} \end{aligned} \quad (39)$$

And

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \quad (41)$$

As discussed in previous section, the switching surfaces are designed as the system equation in (22) is stable that is

$$\begin{bmatrix} x_r^h(i+1, j) \\ x_r^v(i, j+1) \end{bmatrix} = \begin{bmatrix} \frac{13}{20} - \frac{8}{25}c_1 & -\frac{1}{4} - \frac{8}{25}c_2 & \frac{1}{4} - \frac{1}{5}c_3 & -\frac{3}{10} - \frac{1}{5}c_4 \\ -\frac{1}{5} + \frac{3}{20}c_1 & \frac{3}{4} + \frac{3}{20}c_2 & -\frac{3}{10} - \frac{6}{25}c_3 & \frac{3}{20} - \frac{6}{25}c_4 \\ \frac{9}{20} + \frac{3}{20}c_1 & \frac{1}{5} + \frac{3}{20}c_2 & \frac{3}{5} - \frac{9}{50}c_3 & \frac{1}{4} - \frac{9}{50}c_4 \\ \frac{1}{4} - \frac{1}{5}c_1 & -\frac{3}{10} - \frac{1}{5}c_2 & -\frac{3}{4} - \frac{7}{50}c_3 & -\frac{2}{5} - \frac{7}{50}c_4 \end{bmatrix} \begin{bmatrix} x_r^h(i, j) \\ x_r^v(i, j) \end{bmatrix} \quad (42)$$

Where $x_r^h(i, j) \in R^2$, $x_r^v(i, j) \in R^2$ are reduced state in and

$$c_1 = \frac{c_1^h}{c_3^h}, \quad c_2 = \frac{c_2^h}{c_3^h}, \quad c_3 = \frac{c_1^v}{c_3^v}, \quad c_4 = \frac{c_2^v}{c_3^v} \quad (43)$$

It is easily shown that if we choose

$$C^h = [40.3735 \quad -99.1097 \quad 75.3160] \quad (44)$$

$$C^v = [43.6978 \quad 1.3936 \quad -290.8205]$$

Then the reduced system (42) is stable. Therefore, the equivalent control laws are

$$\begin{bmatrix} u_{eq}^h(i, j) \\ u_{eq}^v(i, j) \end{bmatrix} = -(CB)^{-1} CA \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} \quad (45)$$

Where $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, $B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$, $C = \begin{bmatrix} C^h & 0 \\ 0 & C^v \end{bmatrix}$. Also, according to theorem 5 we can obtain the switching laws that are

$$\begin{bmatrix} u_s^h(i, j) \\ u_s^v(i, j) \end{bmatrix} = \begin{bmatrix} 0.0001 & 0 \\ 0 & -0.0004 \end{bmatrix} \begin{bmatrix} s^h(i, j) \\ s^v(i, j) \end{bmatrix} \quad (46)$$

The simulation results are shown in Figs. 2 - 5.

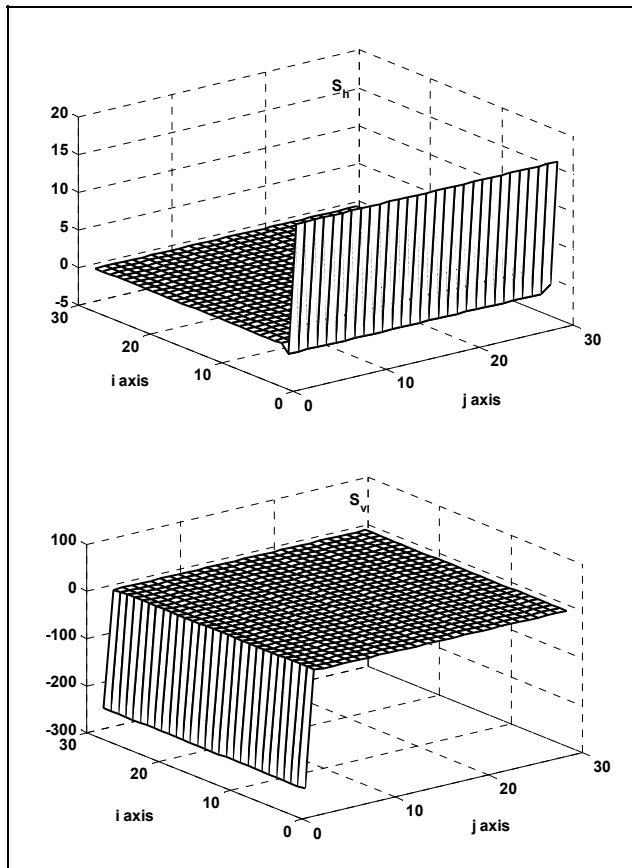


Fig. 4. The horizontal and vertical switching surfaces

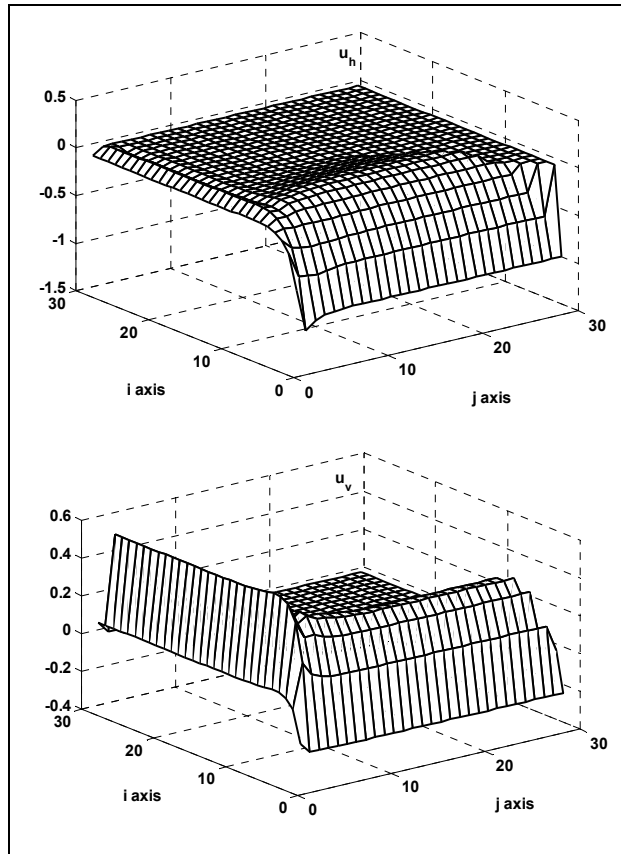


Fig. 5. The horizontal and vertical control inputs

4.2 Let a 2-D uncertain system in RM be given as follows

$$\begin{bmatrix} x_1^h(i+1, j) \\ x_2^h(i+1, j) \\ x_1^v(i, j+1) \\ x_2^v(i, j+1) \end{bmatrix} = (A + \Delta A) \begin{bmatrix} x_1^h(i, j) \\ x_2^h(i, j) \\ x_1^v(i, j) \\ x_2^v(i, j) \end{bmatrix} + (B + \Delta B) \begin{bmatrix} u^h(i, j) \\ u^v(i, j) \end{bmatrix} \quad (47)$$

Where

$$A = \begin{bmatrix} 0.7020 & 0.7846 & 1.1666 & 0.4806 \\ -1.6573 & -0.7190 & -1.7257 & -1.7637 \\ -1.0272 & -0.6165 & 1.6654 & -1.1104 \\ 0.1917 & -0.4467 & 1.0959 & -0.0200 \end{bmatrix} \quad B = \begin{bmatrix} -1.2632 & -0.3524 \\ -1.0438 & -0.2503 \\ 0.5016 & 0.8912 \\ 0.1348 & -0.0587 \end{bmatrix}$$

Suppose $\alpha = 0.5$. For this system, the switching surface is chosen as

$$\begin{bmatrix} s^h(i, j) \\ s^v(i, j) \end{bmatrix} = C \begin{bmatrix} x_1^h(i, j) \\ x_2^h(i, j) \\ x_1^v(i, j) \\ x_2^v(i, j) \end{bmatrix} \quad (48)$$

where

$$C = \begin{bmatrix} c_1^h & c_2^h & 0 & 0 \\ 0 & 0 & c_1^v & c_2^v \end{bmatrix}$$

The constant parameters c_1^h , c_2^h , c_1^v and c_2^v have to be selected such that the augmented system (22) be stable. It can be easily shown that by choosing C as

$$C = \begin{bmatrix} c_1^h & c_2^h & 0 & 0 \\ 0 & 0 & c_1^v & c_2^v \end{bmatrix} = \begin{bmatrix} -0.3608 & -0.2825 & 0 & 0 \\ 0 & 0 & 1.3173 & 0.2140 \end{bmatrix} \quad (49)$$

the augmented system (19) is stable such that

$$\begin{bmatrix} x_1^h(i+1, j) \\ x_2^h(i+1, j) \\ x_1^v(i, j+1) \\ x_2^v(i, j+1) \end{bmatrix} = \begin{bmatrix} 1.6344 & 1.1038 & 1.2997 & 0.9881 \\ -0.8101 & -0.4095 & -1.6595 & -1.2617 \\ 0.0144 & 0.0956 & 0.8075 & 0.2061 \\ -0.0887 & -0.5883 & 1.1849 & -0.2687 \end{bmatrix} \begin{bmatrix} x_1^h(i, j) \\ x_2^h(i, j) \\ x_1^v(i, j) \\ x_2^v(i, j) \end{bmatrix} \quad (50)$$

$$\begin{bmatrix} s^h(i, j) \\ s^v(i, j) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By simplifying (50), we have a reduced stable 2-D system as

$$\begin{bmatrix} x_1^h(i+1, j) \\ x_1^v(i, j+1) \end{bmatrix} = \begin{bmatrix} 0.2248 & -4.7821 \\ -0.1076 & -0.4612 \end{bmatrix} \begin{bmatrix} x_1^h(i, j) \\ x_1^v(i, j) \end{bmatrix} \quad (51)$$

So the control action that has been described in previous section is

$$\begin{bmatrix} u^h(i, j) \\ u^v(i, j) \end{bmatrix} = F \begin{bmatrix} x_1^h(i, j) \\ x_2^h(i, j) \\ x_1^v(i, j) \\ x_2^v(i, j) \end{bmatrix} - \frac{1}{2} \begin{bmatrix} s^h(i, j) \\ s^v(i, j) \end{bmatrix} \quad (52)$$

by selecting $k = 0.5$ the condition in (37) is satisfied such that

$$(1 + \alpha^2)D^T D - 1 = \begin{bmatrix} -0.3636 & -0.258 \\ -0.2580 & -0.768 \end{bmatrix} \quad (53)$$

It is clear that the above matrix is a negative definite matrix. Simulation results of this example have been illustrated in Fig 6 - 8.

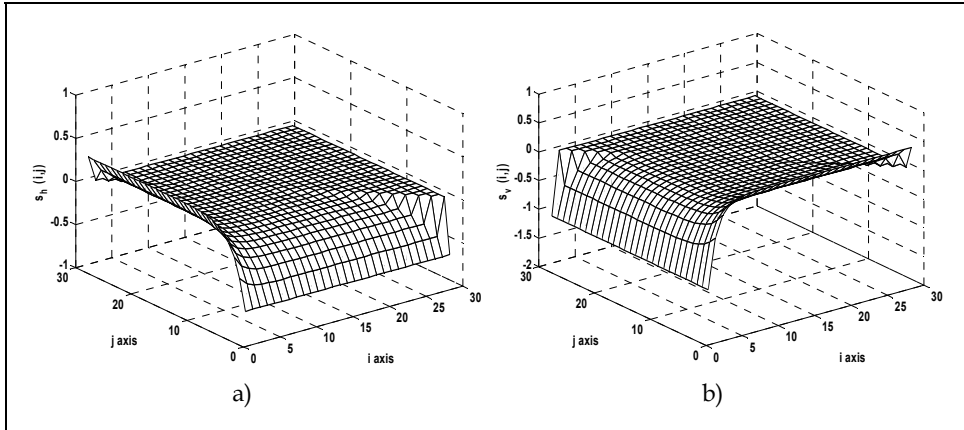


Fig. 6. a) Horizontal sliding surface $s^h(i, j)$ b) Vertical sliding surface $s^v(i, j)$

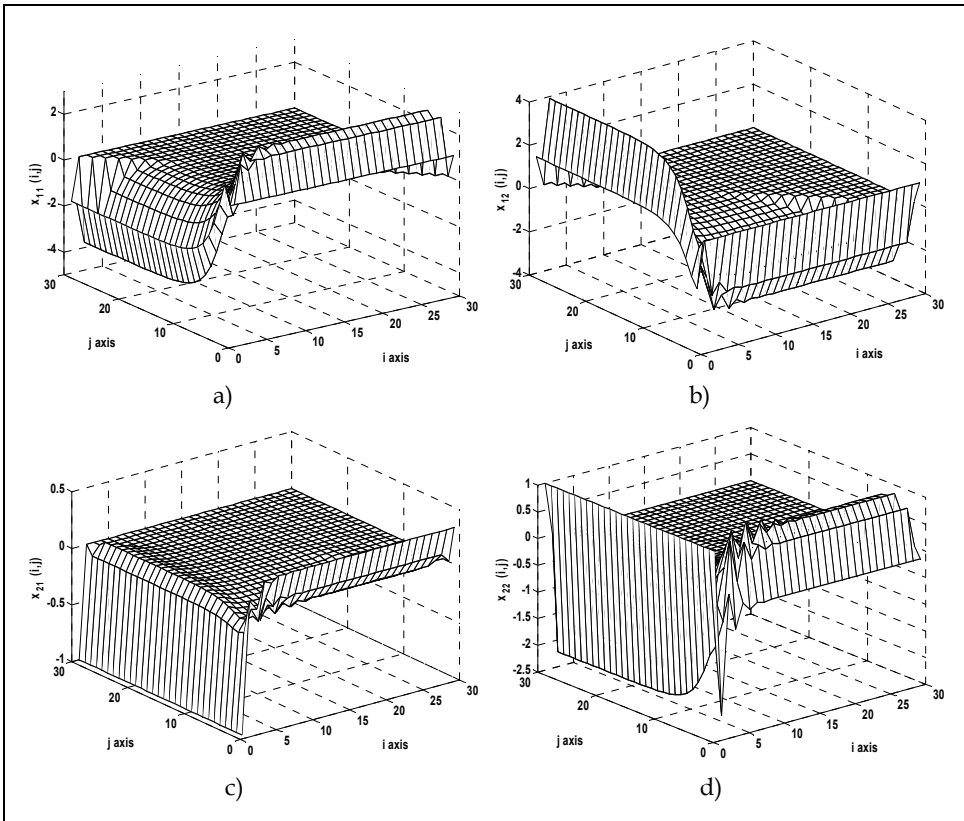


Fig. 7. System states a) $x_1^h(i, j)$, b) $x_2^h(i, j)$, c) $x_1^v(i, j)$ and d) $x_2^v(i, j)$

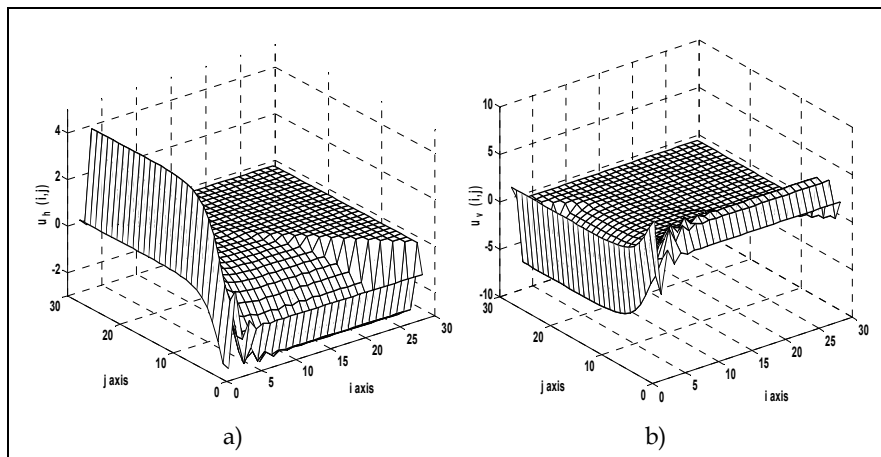


Fig. 8. a) Horizontal input control $u^h(i, j)$, b) Vertical input control $u^v(i, j)$

5. Conclusion

In this Chapter, an extension of 1-D SMC design to the 2-D system in Roesser model has been proposed. Using a 2-D Lyapunov function, we first designed a linear switching surface, and then a feedback control law that satisfies reaching condition was obtained. This method can also be applied to 2-D uncertain systems with matching uncertainty.

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