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On Stabilizability and Detectability of Variational Control Systems

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1. Introduction

The aim of this chapter is to present several interesting connections between the input-output stability properties and the stabilizability and detectability of variational control systems, proposing a new perspective concerning the interference of the interpolation methods in control theory and extending the applicability area of the input-output methods in the stability theory.

Indeed, let X be a Banach space, let (Θ, d) be a locally compact metric space and let $\mathcal{E} = X \times \Theta$. We denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. If Y, U are two Banach spaces, we denote by $\mathcal{B}(U, Y)$ the space of all bounded linear operators from U into Y and by $\mathcal{C}_s(\Theta, \mathcal{B}(U, Y))$ the space of all continuous bounded mappings $H : \Theta \to \mathcal{B}(U, Y)$. With respect to the norm $|||H||| := \sup_{\theta \in \Theta} ||H(\theta)||$, $\mathcal{C}_s(\Theta, \mathcal{B}(U, Y))$ is a Banach space.

If $H \in \mathcal{C}_s(\Theta, \mathcal{B}(U, Y))$ and $Q \in \mathcal{C}_s(\Theta, \mathcal{B}(Y, Z))$ we denote by QH the mapping $\Theta \ni \theta \mapsto Q(\theta)H(\theta)$. It is obvious that $QH \in \mathcal{C}_s(\Theta, \mathcal{B}(U, Z))$.

Definition 1.1. Let $J \in \{\mathbb{R}_+, \mathbb{R}\}$. A continuous mapping $\sigma : \Theta \times J \to \Theta$ is called *a flow* on Θ if $\sigma(\theta, 0) = \theta$ and $\sigma(\theta, s + t) = \sigma(\sigma(\theta, s), t)$, for all $(\theta, s, t) \in \Theta \times J^2$.

Definition 1.2. A pair $\pi = (\Phi, \sigma)$ is called a *linear skew-product flow* on $\mathcal{E} = X \times \Theta$ if σ is a flow on Θ and $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{B}(X)$ satisfies the following conditions:
(i) $\Phi(\theta, 0) = I_d$, the identity operator on X, for all $\theta \in \Theta$;
(ii) $\Phi(\theta, t + s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$, for all $(\theta, t, s) \in \Theta \times \mathbb{R}_+^2$ (*the cocycle identity*);
(iii) $(\theta, t) \mapsto \Phi(\theta, t)x$ is continuous, for every $x \in X$;
(iv) there are $M \geq 1$ and $\omega > 0$ such that $||\Phi(\theta, t)|| \leq Me^{\omega t}$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$.
The mapping Φ is called *the cocycle* associated to the linear skew-product flow $\pi = (\Phi, \sigma)$.

Let $L^1_{loc}(\mathbb{R}_+, X)$ denote the linear space of all locally Bochner integrable functions $u : \mathbb{R}_+ \to X$. Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$. We consider the variational integral system

$$(S_\pi) \qquad x_\theta(t; x_0, u) = \Phi(\theta, t)x_0 + \int_0^t \Phi(\sigma(\theta, s), t - s)u(s)\, ds, \quad t \geq 0, \theta \in \Theta$$

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with $u \in L^1_{loc}(\mathbb{R}_+, X)$ and $x_0 \in X$.

Definition 1.3. The system (S_π) is said to be *uniformly exponentially stable* if there are $N, \nu > 0$ such that

$$||x_\theta(t; x_0, 0)|| \leq Ne^{-\nu t}||x_0||, \quad \forall(\theta, t) \in \Theta \times \mathbb{R}_+, \forall x_0 \in X.$$

Remark 1.4. It is easily seen that the system (S_π) is uniformly exponentially stable if and only if there are $N, \nu > 0$ such that $||\Phi(\theta, t)|| \leq Ne^{-\nu t}$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$.

If $\pi = (\Phi, \sigma)$ is a linear skew-product flow on $\mathcal{E} = X \times \Theta$ and $P \in \mathcal{C}_s(\Theta, \mathcal{B}(X))$, then there exists a unique linear skew-product flow denoted $\pi_P = (\Phi_P, \sigma)$ on $X \times \Theta$ such that this satisfies the variation of constants formula:

$$\Phi_P(\theta, t)x = \Phi(\theta, t)x + \int_0^t \Phi(\sigma(\theta, s), t - s)P(\sigma(\theta, s))\Phi_P(\theta, s)x\, ds \tag{1.1}$$

and respectively

$$\Phi_P(\theta, t)x = \Phi(\theta, t)x + \int_0^t \Phi_P(\sigma(\theta, s), t - s)P(\sigma(\theta, s))\Phi(\theta, s)x\, ds \tag{1.2}$$

for all $(x, \theta, t) \in \mathcal{E} \times \mathbb{R}_+$. Moreover, if M, ω are the exponential growth constants given by Definition 1.2 (iv) for π, then

$$||\Phi_P(\theta, t)|| \leq Me^{(\omega + M\,||P||)t}, \quad \forall(\theta, t) \in \Theta \times \mathbb{R}_+.$$

The perturbed linear skew-product flow $\pi_P = (\Phi_P, \sigma)$ is obtained inductively (see Theorem 2.1 in (Megan et al., 2002)) via the formula

$$\Phi_P(\theta, t) = \sum_{n=0}^{\infty} \Phi_n(\theta, t),$$

where

$$\Phi_0(\theta, t)x = \Phi(\theta, t)x \quad \text{and} \quad \Phi_n(\theta, t)x = \int_0^t \Phi(\sigma(\theta, s), t - s)\, P(\sigma(\theta, s))\, \Phi_{n-1}(\theta, s)x\, ds, n \geq 1$$

for every $(x, \theta) \in \mathcal{E}$ and $t \geq 0$.
Let U, Y be two Banach spaces, let $B \in \mathcal{C}_s(\Theta, \mathcal{B}(U, X))$ and $C \in \mathcal{C}_s(\Theta, \mathcal{B}(X, Y))$. We consider the variational control system (π, B, C) described by the following integral model

$$\begin{cases} x(\theta, t, x_0, u) = \Phi(\theta, t)x_0 + \int_0^t \Phi(\sigma(\theta, s), t - s)B(\sigma(\theta, s))u(s)\, ds \\ y(\theta, t, x_0, u) = C(\sigma(\theta, t))x(\theta, t, x_0, u) \end{cases}$$

where $t \geq 0, (x_0, \theta) \in \mathcal{E}$ and $u \in L^1_{loc}(\mathbb{R}_+, U)$.
Two fundamental concepts related to the asymptotic behavior of the associated perturbed systems (see (Clark et al., 2000), (Curtain & Zwart, 1995), (Sasu & Sasu, 2004)) are described by stabilizability and detectability as follows:

Definition 1.5. The system (π, B, C) is said to be:
(i) *stabilizable* if there exists a mapping $F \in \mathcal{C}_s(\Theta, \mathcal{B}(X, U))$ such that the system $(S_{\pi_{BF}})$ is uniformly exponentially stable;
(ii) *detectable* if there exists a mapping $K \in \mathcal{C}_s(\Theta, \mathcal{B}(Y, X))$ such that the system $(S_{\pi_{KC}})$ is uniformly exponentially stable.

Remark 1.6. (i) The system (π, B, C) is stabilizable if and only if there exists a mapping $F \in \mathcal{C}_s(\Theta, \mathcal{B}(X, U))$ and two constants $N, \nu > 0$ such that the perturbed linear skew-product flow $\pi_{BF} = (\Phi_{BF}, \sigma)$ has the property

$$||\Phi_{BF}(\theta, t)|| \leq Ne^{-\nu t}, \quad \forall (\theta, t) \in \Theta \times \mathbb{R}_+;$$

(ii) The system (π, B, C) is detectable if and only if there exists a mapping $K \in \mathcal{C}_s(\Theta, \mathcal{B}(Y, X))$ and two constants $N, \nu > 0$ such that the perturbed linear skew-product flow $\pi_{KC} = (\Phi_{KC}, \sigma)$ has the property

$$||\Phi_{KC}(\theta, t)|| \leq Ne^{-\nu t}, \quad \forall (\theta, t) \in \Theta \times \mathbb{R}_+.$$

In the present work we will investigate the connections between the stabilizability and the detectability of the variational control system (π, B, C) and the asymptotic properties of the variational integral system (S_π). We propose a new method based on input-output techniques and on the behavior of some associated operators between certain function spaces. We will present a distinct approach concerning the stabilizability and detectability problems for variational control systems, compared with those in the existent literature, working with several representative classes of translations invariant function spaces (see Section 2 in (Sasu, 2008) and also (Bennet & Sharpley, 1988)) and thus we extend the applicability area, providing new perspectives concerning this framework.
A special application of our main results will be the study of the connections between the exponential stability and the stabilizability and detectability of nonautonomous control systems in infinite dimensional spaces. The nonautonomous case treated in this chapter will include as consequences many interesting situations among which we mention the results obtained by Clark, Latushkin, Montgomery-Smith and Randolph (see (Clark et al., 2000)) and the authors (see (Sasu & Sasu, 2004)) concerning the connections between stabilizability, detectability and exponential stability.

2. Preliminaries on Banach function spaces and auxiliary results

In what follows we recall several fundamental properties of Banach function spaces and we introduce the main tools of our investigation. Indeed, let $\mathcal{M}(\mathbb{R}_+, \mathbb{R})$ be the linear space of all Lebesgue measurable functions $u : \mathbb{R}_+ \to \mathbb{R}$, identifying the functions equal a.e.

Definition 2.1. A linear subspace B of $\mathcal{M}(\mathbb{R}_+, \mathbb{R})$ is called *a normed function space*, if there is a mapping $|\cdot|_B : B \to \mathbb{R}_+$ such that:
(i) $|u|_B = 0$ if and only if $u = 0$ a.e.;
(ii) $|\alpha u|_B = |\alpha|\ |u|_B$, for all $(\alpha, u) \in \mathbb{R} \times B$;
(iii) $|u + v|_B \leq |u|_B + |v|_B$, for all $u, v \in B$;
(iv) if $|u(t)| \leq |v(t)|$ a.e. $t \in \mathbb{R}_+$ and $v \in B$, then $u \in B$ and $|u|_B \leq |v|_B$.

If $(B, |\cdot|_B)$ is complete, then B is called a *Banach function space.*

Remark 2.2. If $(B, |\cdot|_B)$ is a Banach function space and $u \in B$ then $|u(\cdot)| \in B$.

A remarkable class of Banach function spaces is represented by the translations invariant spaces. These spaces have a special role in the study of the asymptotic properties of the dynamical systems using control type techniques (see Sasu (2008), Sasu & Sasu (2004)).

Definition 2.3. A Banach function space $(B, |\cdot|_B)$ is said to be *invariant to translations* if for every $u : \mathbb{R}_+ \to \mathbb{R}$ and every $t > 0$, $u \in B$ if and only if the function

$$u_t : \mathbb{R}_+ \to \mathbb{R}, \quad u_t(s) = \begin{cases} u(s-t) & , s \geq t \\ 0 & , s \in [0,t) \end{cases}$$

belongs to B and $|u_t|_B = |u|_B$.

Let $C_c(\mathbb{R}_+, \mathbb{R})$ denote the linear space of all continuous functions $v : \mathbb{R}_+ \to \mathbb{R}$ with compact support contained in $\mathbb{R}_+$ and let $L^1_{loc}(\mathbb{R}_+, \mathbb{R})$ denote the linear space of all locally integrable functions $u : \mathbb{R}_+ \to \mathbb{R}$.

We denote by $\mathcal{T}(\mathbb{R}_+)$ the class of all Banach function spaces B which are invariant to translations and satisfy the following properties:
(i) $C_c(\mathbb{R}_+, \mathbb{R}) \subset B \subset L^1_{loc}(\mathbb{R}_+, \mathbb{R})$;
(ii) if $B \setminus L^1(\mathbb{R}_+, \mathbb{R}) \neq \emptyset$ then there is a continuous function $\delta \in B \setminus L^1(\mathbb{R}_+, \mathbb{R})$.

For every $A \subset \mathbb{R}_+$ we denote by χ_A the characteristic function of the set A.

Remark 2.4. (i) If $B \in \mathcal{T}(\mathbb{R}_+)$, then $\chi_{[0,t)} \in B$, for all $t > 0$.
(ii) Let $B \in \mathcal{T}(\mathbb{R}_+)$, $u \in B$ and $t > 0$. Then, the function $\tilde{u}_t : \mathbb{R}_+ \to \mathbb{R}, \tilde{u}_t(s) = u(s+t)$ belongs to B and $|\tilde{u}_t|_B \leq |u|_B$ (see (Sasu, 2008), Lemma 5.4).

Definition 2.5. (i) Let $u, v \in \mathcal{M}(\mathbb{R}_+, \mathbb{R})$. We say that u and v are *equimeasurable* if for every $t > 0$ the sets $\{s \in \mathbb{R}_+ : |u(s)| > t\}$ and $\{s \in \mathbb{R}_+ : |v(s)| > t\}$ have the same measure.
(ii) A Banach function space $(B, |\cdot|_B)$ is *rearrangement invariant* if for every equimeasurable functions $u, v : \mathbb{R}_+ \to \mathbb{R}_+$ with $u \in B$ we have that $v \in B$ and $|u|_B = |v|_B$.

We denote by $\mathcal{R}(\mathbb{R}_+)$ the class of all Banach function spaces $B \in \mathcal{T}(\mathbb{R}_+)$ which are rearrangement invariant.

A remarkable class of rearrangement invariant function spaces is represented by the so-called *Orlicz spaces* which are introduced in the following remark:

Remark 2.6. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing left-continuous function, which is not identically zero on $(0, \infty)$. The *Young function* associated with φ is defined by $Y_\varphi(t) = \int_0^t \varphi(s)\, ds$. For every $u \in \mathcal{M}(\mathbb{R}_+, \mathbb{R})$ let $M_\varphi(u) := \int_0^\infty Y_\varphi(|u(s)|)\, ds$. The set O_φ of all $u \in \mathcal{M}(\mathbb{R}_+, \mathbb{R})$ with the property that there is $k > 0$ such that $M_\varphi(ku) < \infty$, is a linear space. With respect to the norm $|u|_\varphi := \inf\{k > 0 : M_\varphi(u/k) \leq 1\}$, O_φ is a Banach space, called *the Orlicz space* associated with φ.
The Orlicz spaces are rearrangement invariant (see (Bennet & Sharpley, 1988), Theorem 8.9). Moreover, it is well known that, for every $p \in [1, \infty]$, the space $L^p(\mathbb{R}_+, \mathbb{R})$ is a particular case of Orlicz space.

Let now $(X, ||\cdot||)$ be a real or complex Banach space. For every $B \in \mathcal{T}(\mathbb{R}_+)$ we denote by $B(\mathbb{R}_+, X)$, the linear space of all Bochner measurable functions $u : \mathbb{R}_+ \to X$ with the property that the mapping $N_u : \mathbb{R}_+ \to \mathbb{R}_+, N_u(t) = ||u(t)||$ lies in B. Endowed with the norm $||u||_{B(\mathbb{R}_+,X)} := |N_u|_B$, $B(\mathbb{R}_+, X)$ is a Banach space.
Let (Θ, d) be a metric space and let $\mathcal{E} = X \times \Theta$. Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$. We consider the variational integral system

$$(S_\pi) \qquad x_\theta(t; x_0, u) = \Phi(\theta, t)x_0 + \int_0^t \Phi(\sigma(\theta, s), t-s)u(s)\, ds, \quad t \geq 0, \theta \in \Theta$$

with $u \in L^1_{loc}(\mathbb{R}_+, X)$ and $x_0 \in X$.
An important stability concept related with the asymptotic behavior of dynamical systems is described by the following concept:

Definition 2.7. Let $W \in \mathcal{T}(\mathbb{R}_+)$. The system (S_π) is said to be *completely* $(W(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$*-stable* if the following assertions hold:
(i) for every $u \in W(\mathbb{R}_+, X)$ and every $\theta \in \Theta$ the solution $x_\theta(\cdot\,; 0, u) \in W(\mathbb{R}_+, X)$;
(ii) there is $\lambda > 0$ such that $||x_\theta(\cdot\,; 0, u)||_{W(\mathbb{R}_+,X)} \leq \lambda ||u||_{W(\mathbb{R}_+,X)}$, for all $(u, \theta) \in W(\mathbb{R}_+, X) \times \Theta$.

A characterization of uniform exponential stability of variational systems in terms of the complete stability of a pair of function spaces has been obtained in (Sasu, 2008) (see Corollary 3.19) and this is given by:

Theorem 2.8. *Let $W \in \mathcal{R}(\mathbb{R}_+)$. The system (S_π) is uniformly exponentially stable if and only if (S_π) is completely $(W(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$-stable.*

The problem can be also treated in the setting of the continuous functions. Indeed, let $C_b(\mathbb{R}_+, \mathbb{R})$ be the space of all bounded continuous functions $u : \mathbb{R}_+ \to \mathbb{R}$. Let $C_0(\mathbb{R}_+, \mathbb{R})$ be the space of all continuous functions $u : \mathbb{R}_+ \to \mathbb{R}$ with $\lim\limits_{t\to\infty} u(t) = 0$ and let $C_{00}(\mathbb{R}_+, \mathbb{R}) := \{u \in C_0(\mathbb{R}_+, \mathbb{R}) : u(0) = 0\}$.

Definition 2.9. Let $V \in \{C_b(\mathbb{R}_+, \mathbb{R}), C_0(\mathbb{R}_+, \mathbb{R}), C_{00}(\mathbb{R}_+, \mathbb{R})\}$. The system (S_π) is said to be *completely* $(V(\mathbb{R}_+, X), V(\mathbb{R}_+, X))$*-stable* if the following assertions hold:
(i) for every $u \in V(\mathbb{R}_+, X)$ and every $\theta \in \Theta$ the solution $x_\theta(\cdot\,; 0, u) \in V(\mathbb{R}_+, X)$;
(ii) there is $\lambda > 0$ such that $||x_\theta(\cdot\,; 0, u)||_{V(\mathbb{R}_+,X)} \leq \lambda ||u||_{V(\mathbb{R}_+,X)}$, for all $(u, \theta) \in V(\mathbb{R}_+, X) \times \Theta$.

For the proof of the next result we refer to Corollary 3.24 in (Sasu, 2008) or, alternatively, to Theorem 5.1 in (Megan et al., 2005).

Theorem 2.10. *Let $V \in \{C_b(\mathbb{R}_+, \mathbb{R}), C_0(\mathbb{R}_+, \mathbb{R}), C_{00}(\mathbb{R}_+, \mathbb{R})\}$. The system (S_π) is uniformly exponentially stable if and only if (S_π) is completely $(V(\mathbb{R}_+, X), V(\mathbb{R}_+, X))$-stable.*

Remark 2.11. Let $W \in \mathcal{R}(\mathbb{R}_+) \cup \{C_0(\mathbb{R}_+, X), C_{00}(\mathbb{R}_+, X), C_b(\mathbb{R}_+, X)\}$. If the system (S_π) is uniformly exponentially stable then for every $\theta \in \Theta$ the linear operator

$$P_W^\theta : W(\mathbb{R}_+, X) \to W(\mathbb{R}_+, X), \quad (P_W^\theta u)(t) = \int_0^t \Phi(\sigma(\theta, s), t - s)u(s)\, ds$$

is correctly defined and bounded. Moreover, if $\lambda > 0$ is given by Definition 2.7 or respectively by Definition 2.9, then we have that $\sup_{\theta\in\Theta} ||P_W^\theta|| \leq \lambda$.

These results have several interesting applications in control theory among we mention those concerning the robustness problems (see (Sasu, 2008)) which lead to an inedit estimation of the lower bound of the stability radius, as well as to the study of the connections between stability and stabilizability and detectability of associated control systems, as we will see in what follows. It worth mentioning that these aspects were studied for the very first time for the case of systems associated to evolution operators in (Clark et al., 2000) and were extended for linear skew-product flows in (Megan et al., 2002).

3. Stabilizability and detectability of variational control systems

As stated from the very beginning, in this section our attention will focus on the connections between stabilizability, detectability and the uniform exponential stability. Let X be a Banach space, let (Θ, d) be a metric space and let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$. We consider the variational integral system

$$(S_\pi) \qquad x_\theta(t; x_0, u) = \Phi(\theta, t)x_0 + \int_0^t \Phi(\sigma(\theta, s), t-s)u(s)\, ds, \quad t \geq 0, \theta \in \Theta$$

with $u \in L^1_{loc}(\mathbb{R}_+, X)$ and $x_0 \in X$.

Let U, Y be Banach spaces and let $B \in \mathcal{C}_s(\Theta, \mathcal{B}(U, X))$, $C \in \mathcal{C}_s(\Theta, \mathcal{B}(X, Y))$. We consider the variational control system (π, B, C) described by the following integral model

$$\begin{cases} x(\theta, t, x_0, u) = \Phi(\theta, t)x_0 + \int_0^t \Phi(\sigma(\theta, s), t-s)B(\sigma(\theta, s))u(s)\, ds \\ \\ y(\theta, t, x_0, u) = C(\sigma(\theta, t))x(\theta, t, x_0, u) \end{cases}$$

where $t \geq 0$, $(x_0, \theta) \in \mathcal{E}$ and $u \in L^1_{loc}(\mathbb{R}_+, U)$.
According to Definition 1.5 it is obvious that if the system (S_π) is uniformly exponentially stable, then the control system (π, B, C) is stabilizable (via the trivial feedback $F \equiv 0$) and this is also detectable (via the trivial feedback $K \equiv 0$). The natural question arises whether the converse implication holds.

Example 3.1. Let $X = \mathbb{R}, \Theta = \mathbb{R}$ and let $\sigma(\theta, t) = \theta + t$. Let (S_π) be a variational integral system such that $\Phi(\theta, t) = I_d$ (the identity operator on X), for all $(\theta, t) \in \Theta \times \mathbb{R}_+$. Let $U = Y = X$ and let $B(\theta) = C(\theta) = I_d$, for all $\theta \in \Theta$. Let $\delta > 0$. By considering $F(\theta) = -\delta\, I_d$, for all $\theta \in \Theta$, from relation (1.1), we obtain that

$$\Phi_{BF}(\theta, t)x = x - \delta \int_0^t \Phi_{BF}(\theta, s)x\, ds, \quad \forall t \geq 0$$

for every $(x, \theta) \in \mathcal{E}$. This implies that $\Phi_{BF}(\theta, t)x = e^{-\delta t}x$, for all $t \geq 0$ and all $(x, \theta) \in \mathcal{E}$, so the perturbed system $(S_{\pi_{BF}})$ is uniformly exponentially stable. This shows that the system (π, B, C) is stabilizable.
Similarly, if $\delta > 0$, for $K(\theta) = -\delta\, I_d$, for all $\theta \in \Theta$, we deduce that the variational control system (π, B, C) is also detectable.
In conclusion, the variational control system (π, B, C) is both stabilizable and detectable, but for all that, the variational integral system (S_π) is not uniformly exponentially stable.

It follows that the stabilizability or/and the detectability of the control system (π, B, C) are not sufficient conditions for the uniform exponential stability of the system (S_π). Naturally, additional hypotheses are required. In what follows we shall prove that certain input-output conditions assure a complete resolution to this problem. The answer will be given employing new methods based on function spaces techniques.
Indeed, for every $\theta \in \Theta$, we define

$$P^\theta : L^1_{loc}(\mathbb{R}_+, X) \to L^1_{loc}(\mathbb{R}_+, X), \quad (P^\theta w)(t) = \int_0^t \Phi(\sigma(\theta, s), t-s)w(s)\, ds$$

and respectively

$$B^\theta : L^1_{loc}(\mathbb{R}_+, U) \to L^1_{loc}(\mathbb{R}_+, X), \quad (B^\theta u)(t) = B(\sigma(\theta, t))u(t)$$

$$C^\theta : L^1_{loc}(\mathbb{R}_+, X) \to L^1_{loc}(\mathbb{R}_+, Y), \quad (C^\theta v)(t) = C(\sigma(\theta, t))v(t).$$

We also associate with the control system $S = (\pi, B, C)$ three families of input-output mappings, as follows: *the left input-output operators* $\{L^\theta\}_{\theta\in\Theta}$ defined by

$$L^\theta : L^1_{loc}(\mathbb{R}_+, U) \to L^1_{loc}(\mathbb{R}_+, X), \quad L^\theta := P^\theta B^\theta$$

the right input-output operators $\{R^\theta\}_{\theta\in\Theta}$ given by

$$R^\theta : L^1_{loc}(\mathbb{R}_+, X) \to L^1_{loc}(\mathbb{R}_+, Y), \quad R^\theta := C^\theta P^\theta$$

and respectively *the global input-output operators* $\{G^\theta\}_{\theta\in\Theta}$ defined by

$$G^\theta : L^1_{loc}(\mathbb{R}_+, U) \to L^1_{loc}(\mathbb{R}_+, Y), \quad G^\theta := C^\theta P^\theta B^\theta.$$

A fundamental stability concept for families of linear operators is given by the following:

Definition 3.2. Let Z_1, Z_2 be two Banach spaces and let $W \in \mathcal{T}(\mathbb{R}_+)$ be a Banach function space. A family of linear operators $\{O^\theta : L^1_{loc}(\mathbb{R}_+, Z_1) \to L^1_{loc}(\mathbb{R}_+, Z_2)\}_{\theta\in\Theta}$ is said to be $(W(\mathbb{R}_+, Z_1), W(\mathbb{R}_+, Z_2))$*-stable* if the following conditions are satisfied:
(i) for every $\alpha_1 \in W(\mathbb{R}_+, Z_1)$ and every $\theta \in \Theta$, $O^\theta\alpha_1 \in W(\mathbb{R}_+, Z_2)$;
(ii) there is $m > 0$ such that $||O^\theta\alpha_1||_{W(\mathbb{R}_+,Z_2)} \le m\, ||\alpha_1||_{W(\mathbb{R}_+,Z_1)}$, for all $\alpha_1 \in W(\mathbb{R}_+, Z_1)$ and all $\theta \in \Theta$.

Thus, we observe that if $W \in \mathcal{R}(\mathbb{R}_+)$, then the variational integral system (S_π) is uniformly exponentially stable if and only if the family $\{P^\theta\}_{\theta\in\Theta}$ is $(W(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$-stable (see also Remark 2.11).

Remark 3.3. Let Z_1, Z_2 be two Banach spaces and let $W \in \mathcal{T}(\mathbb{R}_+)$ be a Banach function space. If $Q \in \mathcal{C}_s(\Theta, \mathcal{B}(Z_1, Z_2))$ then the family $\{Q^\theta\}_{\theta\in\Theta}$ defined by

$$Q^\theta : L^1_{loc}(\mathbb{R}_+, Z_1) \to L^1_{loc}(\mathbb{R}_+, Z_2), \quad (Q^\theta\alpha)(t) = Q(\sigma(\theta, t))\alpha(t)$$

is $(W(\mathbb{R}_+, Z_1), W(\mathbb{R}_+, Z_2))$-stable. Indeed, this follows from Definition 2.1 (iv) by observing that

$$||(Q^\theta\alpha)(t)|| \le |||Q|||\, ||\alpha(t)||, \quad \forall t \ge 0, \forall\alpha \in W(\mathbb{R}_+, Z_1), \forall\theta \in \Theta.$$

The main result of this section is:

Theorem 3.4. *Let W be a Banach function space such that* $W \in \mathcal{R}(\mathbb{R}_+)$. *The following assertions are equivalent:*
(i) *the variational integral system* (S_π) *is uniformly exponentially stable;*
(ii) *the variational control system* (π, B, C) *is stabilizable and the family of the left input-output operators* $\{L^\theta\}_{\theta\in\Theta}$ *is* $(W(\mathbb{R}_+, U), W(\mathbb{R}_+, X))$*-stable;*
(iii) *the variational control system* (π, B, C) *is detectable and the family of the right input-output operators* $\{R^\theta\}_{\theta\in\Theta}$ *is* $(W(\mathbb{R}_+, X), W(\mathbb{R}_+, Y))$*-stable*
(iv) *the variational control system* (π, B, C) *is stabilizable, detectable and the family of the global input-output operators* $\{G^\theta\}_{\theta\in\Theta}$ *is* $(W(\mathbb{R}_+, U), W(\mathbb{R}_+, Y))$*-stable.*

Proof. We will independently prove each equivalence $(i) \iff (ii)$, $(i) \iff (iii)$ and respectively $(i) \iff (iv)$. Indeed, we start with the first one and we prove that $(i) \implies (ii)$. Taking into account that (S_π) is uniformly exponentially stable, we have that the family $\{P^\theta\}_{\theta\in\Theta}$ is $(W(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$-stable. In addition, observing that

$$||(L^\theta u)(t)|| \leq \sup_{\theta\in\Theta} ||P^\theta||\; |||B|||\; ||u(t)||, \quad \forall u \in W(\mathbb{R}_+, U), \forall\theta \in \Theta$$

from Definition 2.1 (iv) we deduce that that the family $\{L^\theta\}_{\theta\in\Theta}$ is $(W(\mathbb{R}_+, U), W(\mathbb{R}_+, X))$-stable.

To prove the implication $(ii) \implies (i)$, let $F \in \mathcal{C}_s(\Theta, \mathcal{B}(X, U))$ be such that the system $(S_{\pi_{BF}})$ is uniformly exponentially stable. It follows that the family $\{H^\theta\}_{\theta\in\Theta}$ is $(W(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$-stable, where

$$H^\theta : L^1_{loc}(\mathbb{R}_+, X) \to L^1_{loc}(\mathbb{R}_+, X), \quad (H^\theta u)(t) = \int_0^t \Phi_{BF}(\sigma(\theta, s), t-s)u(s)\, ds, \quad t \geq 0, \theta \in \Theta.$$

For every $\theta \in \Theta$ let

$$F^\theta : L^1_{loc}(\mathbb{R}_+, X) \to L^1_{loc}(\mathbb{R}_+, U), \quad (F^\theta u)(t) = F(\sigma(\theta, t))u(t).$$

Then from Remark 3.3 we have that the family $\{F^\theta\}_{\theta\in\Theta}$ is $(W(\mathbb{R}_+, X), W(\mathbb{R}_+, U))$-stable.
Let $\theta \in \Theta$ and let $u \in L^1_{loc}(\mathbb{R}_+, X)$. Using Fubini's theorem and formula (1.1), we successively deduce that

$$(L^\theta F^\theta H^\theta u)(t) = \int_0^t \int_0^s \Phi(\sigma(\theta, s), t-s)B(\sigma(\theta, s))F(\sigma(\theta, s))\Phi_{BF}(\sigma(\theta, \tau), s-\tau)u(\tau)\, d\tau\, ds =$$

$$= \int_0^t \int_\tau^t \Phi(\sigma(\theta, s), t-s)B(\sigma(\theta, s))F(\sigma(\theta, s))\Phi_{BF}(\sigma(\theta, \tau), s-\tau)u(\tau)\, ds\, d\tau =$$

$$= \int_0^t \int_0^{t-\tau} \Phi(\sigma(\theta, \tau+\xi), t-\tau-\xi)B(\sigma(\theta, \tau+\xi))F(\sigma(\theta, \tau+\xi))\Phi_{BF}(\sigma(\theta, \tau), \xi)u(\tau)\, d\xi\, d\tau =$$

$$= \int_0^t [\Phi_{BF}(\sigma(\theta, \tau), t-\tau)u(\tau) - \Phi(\sigma(\theta, \tau), t-\tau)u(\tau)]\, d\tau =$$

$$= (H^\theta u)(t) - (P^\theta u)(t), \quad \forall t \geq 0.$$

This shows that

$$P^\theta u = H^\theta u - L^\theta F^\theta H^\theta u, \quad \forall u \in L^1_{loc}(\mathbb{R}_+, X), \forall\theta \in \Theta. \tag{3.1}$$

Let m_1 and m_2 be two constants given by Definition 3.2 (ii) for $\{H^\theta\}_{\theta\in\Theta}$ and for $\{L^\theta\}_{\theta\in\Theta}$, respectively. From relation (3.1) we deduce that $P^\theta u \in W(\mathbb{R}_+, X)$, for every $u \in W(\mathbb{R}_+, X)$ and

$$||P^\theta u||_{W(\mathbb{R}_+,X)} \leq m_1(1 + m_2|||F|||)\, ||u||_{W(\mathbb{R}_+,X)}, \quad \forall u \in W(\mathbb{R}_+, X), \forall\theta \in \Theta.$$

From the above relation we obtain that the family $\{P^\theta\}_{\theta\in\Theta}$ is $(W(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$-stable, so the system (S_π) is uniformly exponentially stable.

The implication $(i) \implies (iii)$ follows using similar arguments with those used in the proof of $(i) \implies (ii)$. To prove $(iii) \implies (i)$, let $K \in \mathcal{C}_s(\Theta, \mathcal{B}(Y, X))$ be such that the system $(S_{\pi_{KC}})$ is uniformly exponentially stable. Then, the family $\{\Gamma^\theta\}_{\theta\in\Theta}$ is $(W(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$-stable, where

$$\Gamma^\theta : L^1_{loc}(\mathbb{R}_+, X) \to L^1_{loc}(\mathbb{R}_+, X), \quad (\Gamma^\theta u)(t) = \int_0^t \Phi_{KC}(\sigma(\theta, s), t-s)u(s)\, ds.$$

For every $\theta \in \Theta$ we define

$$K^\theta : L^1_{loc}(\mathbb{R}_+, Y) \to L^1_{loc}(\mathbb{R}_+, X), \quad (K^\theta u)(t) = K(\sigma(\theta, t))u(t).$$

From Remark 3.3 we have that the family $\{K^\theta\}_{\theta\in\Theta}$ is $(W(\mathbb{R}_+, Y), W(\mathbb{R}_+, X))$-stable.
Using Fubini's theorem and the relation (1.2), by employing similar arguments with those from the proof of the implication $(ii) \Longrightarrow (i)$, we deduce that

$$P^\theta u = \Gamma^\theta u - \Gamma^\theta K^\theta R^\theta u, \quad \forall u \in L^1_{loc}(\mathbb{R}_+, X), \forall \theta \in \Theta. \tag{3.2}$$

Denoting by q_1 and by q_2 some constants given by Definition 3.2 (ii) for $\{\Gamma^\theta\}_{\theta\in\Theta}$ and for $\{R^\theta\}_{\theta\in\Theta}$, respectively, from relation (3.2) we have that $P^\theta u \in W(\mathbb{R}_+, X)$, for every $u \in W(\mathbb{R}_+, X)$ and

$$||P^\theta u||_{W(\mathbb{R}_+,X)} \le q_1(1 + q_2|||K|||)\, ||u||_{W(\mathbb{R}_+,X)}, \quad \forall u \in W(\mathbb{R}_+, X), \forall \theta \in \Theta.$$

Hence we deduce that the family $\{P^\theta\}_{\theta\in\Theta}$ is $(W(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$-stable, which shows that the system (S_π) is uniformly exponentially stable.
The implication $(i) \Longrightarrow (iv)$ is obvious, taking into account the above items. To prove that $(iv) \Longrightarrow (i)$, let $K \in \mathcal{C}_s(\Theta, \mathcal{B}(Y, X))$ be such that the system $(S_{\pi_{KC}})$ is uniformly exponentially stable and let $\{K^\theta\}_{\theta\in\Theta}$ and $\{\Gamma^\theta\}_{\theta\in\Theta}$ be defined in the same manner like in the previous stage.
Then, following the same steps as in the previous implications, we obtain that

$$L^\theta u = \Gamma^\theta B^\theta u - \Gamma^\theta K^\theta G^\theta u, \quad \forall u \in L^1_{loc}(\mathbb{R}_+, X), \forall \theta \in \Theta. \tag{3.3}$$

From relation (3.3) we deduce that the family $\{L^\theta\}_{\theta\in\Theta}$ is $(W(\mathbb{R}_+, U), W(\mathbb{R}_+, X))$-stable.
Taking into account that the system (π, B, C) is stabilizable and applying the implication $(ii) \Longrightarrow (i)$, we conclude that the system (S_π) is uniformly exponentially stable. □

Corollary 3.5. *Let* $V \in \{C_b(\mathbb{R}_+, \mathbb{R}), C_0(\mathbb{R}_+, \mathbb{R}), C_{00}(\mathbb{R}_+, \mathbb{R})\}$. *The following assertions are equivalent:*
(i) *the variational integral system* (S_π) *is uniformly exponentially stable;*
(ii) *the variational control system* (π, B, C) *is stabilizable and the family of the left input-output operators* $\{L^\theta\}_{\theta\in\Theta}$ *is* $(V(\mathbb{R}_+, U), V(\mathbb{R}_+, X))$*-stable;*
(iii) *the variational control system* (π, B, C) *is detectable and the family of the right input-output operators* $\{R^\theta\}_{\theta\in\Theta}$ *is* $(V(\mathbb{R}_+, X), V(\mathbb{R}_+, Y))$*-stable*
(iv) *the variational control system* (π, B, C) *is stabilizable, detectable and the family of the global input-output operators* $\{G^\theta\}_{\theta\in\Theta}$ *is* $(V(\mathbb{R}_+, U), V(\mathbb{R}_+, Y))$*-stable.*

Proof. This follows using similar arguments and estimations with those from the proof of Theorem 3.4, by applying Theorem 2.10. □

4. Applications to nonautonomous systems

An interesting application of the main results from the previous section is to deduce necessary and sufficient conditions for uniform exponential stability of nonautonomous systems in terms of stabilizability and detectability. For the first time this topic was considered in (Clark et al., 2000)). We propose in what follows a new method for the resolution of this problem based on the application of the conclusions from the variational case, using arbitrary Banach function spaces.
Let X be a Banach space and let I_d denote the identity operator on X.

Definition 4.1. A family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0} \subset \mathcal{B}(X)$ is called an *evolution family* if the following properties hold:
(i) $U(t,t) = I_d$ and $U(t,s)U(s,t_0) = U(t,t_0)$, for all $t \geq s \geq t_0 \geq 0$;
(ii) there are $M \geq 1$ and $\omega > 0$ such that $||U(t,s)|| \leq Me^{\omega(t-s)}$, for all $t \geq s \geq t_0 \geq 0$;
(iii) for every $x \in X$ the mapping $(t,s) \mapsto U(t,s)x$ is continuous.

Remark 4.2. For every $P \in \mathcal{C}_s(\mathbb{R}_+, \mathcal{B}(X))$ (see e.g. (Curtain & Zwart, 1995)) there is a unique evolution family $\mathcal{U}_P = \{U_P(t,s)\}_{t\geq s\geq 0}$ such that the variation of constants formulas hold:

$$U_P(t,s)x = U(t,s)x + \int_s^t U(t,\tau)P(\tau)U_P(\tau,s)x\, d\tau, \quad \forall t \geq s \geq 0, \forall x \in X$$

and respectively

$$U_P(t,s)x = U(t,s)x + \int_s^t U_P(t,\tau)P(\tau)U(\tau,s)x\, d\tau, \quad \forall t \geq s \geq 0, \forall x \in X.$$

Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family on X. We consider the nonautonomous integral system

$$(S_{\mathcal{U}}) \qquad x_s(t;x_0,u) = U(t,s)x_0 + \int_s^t U(t,\tau)u(\tau)\, d\tau, \quad t \geq s,\ s \geq 0$$

with $u \in L^1_{loc}(\mathbb{R}_+, X)$ and $x_0 \in X$.

Definition 4.3. The system $(S_{\mathcal{U}})$ is said to be *uniformly exponentially stable* if there are $N, \nu > 0$ such that $||x_s(t;x_0,0)|| \leq Ne^{-\nu(t-s)}||x_0||$, for all $t \geq s \geq 0$ and all $x_0 \in X$.

Remark 4.4. The system $(S_{\mathcal{U}})$ is uniformly exponentially stable if and only if there are $N, \nu > 0$ such that $||U(t,s)|| \leq Ne^{-\nu(t-s)}$, for all $t \geq s \geq 0$.

Definition 4.5. Let $W \in \mathcal{T}(\mathbb{R}_+)$. The system $(S_{\mathcal{U}})$ is said to be *completely* $(W(\mathbb{R}_+, X),$ $W(\mathbb{R}_+, X))$*-stable* if for every $u \in W(\mathbb{R}_+, X)$, the solution $x_0(\cdot;0,u) \in W(\mathbb{R}_+, X)$.

Remark 4.6. If the system $(S_{\mathcal{U}})$ is completely $(W(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$-stable, then it makes sense to consider the linear operator

$$\mathcal{P} : W(\mathbb{R}_+, X) \to W(\mathbb{R}_+, X), \quad \mathcal{P}(u) = x_0(\cdot;0,u).$$

It is easy to see that $\mathcal{P}$ is closed, so it is bounded.

Let now U, Y be Banach spaces, let $B \in \mathcal{C}_s(\mathbb{R}_+, \mathcal{B}(U,X))$ and let $C \in \mathcal{C}_s(\mathbb{R}_+, \mathcal{B}(X,Y))$. We consider the nonautonomous control system $(\mathcal{U}, B, C)$ described by the following integral model

$$\begin{cases} x_s(t;x_0,u) = U(t,s)x_0 + \int_s^t U(t,\tau)B(\tau)u(\tau)\, d\tau, & t \geq s,\ s \geq 0 \\ y_s(t;x_0,u) = C(t)x_s(t;x_0,u), & t \geq s,\ s \geq 0 \end{cases}$$

with $u \in L^1_{loc}(\mathbb{R}_+, U), x_0 \in X$.

Definition 4.7. The system $(\mathcal{U}, B, C)$ is said to be:
(i) *stabilizable* if there exists $F \in \mathcal{C}_s(\mathbb{R}_+, \mathcal{B}(X,U))$ such that the system $(S_{\mathcal{U}_{BF}})$ is uniformly exponentially stable;
(ii) *detectable* if there exists $G \in \mathcal{C}_s(\mathbb{R}_+, \mathcal{B}(Y,X))$ such that the system $(S_{\mathcal{U}_{GC}})$ is uniformly exponentially stable.

We consider the operators

$$\mathcal{B} : L^1_{loc}(\mathbb{R}_+, U) \to L^1_{loc}(\mathbb{R}_+, X), \quad (\mathcal{B}u)(t) = B(t)u(t)$$

$$\mathcal{C} : L^1_{loc}(\mathbb{R}_+, X) \to L^1_{loc}(\mathbb{R}_+, Y), \quad (\mathcal{C}u)(t) = B(t)u(t)$$

and we associate with the system $(\mathcal{U}, B, C)$ three input-output operators: *the left input-output operator* defined by

$$\mathcal{L} : L^1_{loc}(\mathbb{R}_+, U) \to L^1_{loc}(\mathbb{R}_+, X), \quad \mathcal{L} = \mathcal{P}\mathcal{B}$$

the right input-output operator given by

$$\mathcal{R} : L^1_{loc}(\mathbb{R}_+, X) \to L^1_{loc}(\mathbb{R}_+, Y), \quad \mathcal{R} = \mathcal{C}\mathcal{P}$$

and respectively *the global input-output operator* defined by

$$\mathcal{G} : L^1_{loc}(\mathbb{R}_+, U) \to L^1_{loc}(\mathbb{R}_+, Y), \quad \mathcal{G} = \mathcal{C}\mathcal{P}\mathcal{B}.$$

Definition 4.8. Let Z_1, Z_2 be two Banach spaces and let $W \in \mathcal{T}(\mathbb{R}_+)$ be a Banach function space. An operator $Q : L^1_{loc}(\mathbb{R}_+, Z_1) \to L^1_{loc}(\mathbb{R}_+, Z_2)$ is said to be $(W(\mathbb{R}_+, Z_1), W(\mathbb{R}_+, Z_2))$-*stable* if for every $\lambda \in W(\mathbb{R}_+, Z_1)$ the function $Q\lambda \in W(\mathbb{R}_+, Z_2)$.

The main result of this section is:

Theorem 4.9. *Let W be a Banach function space such that $B \in \mathcal{R}(\mathbb{R}_+)$. The following assertions are equivalent:*
(i) the integral system $(S_{\mathcal{U}})$ is uniformly exponentially stable;
(ii) the control system $(\mathcal{U}, B, C)$ is stabilizable and the left input-output operator $\mathcal{L}$ is $(W(\mathbb{R}_+, U), W(\mathbb{R}_+, X))$-stable;
(iii) the control system $(\mathcal{U}, B, C)$ is detectable and the right input-output operator $\mathcal{R}$ is $(W(\mathbb{R}_+, X), W(\mathbb{R}_+, Y))$-stable;
(iv) the control system $(\mathcal{U}, B, C)$ is stabilizable, detectable and the global input-output operator $\mathcal{G}$ is $(W(\mathbb{R}_+, U), W(\mathbb{R}_+, Y))$-stable.

Proof. We prove the equivalence $(i) \Longleftrightarrow (ii)$, the other equivalences: $(i) \Longleftrightarrow (iii)$ and $(i) \Longleftrightarrow (iv)$ being similar.
Indeed, the implication $(i) \Longrightarrow (ii)$ is immediate. To prove that $(ii) \Longrightarrow (i)$ let $\Theta = \mathbb{R}_+$, $\sigma : \Theta \times \mathbb{R}_+ \to \Theta, \sigma(\theta, t) = \theta + t$ and let $\Phi(\theta, t) = U(t + \theta, \theta)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$. Then $\pi = (\Phi, \sigma)$ is a linear skew-product flow and it makes sense to associate with π the following integral system

$$(S_\pi) \qquad x_\theta(t; x_0, u) = \Phi(\theta, t)x_0 + \int_0^t \Phi(\sigma(\theta, s), t - s)u(s)\, ds, \quad t \geq 0, \theta \in \Theta$$

with $u \in L^1_{loc}(\mathbb{R}_+, X)$ and $x_0 \in X$.
We also consider the control system (π, B, C) given by

$$\begin{cases} x(\theta, t, x_0, u) = \Phi(\theta, t)x_0 + \int_0^t \Phi(\sigma(\theta, s), t - s)B(\sigma(\theta, s))u(s)\, ds \\ y(\theta, t, x_0, u) = C(\sigma(\theta, t))x(\theta, t, x_0, u) \end{cases}$$

where $t \geq 0, (x_0, \theta) \in \mathcal{E}$ and $u \in L^1_{loc}(\mathbb{R}_+, U)$. For every $\theta \in \Theta$ we associate with the system (π, B, C) the operators P^θ, B^θ and L^θ using their definitions from Section 3.

We prove that the family $\{L^\theta\}_{\theta\in\Theta}$ is $(W(\mathbb{R}_+,U),W(\mathbb{R}_+,X))$-stable. Let $\theta \in \Theta$ and let $\alpha \in W(\mathbb{R}_+,U)$. Since W is invariant to translations the function

$$\alpha_\theta : \mathbb{R}_+ \to U, \quad \alpha_\theta(t) = \begin{cases} \alpha(t-\theta), & t \geq \theta \\ 0, & t \in [0,\theta) \end{cases}$$

belongs to $W(\mathbb{R}_+,U)$ and $||\alpha_\theta||_{W(\mathbb{R}_+,U)} = ||\alpha||_{W(\mathbb{R}_+,U)}$. Since the operator $\mathcal{L}$ is $(W(\mathbb{R}_+,U),W(\mathbb{R}_+,X))$-stable we obtain that the function

$$\varphi : \mathbb{R}_+ \to X, \quad \varphi(t) = (\mathcal{L}\alpha_\theta)(t)$$

belongs to $W(\mathbb{R}_+,X)$. Using Remark 2.4 (ii) we deduce that the function

$$\gamma : \mathbb{R}_+ \to X, \quad \gamma(t) = \varphi(t+\theta)$$

belongs to $W(\mathbb{R}_+,X)$ and $||\gamma||_{W(\mathbb{R}_+,X)} \leq ||\varphi||_{W(\mathbb{R}_+,X)}$. We observe that

$$(L^\theta\alpha)(t) = \int_0^t U(\theta+t,\theta+s)B(\theta+s)\alpha(s)\,ds = \int_\theta^{\theta+t} U(\theta+t,\tau)B(\tau)\alpha(\tau-\theta)\,d\tau =$$

$$= \int_\theta^{\theta+t} U(\theta+t,\tau)B(\tau)\alpha_\theta(\tau)\,d\tau = (\mathcal{L}\alpha_\theta)(\theta+t) = \gamma(t), \quad \forall t \geq 0.$$

This implies that $L^\theta\alpha$ belongs to $W(\mathbb{R}_+,X)$ and

$$||L^\theta\alpha||_{W(\mathbb{R}_+,X)} = ||\gamma||_{W(\mathbb{R}_+,X)} \leq ||\varphi||_{W(\mathbb{R}_+,X)} \leq$$

$$\leq ||\mathcal{L}||\,||\alpha_\theta||_{W(\mathbb{R}_+,U)} = ||\mathcal{L}||\,||\alpha||_{W(\mathbb{R}_+,U)}. \tag{4.1}$$

Since $\theta \in \Theta$ and $\alpha \in W(\mathbb{R}_+,U)$ were arbitrary from (4.1) we deduce that the family $\{L^\theta\}_{\theta\in\Theta}$ is $(W(\mathbb{R}_+,U),W(\mathbb{R}_+,X))$-stable.
According to our hypothesis we have that the system $(\mathcal{U},B,C)$ is stabilizable. Then there is $F \in \mathcal{C}_s(\mathbb{R}_+,\mathcal{B}(X,U))$ such that the (unique) evolution family $\mathcal{U}_{BF} = \{U_{BF}(t,s)\}_{t\geq s\geq 0}$ which satisfies the equation

$$U_{BF}(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)F(\tau)U_{BF}(\tau,s)x\,d\tau, \quad \forall t \geq s \geq 0, \forall x \in X \tag{4.2}$$

has the property that there are $N,\nu > 0$ such that

$$||U_{BF}(t,s)|| \leq Ne^{-\nu(t-s)}, \quad \forall t \geq s \geq 0. \tag{4.3}$$

For every $(\theta,t) \in \Theta \times \mathbb{R}_+$, let $\tilde{\Phi}(\theta,t) := U_{BF}(\theta+t,\theta)$. Then, we have that $\tilde{\pi} = (\tilde{\Phi},\sigma)$ is a linear skew-product flow. Moreover, using relation (4.2) we deduce that

$$\int_0^t \Phi(\sigma(\theta,s),t-s)B(\sigma(\theta,s))F(\sigma(\theta,s))\tilde{\Phi}(\theta,s)x\,ds =$$

$$= \int_0^t U(\theta+t,\theta+s)B(\theta+s)F(\theta+s)U_{BF}(\theta+s,\theta)x\,ds =$$

$$= \int_\theta^{\theta+t} U(\theta+t,\tau)B(\tau)F(\tau)U_{BF}(\tau,\theta)x\,d\tau =$$

$$= U_{BF}(\theta + t, \theta)x - U(\theta + t, \theta)x = \tilde{\Phi}(\theta, t)x - \Phi(\theta, t)x \quad (4.4)$$

for all $(\theta, t) \in \Theta \times \mathbb{R}_+$ and all $x \in X$. According to Theorem 2.1 in (Megan et al., 2002), from relation (4.4) it follows that

$$\tilde{\Phi}(\theta, t) = \Phi_{BF}(\theta, t), \quad \forall (\theta, t) \in \Theta \times \mathbb{R}_+$$

so $\tilde{\pi} = \pi_{BF}$. Hence from relation (4.3) we have that

$$||\Phi_{BF}(\theta, t)|| = ||U_{BF}(\theta + t, \theta)|| \leq Ne^{-\nu t}, \quad \forall t \geq 0, \forall \theta \in \Theta$$

which shows that the system $(S_{\pi_{BF}})$ is uniformly exponentially stable. So the system (π, B, C) is stabilizable.
In this way we have proved that the system (S_π) is stabilizable and the associated left input-output family $\{L^\theta\}_{\theta \in \Theta}$ is $(W(\mathbb{R}_+, U), W(\mathbb{R}_+, X))$-stable. By applying Theorem 3.4 we deduce that the system (S_π) is uniformly exponentially stable. Then, there are $\tilde{N}, \delta > 0$ such that

$$||\Phi(\theta, t)|| \leq \tilde{N}e^{-\delta t}, \quad \forall t \geq 0, \forall \theta \in \Theta.$$

This implies that

$$||U(t, s)|| = ||\Phi(s, t - s)|| \leq \tilde{N}e^{-\delta(t-s)}, \quad \forall t \geq s \geq 0. \quad (4.5)$$

From inequality (4.5) and Remark 4.4 we obtain that the system $(S_{\mathcal{U}})$ is uniformly exponentially stable. □

Remark 4.10. The version of the above result, for the case when $W = L^p(\mathbb{R}_+, \mathbb{R})$ with $p \in [1, \infty)$, was proved for the first time by Clark, Latushkin, Montgomery-Smith and Randolph in (Clark et al., 2000) employing evolution semigroup techniques.

The method may be also extended for spaces of continuous functions, as the following result shows:

Corollary 4.11. *Let $V \in \{C_b(\mathbb{R}_+, \mathbb{R}), C_0(\mathbb{R}_+, \mathbb{R}), C_{00}(\mathbb{R}_+, \mathbb{R})\}$. The following assertions are equivalent:*
(i) the system $(S_{\mathcal{U}})$ is uniformly exponentially stable;
(ii) the system $(\mathcal{U}, B, C)$ is stabilizable and the left input-output operator $\mathcal{L}$ is $(V(\mathbb{R}_+, U), V(\mathbb{R}_+, X))$-stable;
(iii) the system $(\mathcal{U}, B, C)$ is detectable and the right input-output operator $\mathcal{R}$ is $(V(\mathbb{R}_+, X), V(\mathbb{R}_+, Y))$-stable;
(iv) the system $(\mathcal{U}, B, C)$ is stabilizable, detectable and the global input-output operator $\mathcal{G}$ is $(V(\mathbb{R}_+, U), V(\mathbb{R}_+, Y))$-stable.

Proof. This follows using Corollary 3.5 and similar arguments with those from the proof of Theorem 4.9. □

5. Conclusions

Stabilizability and detectability of variational/nonautonomous control systems are two properties which are strongly related with the stable behavior of the initial integral system. These two properties (not even together) cannot assure the uniform exponential stability of the initial system, as Example 3.1 shows. But, in association with a stability of certain input-output operators the stabilizability or/and the detectability of the control system (π, B, C) imply the existence of the exponentially stable behavior of the initial system (S_π). Here we have extended the topic from evolution families to variational systems and the obtained results are given in a more general context. As we have shown in Remark 2.6 the spaces involved in the stability properties of the associated input-output operators may be not only L^p-spaces but also general Orlicz function spaces which is an aspect that creates an interesting link between the modern control theory of dynamical systems and the classical interpolation theory.

It worth mentioning that the framework presented in this chapter may be also extended to some slight weak concepts, taking into account the main results concerning the uniform stability concept from Section 3 in (Sasu, 2008) (see Definition 3.3 and Theorem 3.6 in (Sasu, 2008)). More precisely, considering that the system (π, B, C) is weak stabilizable (respectively weak detectable) if there exists a mapping $F \in \mathcal{C}_s(\Theta, \mathcal{B}(X, U))$ (respectively $K \in \mathcal{C}_s(\Theta, \mathcal{B}(Y, X))$) such that the system $(S_{\pi_{BF}})$ (respectively $(S_{\pi_{KC}})$) is uniformly stable, then starting with the result provided by Theorem 3.6 in (Sasu, 2008), the methods from the present chapter may be applied to the study of the uniform stability in terms of weak stabilizability and weak detectability. In authors opinion, the technical trick of the new study will rely on the fact that in this case the families of the associated input-output operators will have to be (L^1, L^∞)-stable.

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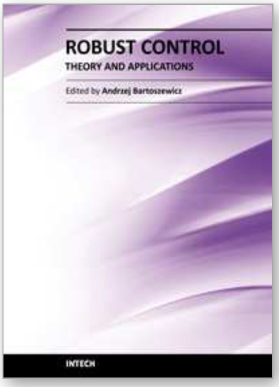

Robust Control, Theory and Applications

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The main objective of this monograph is to present a broad range of well worked out, recent theoretical and application studies in the field of robust control system analysis and design. The contributions presented here include but are not limited to robust PID, H-infinity, sliding mode, fault tolerant, fuzzy and QFT based control systems. They advance the current progress in the field, and motivate and encourage new ideas and solutions in the robust control area.

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