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Electromagnetic Waves in Plasma

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1. Introduction

No less than 99.9% of the matter in the visible Universe is in the plasma state. The plasma is a gas in which a certain portion of the particles are ionized, and is considered to be the "fourth" state of the matter. The Universe is filled with plasma particles ejected from the upper atmosphere of stars. The stream of plasma is called the stellar wind, which also carries the intrinsic magnetic field of the stars. Our solar system is filled with solar-wind-plasma particles. Neutral gases in the upper atmosphere of the Earth are also ionized by a photoelectric effect due to absorption of energy from sunlight. The number density of plasma far above the Earth's ionosphere is very low ($\sim$100cm^{-3} or much less). A typical mean-free path of solar-wind plasma is about 1AU[1] (Astronomical Unit: the distance from the Sun to the Earth). Thus plasma in Geospace can be regarded as collisionless.
Motion of plasma is affected by electromagnetic fields. The change in the motion of plasma results in an electric current, and the surrounding electromagnetic fields are then modified by the current. The plasma behaves as a dielectric media. Thus the linear dispersion relation of electromagnetic waves in plasma is strongly modified from that in vacuum, which is simply $\tilde{\omega} = kc$ where $\tilde{\omega}$, k, and c represent angular frequency, wavenumber, and the speed of light, respectively. This chapter gives an introduction to electromagnetic waves in collisionless plasma[2], because it is important to study electromagnetic waves in plasma for understanding of electromagnetic environment around the Earth.
Section 2 gives basic equations for electromagnetic waves in collisionless plasma. Then, the linear dispersion relation of plasma waves is derived. It should be noted that there are many good textbooks for linear dispersion relation of plasma waves. However, detailed derivation of the linear dispersion relation is presented only in a few textbooks (e.g., Stix, 1992; Swanson, 2003; 2008). Thus Section 2 aims to revisit the derivation of the linear dispersion relation.
Section 3 discusses excitation of plasma waves, by providing examples on the excitation of plasma waves based on the linear dispersion analysis.
Section 4 gives summary of this chapter. It is noted that the linear dispersion relation can be applied for small-amplitude plasma waves only. Large-amplitude plasma waves sometimes result in nonlinear processes. Nonlinear processes are so complex that it is difficult to provide their analytical expressions, and computer simulations play important roles in studies of nonlinear processes, which should be left as a future study.

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2. Linear dispersion relation

2.1 Basic equations

The starting point is Maxwell's equations (1-4)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{1}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}, \tag{2}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \tag{3}$$

$$\nabla \cdot \vec{B} = 0, \tag{4}$$

where $\vec{E}$, $\vec{B}$, $\vec{J}$, and ρ represent electric field, magnetic field, current density, and charge density, respectively. Here a useful relation $\epsilon_0\mu_0 = 1/c^2$ is used where ϵ_0 and μ_0 are dielectric constant and magnetic permeability in vacuum, respectively.
The motion of charged particles is described by the Newton-Lorentz equations (5,6)

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{v}, \tag{5}$$

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{q}{m}\left(\vec{E} + \vec{v} \times \vec{B}\right), \tag{6}$$

where $\vec{x}$ and $\vec{v}$ represent the position and velocity of a charged particle with q and m being its charge and mass. The motion of charged particles is also expressed in terms of microscopic distribution functions

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{q}{m}\left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \frac{\partial f}{\partial \vec{v}} = 0, \tag{7}$$

where $f[\vec{x}, \vec{v}, t]$ represents distribution function of plasma particles in a position-velocity phase space. Equation (7) is called the Vlasov equation or the collisionless Boltzmann equation (collision terms of the Boltzmann equation in right hand side is neglected). The zeroth momentum and the first momentum of the distribution function give the charge density and the current density

$$\rho = q \int f \mathrm{d}^3\vec{v}, \tag{8}$$

$$\vec{J} = q \int \vec{v} f \mathrm{d}^3\vec{v}. \tag{9}$$

2.2 Derivation of linear dispersion equation

Let us "linearize" the Vlasov equation. That is, we divide physical quantities into an equilibrium part and a small perturbation part (for the distribution function $f = n(f_0 + f_1)$ with f_0 and f_1 being the equilibrium and the small perturbation parts normalized to unity, respectively). Then the Vlasov equation (7) becomes

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \frac{\partial f_1}{\partial \vec{x}} + \frac{q}{m}\left(\vec{v} \times \vec{B}_0\right) \cdot \frac{\partial f_1}{\partial \vec{v}} = -\frac{q}{m}\left(\vec{E}_1 + \vec{v} \times \vec{B}_1\right) \cdot \frac{\partial f_0}{\partial \vec{v}}. \tag{10}$$

Here, the electric field has only the perturbed component ($\vec{E}_0 = 0$) and the multiplication of small perturbation parts is neglected ($f_1\vec{E}_1 \to 0$ and $f_1\vec{B}_1 \to 0$). Let us evaluate the term $\left(\vec{v} \times \vec{B}_0\right) \cdot \frac{\partial f_0}{\partial \vec{v}}$ by taking the spatial coordinate relative to the ambient magnetic field and writing the velocity in terms of its Cartesian coordinate $\vec{v} = [v_\perp \cos\phi, v_\perp \sin\phi, v_{||}]$. Here, $v_{||}$ and $v_\perp$ represent velocity components parallel and perpendicular to the ambient magnetic field, and $\phi = \Omega_c t + \phi_0$ represents the phase angle of the gyro-motion where $\Omega_c \equiv \frac{q}{m}B$ is the cyclotron angular frequency (with sign included). Then, we obtain

$$\left(\vec{v} \times \vec{B}_0\right) \cdot \frac{\partial f_0}{\partial \vec{v}} = \left[\begin{array}{c} v_y B_0 \\ -v_x B_0 \\ 0 \end{array}\right] \cdot \left[\begin{array}{c} \frac{\partial v_\perp}{\partial v_x}\frac{\partial f_0}{\partial v_\perp} \\ \frac{\partial v_\perp}{\partial v_y}\frac{\partial f_0}{\partial v_\perp} \\ \frac{\partial f_0}{\partial v_{||}} \end{array}\right] = \left[\begin{array}{c} v_y B_0 \\ -v_x B_0 \\ 0 \end{array}\right] \cdot \left[\begin{array}{c} \frac{v_x}{v_\perp}\frac{\partial f_0}{\partial v_\perp} \\ \frac{v_y}{v_\perp}\frac{\partial f_0}{\partial v_\perp} \\ \frac{\partial f_0}{\partial v_{||}} \end{array}\right] = 0.$$

This means that the distribution function must not be changed during the gyration of plasma particles around the ambient magnetic field at an equilibrium state. By using the total derivative, Eq.(10) can be rewritten as

$$\frac{\mathrm{d}f_1}{\mathrm{d}t} = -\frac{q}{m}\left(\vec{E}_1 + \vec{v} \times \vec{B}_1\right) \cdot \frac{\partial f_0}{\partial \vec{v}},$$

and the solution to which can be obtained as

$$f_1[\vec{x},\vec{v},t] = -\frac{q}{m}\int_{-\infty}^{t}\left(\vec{E}_1[\vec{x}',t'] + \vec{v}' \times \vec{B}_1[\vec{x}',t']\right) \cdot \frac{\partial f_0[\vec{v}']}{\partial \vec{v}'}\mathrm{d}t', \tag{11}$$

where $[\vec{x}',\vec{v}']$ is an unperturbed trajectory of a particle which passes through the point $[\vec{x},\vec{v}]$ when $t' = t$.
Let us Fourier analyze electromagnetic fields,

$$\begin{aligned} E_1(\vec{x},t) &\equiv E_1 \exp[i\vec{k}\cdot\vec{x} - i\tilde{\omega}t], \\ B_1(\vec{x},t) &\equiv B_1 \exp[i\vec{k}\cdot\vec{x} - i\tilde{\omega}t]. \end{aligned}$$

where $\tilde{\omega} \equiv \omega + i\gamma$ is complex frequency and $\vec{k}$ is wavenumber vector. Then Maxwell's equations yield

$$\vec{k} \times \vec{E}_1 = \tilde{\omega}\vec{B}_1, \tag{12}$$

$$\vec{k} \times \vec{B}_1 = -i\mu_0\vec{J}_1 - \frac{\tilde{\omega}}{c^2}\vec{E}_1. \tag{13}$$

Inserting Eq.(12) into Eq.(13), we obtain

$$\vec{k} \times (\vec{k} \times \vec{E}_1) = (\vec{k}\cdot\vec{E}_1)\vec{k} - |\vec{k}|^2\vec{E}_1 = -i\tilde{\omega}\mu_0\vec{J}_1 - \frac{\tilde{\omega}^2}{c^2}\vec{E}_1,$$

$$0 = \left(\vec{k}\vec{k} - |\vec{k}|^2\overleftrightarrow{I}\right)\frac{c^2}{\tilde{\omega}^2}\vec{E}_1 + \vec{E}_1 + i\frac{c^2}{\tilde{\omega}}\mu_0\vec{J}_1, \tag{14}$$

where $\overleftrightarrow{I}$ represents a unit tensor and $\vec{a}\vec{b}$ denotes a tensor such that

$$\vec{a}\vec{b} = \begin{bmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}^T .$$

By using Eqs.(9), (11) and (12), the last term in the right hand side of Eq.(14) yields

$$i\frac{c^2}{\tilde{\omega}}\mu_0 \vec{J}_1 = -i\frac{\Pi_p^2}{\tilde{\omega}} \int \int_{-\infty}^{t} \left(\vec{E}_1 + \vec{v}' \times \frac{\vec{k} \times \vec{E}_1}{\tilde{\omega}} \right) \cdot \frac{\partial f_0}{\partial \vec{v}'} \exp[i\vec{k}\cdot\vec{x}' - i\tilde{\omega}t'] \mathrm{d}t' \vec{v} \mathrm{d}^3\vec{v}, \tag{15}$$

where $\Pi_p \equiv \sqrt{\frac{q^2 n}{m\epsilon_0}}$ represents the plasma angular frequency. It follows that

$$\begin{aligned} &\left(\vec{E}_1 + \vec{v}' \times \frac{\vec{k} \times \vec{E}_1}{\tilde{\omega}} \right) \cdot \frac{\partial f_0}{\partial \vec{v}'} \\ = &\begin{bmatrix} E_{x1}\left(1 - \frac{v_y' k_y + v_z' k_z}{\tilde{\omega}}\right) + E_{y1}\frac{v_y' k_x}{\tilde{\omega}} + E_{z1}\frac{v_z' k_x}{\tilde{\omega}} \\ E_{x1}\frac{v_x' k_y}{\tilde{\omega}} + E_{y1}\left(1 - \frac{v_x' k_x + v_z' k_z}{\tilde{\omega}}\right) + E_{z1}\frac{v_z' k_y}{\tilde{\omega}} \\ E_{x1}\frac{v_x' k_z}{\tilde{\omega}} + E_{y1}\frac{v_y' k_z}{\tilde{\omega}} + E_{z1}\left(1 - \frac{v_x' k_x + v_y' k_y}{\tilde{\omega}}\right) \end{bmatrix} \cdot \begin{bmatrix} \frac{v_x'}{v_\perp}\frac{\partial f_0}{\partial v_\perp} \\ \frac{v_y'}{v_\perp}\frac{\partial f_0}{\partial v_\perp} \\ \frac{\partial f_0}{\partial v_{||}} \end{bmatrix} \\ = &\; E_{x1}h_x + E_{y1}h_y + E_{z1}h_z \end{aligned}$$

where

$$\left.\begin{aligned} h_x &= \frac{v_x'}{v_\perp}\left(1 - \frac{v_z' k_z}{\tilde{\omega}}\right)\frac{\partial f_0}{\partial v_\perp} + \frac{v_x' k_z}{\tilde{\omega}}\frac{\partial f_0}{\partial v_{||}} \\ h_y &= \frac{v_y'}{v_\perp}\left(1 - \frac{v_z' k_z}{\tilde{\omega}}\right)\frac{\partial f_0}{\partial v_\perp} + \frac{v_y' k_z}{\tilde{\omega}}\frac{\partial f_0}{\partial v_{||}} \\ h_z &= \frac{v_z'(v_x' k_x + v_y' k_y)}{\tilde{\omega} v_\perp}\frac{\partial f_0}{\partial v_\perp} + \left(1 - \frac{v_x' k_x + v_y' k_y}{\tilde{\omega}}\right)\frac{\partial f_0}{\partial v_{||}} \end{aligned}\right\}. \tag{16}$$

Now, let us consider transforming from Lagrangian coordinate along the unperturbed trajectory $[\vec{x}', \vec{v}', t']$ to Eulerian coordinate $[\vec{x}, \vec{v}, t]$ in a stationary frame. We define the velocity as

$$\left.\begin{aligned} v_x' &= v_\perp \cos[\Omega_c(t-t') + \phi_0] \\ v_y' &= v_\perp \sin[\Omega_c(t-t') + \phi_0] \\ v_z' &= v_{||} \end{aligned}\right\},$$

and integrate the velocity in the polar coordinate over time to obtain the position

$$\left.\begin{aligned} x' &= x - \frac{v_\perp}{\Omega_c}\{\sin[\Omega_c(t-t') + \phi_0] - \sin\phi_0\} \\ y' &= y + \frac{v_\perp}{\Omega_c}\{\cos[\Omega_c(t-t') + \phi_0] - \cos\phi_0\} \\ z' &= z - v_{||}(t-t') \end{aligned}\right\}.$$

Further taking the wavenumber vector $k_x = k_\perp \cos\theta, k_y = k_\perp \sin\theta, k_z = k_{||}$, we obtain

$$\begin{aligned}
\exp[i\vec{k}\cdot\vec{x}' - i\tilde{\omega}t'] &= \exp[i\vec{k}\cdot\vec{x} - i\tilde{\omega}t]\exp[i(\tilde{\omega} - v_{||}k_{||})(t-t')] \\
&\quad \times \exp\left[-i\frac{v_\perp k_\perp}{\Omega_c}\left\{\sin[\Omega_c(t-t') + \phi_0 - \theta] - \sin[\phi_0 - \theta]\right\}\right] \\
&= \exp[i\vec{k}\cdot\vec{x} - i\tilde{\omega}t]\sum_{l,n=-\infty}^{\infty} J_l\left[\frac{v_\perp k_\perp}{\Omega_c}\right] J_n\left[\frac{v_\perp k_\perp}{\Omega_c}\right] \\
&\quad \times \exp[i(l-n)(\phi_0 - \theta)]\exp[i(\tilde{\omega} - v_{||}k_{||} - n\Omega_c)(t-t')] \qquad (17)
\end{aligned}$$

where $J_n[x]$ is the Bessel function of the first kind of order n with

$$\exp[ia\sin\psi] = \sum_{n=-\infty}^{\infty} J_n[a]\exp[in\psi].$$

Eq.(16) also becomes

$$\left.\begin{aligned}
h_x &= \cos[\Omega_c(t-t') + \phi_0]\left\{\left(1 - \frac{v_{||}k_{||}}{\tilde{\omega}}\right)\frac{\partial f_0}{\partial v_\perp} + \frac{v_\perp k_{||}}{\tilde{\omega}}\frac{\partial f_0}{\partial v_{||}}\right\} \\
h_y &= \sin[\Omega_c(t-t') + \phi_0]\left\{\left(1 - \frac{v_{||}k_{||}}{\tilde{\omega}}\right)\frac{\partial f_0}{\partial v_\perp} + \frac{v_\perp k_{||}}{\tilde{\omega}}\frac{\partial f_0}{\partial v_{||}}\right\} \\
h_z &= \frac{v_{||}k_\perp}{\tilde{\omega}}\cos[\Omega_c(t-t') + \phi_0 - \theta]\frac{\partial f_0}{\partial v_\perp} + \left(1 - \frac{v_\perp k_\perp}{\tilde{\omega}}\cos[\Omega_c(t-t') + \phi_0 - \theta]\right)\frac{\partial f_0}{\partial v_{||}}
\end{aligned}\right\}.$$

For the time integral in Eq.(15), we use the following relationship,

$$\begin{aligned}
&\int_{-\infty}^{t}\sum_{n=-\infty}^{\infty} J_n[\lambda]\begin{bmatrix}\cos[\Omega_c(t-t') + \phi_0] \\ \sin[\Omega_c(t-t') + \phi_0] \\ 1\end{bmatrix}\exp[-in\phi_0]\exp[i(\tilde{\omega} - v_{||}k_{||} - n\Omega_c)(t-t')]\mathrm{d}t' \\
&= \sum_{n=-\infty}^{\infty}\frac{J_n[\lambda]}{2}\begin{bmatrix}\dfrac{i\exp[-i(n-1)\phi_0]}{\tilde{\omega} - v_{||}k_{||} - (n-1)\Omega_c} + \dfrac{i\exp[-i(n+1)\phi_0]}{\tilde{\omega} - v_{||}k_{||} - (n+1)\Omega_c} \\ \dfrac{\exp[-i(n-1)\phi_0]}{\tilde{\omega} - v_{||}k_{||} - (n-1)\Omega_c} - \dfrac{\exp[-i(n+1)\phi_0]}{\tilde{\omega} - v_{||}k_{||} - (n+1)\Omega_c} \\ \dfrac{2i\exp[-in\phi_0]}{\tilde{\omega} - v_{||}k_{||} - n\Omega_c}\end{bmatrix} \\
&= \sum_{n=-\infty}^{\infty}\begin{bmatrix}\dfrac{n\Omega_c}{v_\perp k_\perp}J_n[\lambda]\dfrac{i\exp[-in\phi_0]}{\tilde{\omega} - v_{||}k_{||} - n\Omega_c} \\ -J_n'[\lambda]\dfrac{\exp[-in\phi_0]}{\tilde{\omega} - v_{||}k_{||} - n\Omega_c} \\ J_n[\lambda]\dfrac{i\exp[-in\phi_0]}{\tilde{\omega} - v_{||}k_{||} - n\Omega_c}\end{bmatrix}. \qquad (18)
\end{aligned}$$

Here,

$$
\begin{aligned}
\lambda &\equiv \frac{v_\perp k_\perp}{\Omega_c}, \\
\cos\phi &= \frac{\exp[i\phi] + \exp[-i\phi]}{2}, \\
\sin\phi &= \frac{\exp[i\phi] - \exp[-i\phi]}{2i},
\end{aligned}
$$

and the following Bessel identities are used,

$$
\begin{aligned}
J_{n+1}[\lambda] + J_{n-1}[\lambda] &= \frac{2n}{\lambda} J_n[\lambda] \\
J_{n+1}[\lambda] - J_{n-1}[\lambda] &= -2J_n'[\lambda]
\end{aligned}
$$

with

$$
\sum_{n=-\infty}^{\infty} J_n[\lambda] \left(A[n-1] \pm A[n+1]\right) = \sum_{n=-\infty}^{\infty} \left(J_{n+1}[\lambda] \pm J_{n-1}[\lambda]\right) A[n]
$$

By using Eq.(18), Eq.(15) is rewritten as

$$
\begin{aligned}
& -i\frac{\Pi_p^2}{\tilde{\omega}} \int \int_{-\infty}^{t} \left(E_x h_x + E_y h_y + E_z h_z\right) \exp[i\vec{k}\cdot\vec{x}' - i\tilde{\omega}t'] \mathrm{d}t' \vec{v} \mathrm{d}^3\vec{v} \\
&= -i\frac{\Pi_p^2}{\tilde{\omega}} \int \sum_{l,n=-\infty}^{\infty} \left[\begin{array}{c} i\dfrac{n\Omega_c}{v_\perp k_\perp} J_n[\lambda] \left\{ \left(1 - \dfrac{v_{||}k_{||}}{\tilde{\omega}}\right) \dfrac{\partial f_0}{\partial v_\perp} + \dfrac{v_\perp k_{||}}{\tilde{\omega}} \dfrac{\partial f_0}{\partial v_{||}} \right\} \\ -J_n'[\lambda] \left\{ \left(1 - \dfrac{v_{||}k_{||}}{\tilde{\omega}}\right) \dfrac{\partial f_0}{\partial v_\perp} + \dfrac{v_\perp k_{||}}{\tilde{\omega}} \dfrac{\partial f_0}{\partial v_{||}} \right\} \\ i\dfrac{n\Omega_c v_{||}}{\tilde{\omega} v_\perp} J_n[\lambda] \dfrac{\partial f_0}{\partial v_\perp} + iJ_n[\lambda] \left(1 - \dfrac{n\Omega_c}{\tilde{\omega}}\right) \dfrac{\partial f_0}{\partial v_{||}} \end{array} \right] \\
& \qquad \times \frac{J_l[\lambda]}{\tilde{\omega} - v_{||}k_{||} - n\Omega_c} \exp[i(l-n)(\phi_0 - \theta)] \vec{E}_1 \vec{v} \mathrm{d}^3\vec{v} \\
&= \frac{\Pi_p^2}{\tilde{\omega}^2} \int \sum_{l,n=-\infty}^{\infty} \left[\begin{array}{c} v_\perp \cos\phi_0 \\ v_\perp \sin\phi_0 \\ v_{||} \end{array} \right] \left[\begin{array}{c} \dfrac{n\Omega_c}{v_\perp k_\perp} J_n[\lambda] U \\ iJ_n'[\lambda] U \\ J_n[\lambda] W \end{array} \right]^T \vec{E}_1 \\
& \qquad \times \frac{J_l[\lambda]}{\tilde{\omega} - v_{||}k_{||} - n\Omega_c} \exp[i(l-n)(\phi_0 - \theta)] \mathrm{d}^3\vec{v}, \qquad (19)
\end{aligned}
$$

where

$$
\begin{aligned}
U &\equiv v_\perp k_{||} \frac{\partial f_0}{\partial v_{||}} + (\tilde{\omega} - k_{||}v_{||}) \frac{\partial f_0}{\partial v_\perp}, \\
W &\equiv (\tilde{\omega} - n\Omega_c) \frac{\partial f_0}{\partial v_{||}} + \frac{n\Omega_c v_{||}}{v_\perp} \frac{\partial f_0}{\partial v_\perp}.
\end{aligned}
$$

and

$$\begin{bmatrix} v_x(E_x h_x + E_y h_y + E_z h_z) \\ v_y(E_x h_x + E_y h_y + E_z h_z) \\ v_z(E_x h_x + E_y h_y + E_z h_z) \end{bmatrix} = \begin{bmatrix} v_x h_x & v_x h_y & v_x h_z \\ v_y h_x & v_y h_y & v_y h_z \\ v_z h_x & v_z h_y & v_z h_z \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$
$$= \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}^T \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Let us assume that distribution functions are gyrotropic, i.e., $f_0(\vec{v}) \equiv f_0(v_{||}, v_\perp)$ ($\frac{\partial f_0}{\partial \phi_0} = 0$) and that the wavenumber vector $\vec{k}$ is taken in the $x - z$ plane, i.e., $\theta = 0$. Then, we have

$$\int d^3\vec{v} = \int_0^{2\pi} \int_0^\infty \int_{-\infty}^\infty v_\perp dv_{||} dv_\perp d\phi_0 = 2\pi \int_0^\infty \int_{-\infty}^\infty v_\perp dv_{||} dv_\perp$$

and

$$\begin{aligned} \int_0^{2\pi} J_l[\lambda] \cos\phi_0 \exp[i(l-n)\phi_0] d\phi_0 &= \frac{2\pi}{2} \left(J_{n-1}[\lambda] + J_{n+1}[\lambda]\right) = \frac{2\pi n}{\lambda} J_n[\lambda], \\ \int_0^{2\pi} J_l[\lambda] \sin\phi_0 \exp[i(l-n)\phi_0] d\phi_0 &= \frac{2\pi}{2i} \left(J_{n-1}[\lambda] - J_{n+1}[\lambda]\right) = \frac{2\pi}{i} J_n'[\lambda], \\ \int_0^{2\pi} J_l[\lambda] \exp[i(l-n)\phi_0] d\phi_0 &= 2\pi J_n[\lambda]. \end{aligned}$$

Thus Eq.(14) can be rewritten as

$$0 = \left(\vec{k}\vec{k} - |\vec{k}|^2 \overleftrightarrow{I}\right) \frac{c^2}{\tilde{\omega}^2} + \overleftrightarrow{\epsilon}(\tilde{\omega}, \vec{k}), \tag{20}$$

where

$$\begin{aligned} \overleftrightarrow{\epsilon}(\tilde{\omega}, \vec{k}) \vec{E}_1 &= \vec{E}_1 + i\frac{c^2}{\tilde{\omega}} \mu_0 \vec{J}_1, \\ \overleftrightarrow{\epsilon}(\tilde{\omega}, \vec{k}) &= \overleftrightarrow{I} + \sum_s \frac{\Pi_p^2}{\tilde{\omega}^2} \sum_{n=-\infty}^\infty \int \frac{1}{\tilde{\omega} - k_{||}v_{||} - n\Omega_c} \overleftrightarrow{T_n} d^3\vec{v}, \end{aligned} \tag{21}$$

with

$$\overleftrightarrow{T_n} = \begin{bmatrix} \frac{n^2\Omega_c^2}{k_\perp^2 v_\perp} J_n^2[\lambda] U & i\frac{n\Omega_c}{k_\perp} J_n[\lambda] J_n'[\lambda] U & \frac{n\Omega_c}{k_\perp} J_n^2[\lambda] W \\ -i\frac{n\Omega_c}{k_\perp} J_n[\lambda] J_n'[\lambda] U & v_\perp {J_n'}^2[\lambda] U & -iv_\perp J_n[\lambda] J_n'[\lambda] W \\ \frac{n\Omega_c v_{||}}{k_\perp v_\perp} J_n^2[\lambda] U & iv_{||} J_n[\lambda] J_n'[\lambda] U & v_{||} J_n^2[\lambda] W \end{bmatrix}.$$

Here Eq.(20) is called the linear dispersion relation and Eq.(21) is called the plasma dielectric equation. Note that $\sum_s$ is added in Eq.(21) for treating multi-species (e.g., ions and electrons) plasma $(\Pi_{ps}, \Omega_{cs}, f_{0s})$.
There also exists another expression for the plasma dielectric equation,

$$\overleftrightarrow{\epsilon} = \overleftrightarrow{I} + \sum_s \frac{\Pi_p^2}{\tilde{\omega}^2} \left\{ \sum_{n=-\infty}^{\infty} \int \frac{k_{||}\dfrac{\partial f_0}{\partial v_{||}} + \dfrac{n\Omega_c}{v_\perp}\dfrac{\partial f_0}{\partial v_\perp}}{\tilde{\omega} - k_{||}v_{||} - n\Omega_c} \overleftrightarrow{S}_n \mathrm{d}^3\vec{v} - \overleftrightarrow{I} \right\}, \tag{22}$$

where

$$\overleftrightarrow{S}_n = \begin{bmatrix} \dfrac{n^2\Omega_c^2}{k_\perp^2} J_n^2[\lambda] & i\dfrac{n\Omega_c}{k_\perp} v_\perp J_n[\lambda] J_n'[\lambda] & \dfrac{n\Omega_c}{k_\perp} v_{||} J_n^2[\lambda] \\ -i\dfrac{n\Omega_c}{k_\perp} v_\perp J_n[\lambda] J_n'[\lambda] & v_\perp^2 {J_n'}^2[\lambda] & -i v_{||} v_\perp J_n[\lambda] J_n'[\lambda] \\ \dfrac{n\Omega_c}{k_\perp} v_{||} J_n^2[\lambda] & i v_{||} v_\perp J_n[\lambda] J_n'[\lambda] & v_{||}^2 J_n^2[\lambda] \end{bmatrix}.$$

2.3 Linear dispersion relation for waves in Maxwellian plasma

The Maxwellian (or Maxwell-Boltzmann) distribution is usually regarded as a distribution of particle velocity at an equilibrium state. Plasma or charged particles easily move along an ambient magnetic field, while they do not move across the ambient magnetic field. Distributions of particle velocity often show anisotropy in the direction parallel and perpendicular to the ambient magnetic field. That is, the average drift velocity and the temperature in the direction parallel to the ambient magnetic field differ from those in the direction perpendicular to the ambient magnetic field. Thus the following shifted bi-Maxwellian distribution is used as a velocity distribution at an initial state or an equilibrium state,

$$\left.\begin{array}{rcl} f(v_{||}, v_\perp) &=& f_{||}(v_{||}) f_\perp(v_\perp) \\ f_{||}(v_{||}) &=& \dfrac{1}{\sqrt{2\pi} V_{t||}} \exp\left[-\dfrac{(v_{||} - V_d)^2}{2V_{t||}^2}\right] \\ f_\perp(v_\perp) &=& \dfrac{1}{2\pi V_{t\perp}^2} \exp\left[-\dfrac{v_\perp^2}{2V_{t\perp}^2}\right] \end{array}\right\}, \tag{23}$$

where V_d is the drift velocity in the direction parallel to the ambient magnetic field, and $V_{t||} \equiv \sqrt{T_{||}/m}$ and $V_{t\perp} \equiv \sqrt{T_\perp/m}$ are the thermal velocities in the direction parallel and perpendicular to the ambient magnetic field, respectively, with T being temperature of plasma particles.
When plasma has the Maxwellian velocity distribution (23), we can explicitly perform the velocity-space integral by using the following properties,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{f_{||}\mathrm{d}v_{||}}{\tilde{\omega} - k_{||}v_{||} - n\Omega_c} &= -\frac{1}{\sqrt{2}k_{||}V_{t||}} Z_0\left[\zeta_n\right] \\
\int_{-\infty}^{\infty} \frac{v_{||}f_{||}\mathrm{d}v_{||}}{\tilde{\omega} - k_{||}v_{||} - n\Omega_c} &= -\frac{1}{\sqrt{2}k_{||}V_{t||}} Z_1\left[\zeta_n\right] = -\frac{1}{k_{||}}\left\{1 + \frac{\tilde{\omega} - n\Omega_c}{\sqrt{2}k_{||}V_{t||}} Z_0\left[\zeta_n\right]\right\} \\
\int_{-\infty}^{\infty} \frac{v_{||}^2 f_{||}\mathrm{d}v_{||}}{\tilde{\omega} - k_{||}v_{||} - n\Omega_c} &= -\frac{1}{\sqrt{2}k_{||}V_{t||}} Z_2\left[\zeta_n\right] \\
&= -\frac{\sqrt{2}V_{t||}}{k_{||}}\left\{\frac{\tilde{\omega} + k_{||}V_d - n\Omega_c}{\sqrt{2}k_{||}V_{t||}} + \left(\frac{\tilde{\omega} - n\Omega_c}{\sqrt{2}k_{||}V_{t||}}\right)^2 Z_0\left[\zeta_n\right]\right\} \\
\int_{-\infty}^{\infty} \frac{\frac{\partial f_{||}}{\partial v_{||}}\mathrm{d}v_{||}}{\tilde{\omega} - k_{||}v_{||} - n\Omega_c} &= \frac{1}{\sqrt{2}k_{||}V_{t||}^3}\left\{Z_1\left[\zeta_n\right] - V_d Z_0\left[\zeta_n\right]\right\} \\
&= \frac{1}{k_{||}V_{t||}^2}\left\{1 + \frac{\tilde{\omega} - k_{||}V_d - n\Omega_c}{\sqrt{2}k_{||}V_{t||}} Z_0\left[\zeta_n\right]\right\} \\
\int_{-\infty}^{\infty} \frac{v_{||}\frac{\partial f_{||}}{\partial v_{||}}\mathrm{d}v_{||}}{\tilde{\omega} - k_{||}v_{||} - n\Omega_c} &= \frac{1}{\sqrt{2}k_{||}V_{t||}^3}\left\{Z_2\left[\zeta_n\right] - V_d Z_1\left[\zeta_n\right]\right\} \\
&= \frac{\tilde{\omega} - n\Omega}{k_{||}^2 V_{t||}^2}\left\{1 + \frac{\tilde{\omega} - k_{||}V_d - n\Omega_c}{\sqrt{2}k_{||}V_{t||}} Z_0\left[\zeta_n\right]\right\}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{\infty} J_n^2\left[\lambda\right] f_{\perp} 2\pi v_{\perp}\mathrm{d}v_{\perp} &= \exp\left[-\frac{a^2}{2}\right] I_n\left[\frac{a^2}{2}\right] \\
\int_0^{\infty} J_n^2\left[\lambda\right] \frac{\partial f_{\perp}}{\partial v_{\perp}} 2\pi \mathrm{d}v_{\perp} &= -\frac{1}{V_{t\perp}^2}\exp\left[-\frac{a^2}{2}\right] I_n\left[\frac{a^2}{2}\right]
\end{aligned}$$

$$\begin{aligned}
\int_0^{\infty} J_n\left[\lambda\right] J_n'\left[\lambda\right] f_{\perp} 2\pi v_{\perp}^2\mathrm{d}v_{\perp} &= \frac{k_{\perp}V_{t\perp}^2}{\Omega_c}\exp\left[-\frac{a^2}{2}\right]\left\{I_n'\left[\frac{a^2}{2}\right] - I_n\left[\frac{a^2}{2}\right]\right\} \\
\int_0^{\infty} J_n\left[\lambda\right] J_n'\left[\lambda\right] \frac{\partial f_{\perp}}{\partial v_{\perp}} 2\pi v_{\perp}\mathrm{d}v_{\perp} &= -\frac{k_{\perp}}{\Omega_c}\exp\left[-\frac{a^2}{2}\right]\left\{I_n'\left[\frac{a^2}{2}\right] - I_n\left[\frac{a^2}{2}\right]\right\}
\end{aligned}$$

$$\begin{aligned}
\int_0^{\infty} {J_n'}^2\left[\lambda\right] f_{\perp} 2\pi v_{\perp}^3\mathrm{d}v_{\perp} &= 4V_{t\perp}^2\exp\left[-\frac{a^2}{2}\right]\left\{\frac{n^2}{2a^2} I_n\left[\frac{a^2}{2}\right] - \frac{a^2}{4} I_n'\left[\frac{a^2}{2}\right] + \frac{a^2}{4} I_n\left[\frac{a^2}{2}\right]\right\} \\
\int_0^{\infty} {J_n'}^2\left[\lambda\right] \frac{\partial f_{\perp}}{\partial v_{\perp}} 2\pi v_{\perp}^2\mathrm{d}v_{\perp} &= -4\exp\left[-\frac{a^2}{2}\right]\left\{\frac{n^2}{2a^2} I_n\left[\frac{a^2}{2}\right] - \frac{a^2}{4} I_n'\left[\frac{a^2}{2}\right] + \frac{a^2}{4} I_n\left[\frac{a^2}{2}\right]\right\}
\end{aligned}$$

where

$$\zeta_n \equiv \frac{\tilde{\omega} - k_{||}V_d - n\Omega_c}{\sqrt{2}k_{||}V_{t||}},$$
$$a \equiv \frac{\sqrt{2}k_\perp V_{t\perp}}{\Omega_c} = \sqrt{2}\lambda\frac{V_{t\perp}}{v_\perp}.$$

Here $I_n[x]$ is the modified Bessel function of the first kind of order n with the following properties,

$$\int_0^\infty xJ_n^2[tx]\exp\left[-x^2\right]\mathrm{d}x = \frac{1}{2}\exp\left[-\frac{t^2}{2}\right]I_n\left[\frac{t^2}{2}\right],$$
$$\int_0^\infty x^2J_n[tx]J_n'[tx]\exp\left[-x^2\right]\mathrm{d}x = \frac{t}{4}\exp\left[-\frac{t^2}{2}\right]\left\{I_n'\left[\frac{t^2}{2}\right] - I_n\left[\frac{t^2}{2}\right]\right\},$$
$$\int_0^\infty x^3J_n'^2[tx]\exp\left[-x^2\right]\mathrm{d}x = \exp\left[-\frac{t^2}{2}\right]\left\{\frac{n^2}{2t^2}I_n\left[\frac{t^2}{2}\right] - \frac{t^2}{4}I_n'\left[\frac{t^2}{2}\right] + \frac{t^2}{4}I_n\left[\frac{t^2}{2}\right]\right\},$$

and $Z_p[x]$ is the plasma dispersion function (Fried & Conte, 1961)

$$Z_0[x] \equiv \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{1}{t-x}\mathrm{d}t, \tag{24}$$
$$Z_p[\zeta_n] \equiv -\frac{k_{||}}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{v_{||}^p}{\tilde{\omega} - k_{||}v_{||} - n\Omega_c}\exp\left[-\frac{(v_{||}-V_d)^2}{2V_{t||}^2}\right]\mathrm{d}v_{||},$$
$$Z_1[\zeta_n] = \sqrt{2}V_{t||}\left\{1 + \frac{\tilde{\omega} - n\Omega_c}{\sqrt{2}k_{||}V_{t||}}Z_0[\zeta_n]\right\},$$
$$Z_2[\zeta_n] = 2V_{t||}^2\left\{\frac{\tilde{\omega} + k_{||}V_d - n\Omega_c}{\sqrt{2}k_{||}V_{t||}} + \left(\frac{\tilde{\omega} - n\Omega_c}{\sqrt{2}k_{||}V_{t||}}\right)^2 Z_0[\zeta_n]\right\}.$$

We also use the following identity of the modified Bessel function,

$$\sum_{n=-\infty}^{\infty} nI_n'[\lambda] = \sum_{n=-\infty}^{\infty}\frac{n}{2}\left(I_{n+1}[\lambda] + I_{n-1}[\lambda]\right) = \sum_{n=-\infty}^{\infty} nI_n[\lambda] = 0$$
$$\sum_{n=-\infty}^{\infty}\left\{I_n[\lambda] - I_n'[\lambda]\right\} = \sum_{n=-\infty}^{\infty}\left\{I_n[\lambda] - \frac{1}{2}\left(I_{n+1}[\lambda] + I_{n-1}[\lambda]\right)\right\} = 0$$

Then we obtain

$$\sum_{n=-\infty}^{\infty}\int\frac{1}{\tilde{\omega} - k_{||}v_{||} - n\Omega_c}\overleftrightarrow{T_n}\mathrm{d}^3\vec{v} \equiv \overleftrightarrow{K} = \begin{bmatrix} K_{1,1} & K_{1,2} & K_{1,3} \\ -K_{1,2} & K_{2,2} & K_{2,3} \\ K_{1,3} & -K_{2,3} & K_{3,3}\end{bmatrix}, \tag{25}$$

where

$$
\begin{aligned}
K_{1,1} &= \sum_{n=-\infty}^{\infty} \frac{2n^2}{a^2} \exp\left[-\frac{a^2}{2}\right] I_n\left[\frac{a^2}{2}\right] \left\{ \left(\frac{V_{t\perp}^2}{V_{t||}^2} - 1 \right) + \frac{V_{t\perp}^2}{V_{t||}^2} \xi_n Z_0[\zeta_n] \right\}, \\
K_{1,2} &= i \sum_{n=-\infty}^{\infty} n \exp\left[-\frac{a^2}{2}\right] \left\{ I_n\left[\frac{a^2}{2}\right] - I_n'\left[\frac{a^2}{2}\right] \right\} \frac{V_{t\perp}^2}{V_{t||}^2} \xi_n Z_0[\zeta_n], \\
K_{1,3} &= \sum_{n=-\infty}^{\infty} \frac{2nV_{t\perp}}{aV_{t||}} \exp\left[-\frac{a^2}{2}\right] I_n\left[\frac{a^2}{2}\right] \left\{ -\frac{n\Omega_c \left(1 - \frac{V_{t||}^2}{V_{t\perp}^2}\right)}{\sqrt{2}k_{||}V_{t||}} + \frac{\tilde{\omega} - n\Omega_c}{\sqrt{2}k_{||}V_{t||}} \xi_n Z_0[\zeta_n] \right\}, \\
K_{2,2} &= K_{1,1} - \sum_{n=-\infty}^{\infty} a^2 \exp\left[-\frac{a^2}{2}\right] \left\{ I_n'\left[\frac{a^2}{2}\right] + I_n\left[\frac{a^2}{2}\right] \right\} \frac{V_{t\perp}^2}{V_{t||}^2} \xi_n Z_0[\zeta_n], \\
K_{2,3} &= i \sum_{n=-\infty}^{\infty} \frac{aV_{t\perp}}{V_{t||}} \exp\left[-\frac{a^2}{2}\right] \left\{ I_n\left[\frac{a^2}{2}\right] - I_n'\left[\frac{a^2}{2}\right] \right\} \frac{\tilde{\omega} - n\Omega_c}{\sqrt{2}k_{||}V_{t||}} \xi_n Z_0[\zeta_n], \\
K_{3,3} &= 2 \sum_{n=-\infty}^{\infty} \exp\left[-\frac{a^2}{2}\right] I_n\left[\frac{a^2}{2}\right] \left\{ \frac{\tilde{\omega}^2 + n^2\Omega_c^2 \left(1 - \frac{V_{t||}^2}{V_{t\perp}^2}\right)}{2k_{||}^2 V_{t||}^2} + \left(\frac{\tilde{\omega} - n\Omega_c}{\sqrt{2}k_{||}V_{t||}} \right)^2 \xi_n Z_0[\zeta_n] \right\},
\end{aligned}
$$

with

$$
\xi_n \equiv \frac{\tilde{\omega} - k_{||}V_d - n\Omega_c \left(1 - \frac{V_{t||}^2}{V_{t\perp}^2}\right)}{\sqrt{2}k_{||}V_{t||}}.
$$

The linear dispersion relation is obtained by solving the following equation,

$$
0 = \mathrm{Det}\left[\frac{c^2}{\tilde{\omega}^2} \left(\vec{k}\vec{k} - |\vec{k}|^2 \overleftrightarrow{I} \right) + \overleftrightarrow{I} + \sum_s \frac{\Pi_p^2}{\tilde{\omega}^2} \overleftrightarrow{K} \right] \tag{26}
$$

$$
\equiv \begin{vmatrix}
\sum_s \Pi_p^2 K_{1,1} + \tilde{\omega}^2 - c^2 k_{||}^2 & \sum_s \Pi_p^2 K_{1,2} & \sum_s \Pi_p^2 K_{1,3} + c^2 (k_{||} k_{\perp}) \\
-\sum_s \Pi_p^2 K_{1,2} & \sum_s \Pi_p^2 K_{2,2} + \tilde{\omega}^2 - c^2 (k_{||}^2 + k_{\perp}^2) & \sum_s \Pi_p^2 K_{2,3} \\
\sum_s \Pi_p^2 K_{1,3} + c^2 (k_{||} k_{\perp}) & -\sum_s \Pi_p^2 K_{2,3} & \sum_s \Pi_p^2 K_{3,3} + \tilde{\omega}^2 - c^2 k_{\perp}^2
\end{vmatrix}.
$$

3. Excitation of electromagnetic waves

Eq.(26) tells us what kind of plasma waves grows and damps in arbitrary Maxwellian plasma. This section gives examples on the excitation of plasma waves based on the linear dispersion analysis.

For simplicity, let us assume propagation of plasma waves in the direction parallel to the ambient magnetic field, i.e., $k_\perp = 0$. Then, we have $I_0[0] = 1$ and $I_{n\neq 0}[0] = 0$. These also gives $I'_{\pm 1}[0] = 0.5$. Thus we obtain $K_{1,1} = K_{2,2}$, $K_{1,3} = 0$[3], $K_{2,3} = 0$ and Eq.(26) becomes

$$\left\{\left(\sum_s \Pi_p^2 K_{1,1} + \tilde{\omega}^2 - c^2 k_{||}^2\right)^2 + \sum_s \Pi_p^2 K_{1,2}^2\right\}\left\{\sum_s \Pi_p^2 K_{3,3} + \tilde{\omega}^2\right\} = 0 \tag{27}$$

The first factor is for transverse waves where $\vec{k} \perp \vec{E}$. That is, a wave propagates in the z direction while its electromagnetic fields polarize in the $x - y$ plane. The second factor is for longitudinal waves where $\vec{k} \,||\, \vec{E}$. That is, a wave propagates in the z direction and only its electric fields polarize in the z direction. The longitudinal waves are also referred to as compressional waves or sound waves. Especially in the case of $\vec{k} \,||\, \vec{E}$, waves are called "electrostatic" waves because these waves arise from electric charge and are expressed by the Poisson equation (3).

3.1 Transverse electromagnetic waves

The first factor of Eq.(27) becomes the following equation,

$$\sum_s \Pi_p^2 \left\{\left(\frac{V_{t\perp}^2}{V_{t||}^2} - 1\right) + \frac{V_{t\perp}^2}{V_{t||}^2} \frac{\tilde{\omega} - k_{||} V_d - \Omega_c \left(1 - \frac{V_{t||}^2}{V_{t\perp}^2}\right)}{\sqrt{2} k_{||} V_{t||}} Z_0 \left[\frac{\tilde{\omega} - k_{||} V_d - \Omega_c}{\sqrt{2} k_{||} V_{t||}}\right]\right\} = c^2 k_{||}^2 - \tilde{\omega}^2 \tag{28}$$

The plasma dispersion function $Z_0[x]$ (Fried & Conte, 1961) is approximated for $|x| \gg 1$ as

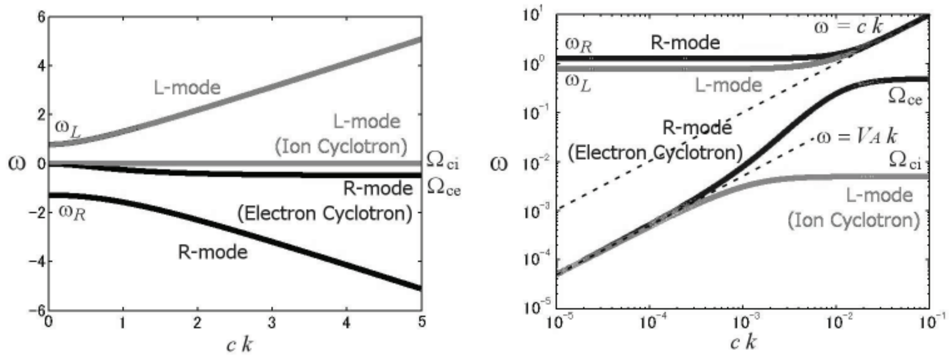
$$Z_0[x] \sim i\sigma\sqrt{\pi}\exp[-x^2] - \frac{1}{x}\left(1 + \frac{1}{2x^2} + \frac{3}{4x^4} + \cdots\right) \tag{29}$$

with

$$\sigma = \begin{cases} 0, & \mathrm{Im}[x] > 1/|\mathrm{Re}[x]| \\ 1, & |\mathrm{Im}[x]| < 1/|\mathrm{Re}[x]| \\ 2, & \mathrm{Im}[x] < -1/|\mathrm{Re}[x]| \end{cases}$$

Here the argument of the dispersion function x is a complex value.
Let us consider that a phase speed of waves is much faster than velocities of plasma particles. Then, the argument of the plasma dispersion function becomes a larger number. Here, the drift velocity of plasma V_d is also neglected. Equation (28) is thus rewritten by using Eq.(29) as

[3] $K_{1,3} = 0$ if $n = 0$, and $\lim_{a\to 0} \frac{I_n\left[\frac{a^2}{2}\right]}{a} = \lim_{a\to 0} \frac{a}{2} \frac{I_n\left[\frac{a^2}{2}\right] - I_n[0]}{\frac{a^2}{2} - 0} = \lim_{a\to 0} \frac{a}{2} I'_n\left[\frac{a^2}{2}\right] = 0$ if $n \neq 0$.



(a) Linear scale. (b) Logarithmic scale.

Fig. 1. Linear dispersion relation (frequency ω versus wavenumber k) for electromagnetic waves in plasma. The quantities ω and ck are normalized by Π_{pe}.

$$\sum_s \Pi_p^2 \left\{ \left(\frac{V_{t\perp}^2}{V_{t||}^2} - 1 \right) - \frac{V_{t\perp}^2}{V_{t||}^2} \frac{\tilde{\omega} - \Omega_c \left(1 - \frac{V_{t||}^2}{V_{t\perp}^2} \right)}{\tilde{\omega} - \Omega_c} \left(1 + \frac{k_{||}^2 V_{t||}^2}{(\tilde{\omega} - \Omega_c)^2} + \cdots \right) \right\} \tag{30}$$

$$+ i\sigma\sqrt{\pi} \sum_s \Pi_p^2 \frac{V_{t\perp}^2}{V_{t||}^2} \frac{\tilde{\omega} - \Omega_c \left(1 - \frac{V_{t||}^2}{V_{t\perp}^2} \right)}{\sqrt{2} k_{||} V_{t||}} \exp\left[-\frac{(\tilde{\omega} - \Omega_c)^2}{2 k_{||}^2 V_{t||}^2} \right] - c^2 k_{||}^2 + \tilde{\omega}^2 = 0$$

The solutions to above equation are simplified when we assume that the temperature of plasma approaches to zero, i.e., $V_{t||} \to 0$ and $V_{t\perp} \to 0$. Note that this approach is known as the "cold plasma approximation." Equation (30) is rewritten by the cold plasma approximation as

$$\Pi_{pe}^2 \frac{\tilde{\omega}}{\tilde{\omega} - \Omega_{ce}} + \Pi_{pi}^2 \frac{\tilde{\omega}}{\tilde{\omega} - \Omega_{ci}} + c^2 k_{||}^2 - \tilde{\omega}^2 = 0 \tag{31}$$

Here an electron-ion pair plasma is assumed ($\Omega_{ci} > 0$ and $\Omega_{ce} < 0$). The solutions to the above equation are shown in Figure 1. Note that the imaginary part of the complex frequency (growth/damping rate) becomes zero in the present case. There exist four dispersion curves. The dispersion curves for $\omega > 0$ are called "L-mode" (left-handed circularly polarized) waves, while the dispersion curves for $\omega < 0$ are called "R-mode" (right-handed circularly polarized) waves. In the present case, a positive frequency corresponds to the direction of ion gyro-motion, which is left-handed (counter-clockwise) circularly polarized against magnetic field lines.

The dispersion curves for the high-frequency R-mode and L-mode waves approach to the following frequencies as $k_{||} \to 0$,

$$\omega_R = \frac{\sqrt{(\Omega_{ce}+\Omega_{ci})^2+4(\Pi_{pe}^2+\Pi_{pi}^2-\Omega_{ce}\Omega_{ci})}-(\Omega_{ce}+\Omega_{ci})}{2} \quad (32)$$

$$\omega_L = \frac{\sqrt{(\Omega_{ce}+\Omega_{ci})^2+4(\Pi_{pe}^2+\Pi_{pi}^2-\Omega_{ce}\Omega_{ci})}+(\Omega_{ce}+\Omega_{ci})}{2} \quad (33)$$

which are known as the R-mode and L-mode cut-off frequencies, respectively. The two high-frequency waves approach to $\omega = ck_{||}$, i.e., electromagnetic light mode waves as $k_{||} \to \infty$. On the other hand, the low-frequency wave approaches to $k_{||}V_A$ as $k_{||} \to 0$, and approaches to Ω_c as $k_{||} \to \infty$. Note that $V_A \equiv c\Omega_{ci}/\Pi_{pi}$ is called the Alfven velocity. The R-mode and L-mode low-frequency waves are called electromagnetic electron and ion cyclotron wave, respectively, or (electron and ion) whistler mode wave.
The temperature of plasma affects the growth/damping rate in the dispersion relation. Assuming $\omega \gg |\gamma|$ (where $\tilde{\omega} \equiv \omega + i\gamma$), the imaginary part of Eq.(30) gives the growth rate γ as

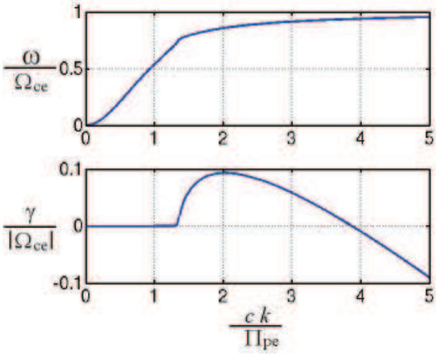
$$\gamma \sim -\frac{1}{2\omega - \dfrac{\Omega_{ce}\Pi_{pe}^2}{(\omega-\Omega_{ce})^2} - \dfrac{\Omega_{ci}\Pi_{pi}^2}{(\omega-\Omega_{ci})^2}} \quad (34)$$
$$\times\sigma\sqrt{\pi}\left\{\Pi_{pe}^2\frac{V_{te\perp}^2}{V_{te||}^2}\frac{\omega-\Omega_{ce}\left(1-\frac{V_{te||}^2}{V_{te\perp}^2}\right)}{\sqrt{2}k_{||}V_{te||}}\exp\left[-\frac{(\omega-\Omega_{ce})^2}{2k_{||}^2V_{te||}^2}\right]\right.$$
$$\left.+\Pi_{pi}^2\frac{V_{ti\perp}^2}{V_{ti||}^2}\frac{\omega-\Omega_{ci}\left(1-\frac{V_{ti||}^2}{V_{te\perp}^2}\right)}{\sqrt{2}k_{||}V_{ti||}}\exp\left[-\frac{(\omega-\Omega_{ci})^2}{2k_{||}^2V_{ti||}^2}\right]\right\}$$

Here the higher-order terms are neglected for simplicity. One can find that the growth rate becomes always negative $\gamma \le 0$ for light mode waves ($|\omega| > \omega_{R,L}$). On the other hand, γ becomes positive for electromagnetic electron cyclotron waves ($\Omega_{ce} < \omega < 0$) with

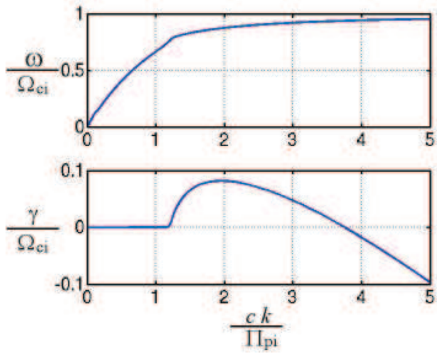
$$\frac{\omega}{\Omega_{ce}} > \left(1-\frac{V_{te||}^2}{V_{te\perp}^2}\right), \quad 2\frac{\omega}{\Omega_{ce}} < \frac{\Pi_{pe}^2}{(\omega-\Omega_{ce})^2} \quad (35)$$

Here ion terms are neglected by assuming $|\omega| \gg \Omega_{ci}$ and $|\omega| \gg \Pi_{pi}$. This condition is achieved when $V_{te\perp} > V_{te||}$, which is known as electron temperature anisotropy instability.
As a special case, electromagnetic electron cyclotron waves are also excited if the ion contribution (the third line in Eq.(34)) becomes larger than the electron contribution (the second line in Eq.(34)) around $|\omega| \sim \Omega_{ci}$. The growth rare becomes positive when

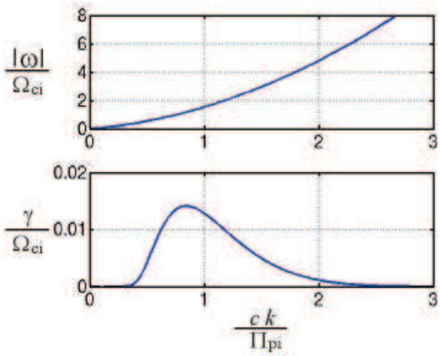
$$\frac{\omega}{\Omega_{ci}} > \left(1-\frac{V_{ti||}^2}{V_{ti\perp}^2}\right), \quad -\frac{\Pi_{pe}^2}{\Omega_{ce}} < \frac{\Omega_{ci}\Pi_{pi}^2}{(\omega-\Omega_{ci})^2} \quad (36)$$



(a) Electron temperature anisotropy instability with $V_{te\perp} = 4V_{te||}$, $T_{e||} = T_{i||} = T_{i\perp}$, $\Pi_{pe} = 10|\Omega_{ce}|$, $c = 200V_{te||}$.



(b) Ion temperature anisotropy instability with $V_{ti\perp} = 4V_{ti||}$, $T_{i||} = T_{e||} = T_{e\perp}$, $\Pi_{pe} = 10|\Omega_{ce}|$, $c = 200V_{te||}$.



(c) Firehose instability with $V_{ti||} = 16V_{ti\perp}$, $T_{i\perp} = T_{e||} = T_{e\perp}$, $\Pi_{pe} = 10|\Omega_{ce}|$, $c = 200V_{te||}$.

Fig. 2. Linear dispersion relation for electromagnetic instabilities.

Here $|\omega| \ll |\Omega_{ce}|$ and $|\omega| \ll \Pi_{pe}$ are assumed. This condition is achieved when $V_{ti||} \gg V_{ti\perp}$, which is known as firehose instability.
For electromagnetic ion cyclotron waves ($0 < \omega < \Omega_{ci}$), the growth rare becomes positive when

$$\frac{\omega}{\Omega_{ci}} < \left(1 - \frac{V_{ti||}^2}{V_{ti\perp}^2}\right), \quad -\frac{\Pi_{pe}^2}{\Omega_{ce}} > \frac{\Omega_{ci}\Pi_{pi}^2}{(\omega - \Omega_{ci})^2} \tag{37}$$

Here $\omega \ll |\Omega_{ce}|$ and $\omega \ll \Pi_{pe}$ are used. This condition is achieved when $V_{ti\perp} > V_{ti||}$, which is known as ion temperature anisotropy instability.
Examples of these electromagnetic linear instabilities are shown in Figure 2, which are obtained by numerically solving Eq.(28).

3.2 Longitudinal electrostatic waves

The second factor of Eq.(27) becomes

$$1+\sum_s \frac{\Pi_p^2}{k_{||}^2 V_{t||}^2}\left\{1+\frac{\tilde{\omega}-k_{||}V_d}{\sqrt{2}k_{||}V_{t||}}Z_0\left[\frac{\tilde{\omega}-k_{||}V_d}{\sqrt{2}k_{||}V_{t||}}\right]\right\}=0 \tag{38}$$

By using the similar approach, the above equation is rewritten by using Eq.(29) as

$$\begin{aligned}1-\sum_s \frac{\Pi_p^2}{(\tilde{\omega}-k_{||}V_d)^2}\left\{1+\frac{3k_{||}^2V_{t||}^2}{(\tilde{\omega}-k_{||}V_d)^2}+\cdots\right\}\\ +i\sigma\sqrt{\pi}\sum_s \frac{\Pi_p^2}{k_{||}^2V_{t||}^2}\frac{\tilde{\omega}-k_{||}V_d}{\sqrt{2}k_{||}V_{t||}}\exp\left[-\frac{(\tilde{\omega}-k_{||}V_d)^2}{2k_{||}^2V_{t||}^2}\right]=0\end{aligned} \tag{39}$$

For a simple case with $V_d=0$ and $\omega\sim\Pi_{pe}$, Eq.(39) gives

$$\omega^2=\frac{\Pi_{pe}^2+\sqrt{\Pi_{pe}^4+12\Pi_{pe}^2k_{||}V_{te||}^2}}{2}\sim\Pi_{pe}^2+3k_{||}^2V_{te||}^2 \tag{40}$$

which is known as the dispersion relation of Langmuir (electron plasma) waves. The imaginary part in Eq.(39) gives the damping rate as

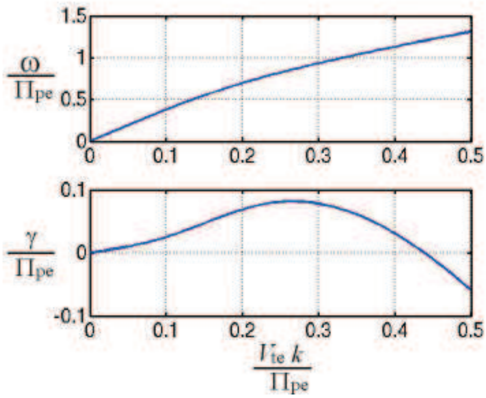
$$\frac{\gamma}{\omega}\sim-\frac{\sigma\sqrt{\pi}}{2}\frac{\omega^5}{\sqrt{2}k_{||}^3V_{te||}^3(\omega^2+6k_{||}^2V_{te||}^2)}\exp\left[-\frac{\omega^2}{2k_{||}^2V_{te||}^2}\right] \tag{41}$$

This means that the damping of the Langmuir waves becomes largest at $k_{||}\sim\Pi_{pe}/V_{te||}$, which is known as the Landau damping. Note that the second line in Eq.(39) comes from the gradient in the velocity distribution function, i.e., $\partial f_0/\partial v_{||}$. Thus electrostatic waves are known to be most unstable where the velocity distribution function has the maximum positive gradient. As an example for the growth of electrostatic waves, let us assume a two-species plasma, and one species drift against the other species at rest. Then, Eq.(39) becomes

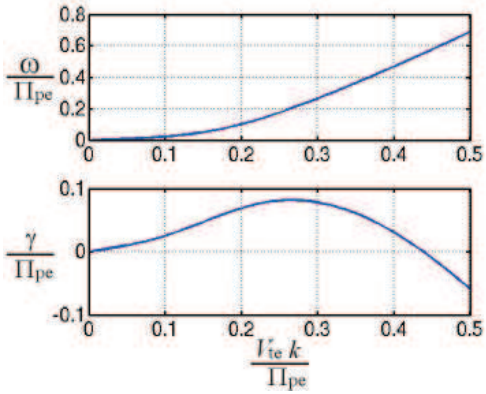
$$\begin{aligned}0=1-\frac{\Pi_{p1}^2}{(\tilde{\omega}-k_{||}V_{d1})^2}\left\{1+\frac{3k_{||}^2V_{t1||}^2}{(\tilde{\omega}-k_{||}V_{d1})^2}\right\}-\frac{\Pi_{p2}^2}{\tilde{\omega}^2}\left\{1+\frac{3k_{||}^2V_{t2||}^2}{\tilde{\omega}^2}\right\}\\ +i\frac{\sigma\sqrt{\pi}}{k_{||}^2}\left\{\frac{\Pi_{p1}^2}{V_{t1||}^2}\frac{\tilde{\omega}-k_{||}V_{d1}}{\sqrt{2}k_{||}V_{t1||}}\exp\left[-\frac{(\tilde{\omega}-k_{||}V_{d1})^2}{2k_{||}^2V_{t1||}^2}\right]+\frac{\Pi_{p2}^2}{V_{t2||}^2}\frac{\tilde{\omega}}{\sqrt{2}k_{||}V_{t2||}}\exp\left[-\frac{\tilde{\omega}^2}{2k_{||}^2V_{t2||}^2}\right]\right\}\end{aligned} \tag{42}$$

By neglecting higher-order terms, the growth rate is obtained as

$$\begin{aligned}\gamma\sim-\frac{\omega^3(\omega-k_{||}V_{d1})^3}{(\omega-k_{||}V_{d1})^3\Pi_{p2}^2+\omega^3\Pi_{p1}^2}\\ \times\frac{\sigma\sqrt{\pi}}{2k_{||}^2}\left\{\frac{\Pi_{p1}^2}{V_{t1||}^2}\frac{\omega-k_{||}V_{d1}}{\sqrt{2}k_{||}V_{t1||}}\exp\left[-\frac{(\omega-k_{||}V_{d1})^2}{2k_{||}^2V_{t1||}^2}\right]+\frac{\Pi_{p2}^2}{V_{t2||}^2}\frac{\omega}{\sqrt{2}k_{||}V_{t2||}}\exp\left[-\frac{\omega^2}{2k_{||}^2V_{t2||}^2}\right]\right\}\end{aligned} \tag{43}$$



(a) Electron beam-plasma instability with $V_{te} \equiv V_{t1||} = V_{t2||}$, $V_{d1} = 4T_{te}$, $\Pi_{p2} = 9\Pi_{p1}$.



(b) Electron beam-plasma instability with $V_{te} \equiv V_{t1||} = V_{t2||}$, $V_{d1} = 4T_{te}$, $\Pi_{p1} = 9\Pi_{p2}$.

Fig. 3. Linear dispersion relation for electrostatic instabilities. The frequency is normalized by $\Pi_{pe}^2 \equiv \Pi_{p1}^2 + \Pi_{p2}^2$. Note that these two cases have the same growth rate, but the maximum growth rate is given at $\omega \sim \Pi_{p2}$.

One can find that the growth rate becomes positive when $\omega - k_{||}V_{d1} < 0$, $(\omega - k_{||}V_{d1})^3\Pi_{p2}^2 + \omega^3\Pi_{p1}^2 < 0$, and the second line in Eq.(43) is negative. If $V_{d1} \gg V_{t1} + V_{t2}$, the growth rate becomes maximum at $\omega/k_{||} \sim V_{d1} - V_{t1}$. It is again noted that the second line in Eq.(43) comes from derivative of the velocity distribution function with respect to velocity with $v = \omega/k_{||}$. Electrostatic waves are excited when the velocity distribution function has positive gradient. This condition is also called the Landau resonance. Since the positive gradient in the velocity distribution function is due to drifting plasma (or beam), the instability is known as the beam-plasma instability.
Examples of the beam-plasma instability are shown in Figure 3, which are obtained by numerically solving Eq.(38).

3.3 Cyclotron resonance

Since the Newton-Lorentz equation (6) and the Vlasov equation (7) cannot treat the relativism (such that $c \gg V_d$), plasma particles cannot interact with electromagnetic light mode waves. On the other hand, drifting plasma can interact with electromagnetic cyclotron waves when a velocity of particles is faster than the Alfven speed V_A but is slow enough such that electrostatic instabilities do not take place.

For isotropic but drifting plasma, Eq.(27) is rewritten as

$$\begin{aligned} 0 = \tilde{\omega}^2 - c^2 k_{||}^2 - \Pi_{pe}^2 \frac{\tilde{\omega} - k_{||} V_{de}}{\tilde{\omega} - k_{||} V_{de} - \Omega_{ce}} - \Pi_{pi}^2 \frac{\tilde{\omega} - k_{||} V_{di}}{\tilde{\omega} - k_{||} V_{di} - \Omega_{ci}} \\ + i\sigma\sqrt{\pi} \left\{ \Pi_{pe}^2 \frac{\tilde{\omega} - k_{||} V_{de}}{\sqrt{2} k_{||} V_{te||}} \exp\left[-\frac{(\tilde{\omega} - k_{||} V_{de} - \Omega_{ce})^2}{2k_{||}^2 V_{te||}^2} \right] \right. \\ \left. + \Pi_{pi}^2 \frac{\tilde{\omega} - k_{||} V_{di}}{\sqrt{2} k_{||} V_{ti||}} \exp\left[-\frac{(\tilde{\omega} - k_{||} V_{di} - \Omega_{ci})^2}{2k_{||}^2 V_{ti||}^2} \right] \right\} \end{aligned} \tag{44}$$

Here higher-order terms are neglected. The imaginary part of the above equation gives the growth rate as

$$\begin{aligned} \gamma \sim -\frac{1}{2\omega - \frac{\Omega_{ce}\Pi_{pe}^2}{(\omega - k_{||} V_{de} - \Omega_{ce})^2} - \frac{\Omega_{ci}\Pi_{pi}^2}{(\omega - k_{||} V_{di} - \Omega_{ci})^2}} \\ \times \sigma\sqrt{\pi} \left\{ \Pi_{pe}^2 \frac{\omega - k_{||} V_{de}}{\sqrt{2} k_{||} V_{te||}} \exp\left[-\frac{(\omega - k_{||} V_{de} - \Omega_{ce})^2}{2k_{||}^2 V_{te||}^2} \right] \right. \\ \left. + \Pi_{pi}^2 \frac{\omega - k_{||} V_{di}}{\sqrt{2} k_{||} V_{ti||}} \exp\left[-\frac{(\omega - k_{||} V_{di} - \Omega_{ci})^2}{2k_{||}^2 V_{ti||}^2} \right] \right\} \end{aligned} \tag{45}$$

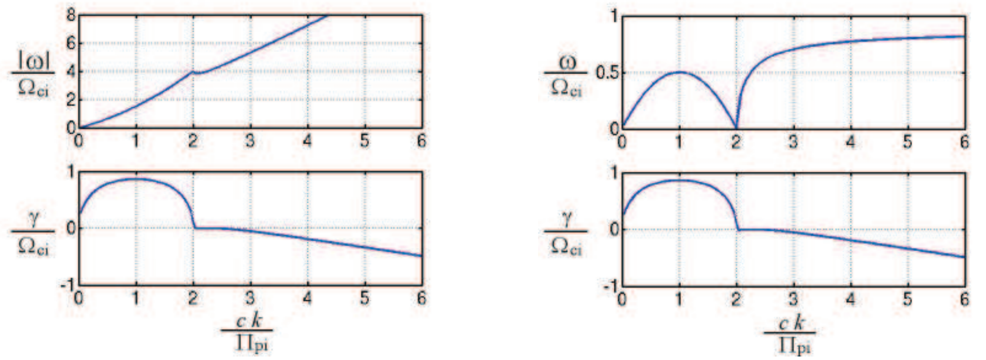
In the case of $V_{de} \neq 0$ and $V_{di} = 0$, ion cyclotron waves have positive growth rate at $\omega \sim \Omega_{ci}$ when

$$\frac{\omega}{k_{||}} - V_{de} < 0, \quad -\frac{\Pi_{pe}^2}{\Omega_{ce}} > \frac{\Omega_{ci}\Pi_{pi}^2}{(\omega - \Omega_{ci})^2} \tag{46}$$

Here $\omega \ll |\Omega_{ce}|$ and $\omega \ll \Pi_{pe}$ are assumed. The maximum growth rate is obtained at $\omega \sim k_{||}(V_{de} - V_A)$.

In the case of $V_{de} = 0$ and $V_{de} \neq 0$, electron cyclotron waves have positive growth rate at $\omega \sim -\Omega_{ci}$ when

$$\frac{\omega}{k_{||}} - V_{di} > 0, \quad -\frac{\Pi_{pe}^2}{\Omega_{ce}} < \frac{\Omega_{ci}\Pi_{pi}^2}{(\omega - k_{||} V_{di} - \Omega_{ci})^2} \tag{47}$$



(a) Ion beam-cyclotron instability with $V_{di} = 2V_A = 0.005c$, $T_{e||} = T_{e\perp} = T_{ti||} = T_{i||}$, $\Pi_{pe} = 10|\Omega_{ce}|$.

(b) Electron beam-cyclotron instability with $V_{de} = 2V_A = 0.005c$, $T_{e||} = T_{e\perp} = T_{ti||} = T_{i||}$, $\Pi_{pe} = 10|\Omega_{ce}|$.

Fig. 4. Linear dispersion relation for beam-cyclotron instabilities.

Here $|\omega| \ll |\Omega_{ce}|$ and $|\omega| \ll \Pi_{pe}$ are assumed. The maximum growth rate is obtained at $\omega \sim k_{||}(V_{di} + V_A)$.
These conditions are called the cyclotron resonance. Note that electron cyclotron waves are excited by drifting ions while ion cyclotron waves are excited by drifting electrons. These instabilities are known as the beam-cyclotron instability.
Examples of the beam-cyclotron instability are shown in Figure 4, which are obtained by numerically solving Eq.(28).

4. Summary

In this chapter, electromagnetic waves in plasma are discussed. The basic equations for electromagnetic waves and charged particle motions are given, and linear dispersion relations of waves in plasma are derived. Then excitation of electromagnetic waves in plasma is discussed by using simplified linear dispersion relations. It is shown that electromagnetic cyclotron waves are excited when the plasma temperature in the direction perpendicular to an ambient magnetic field is not equal to the parallel temperature. Electrostatic waves are excited when a velocity distribution function in the direction parallel to an ambient magnetic field has positive gradient. Note that the former condition is called the temperature anisotropy instability. The latter condition is achieved when a high-speed charged-particle beam propagates along the ambient magnetic field, and is called the beam-plasma instability. Charged-particle beams can also interact with electromagnetic cyclotron waves, which is called the beam-cyclotron instability. These linear instabilities take place by free energy sources existing in velocity space.
It is noted that plasma is highly nonlinear media, and the linear dispersion relation can be applied for small-amplitude plasma waves only. Large-amplitude plasma waves sometimes result in nonlinear processes, which are so complex that it is difficult to provide their analytical expressions. Therefore computer simulations play essential roles in studies of nonlinear processes. One can refer to textbooks on kinetic plasma simulations (e.g., Birdsall & Langdon, 2004; Hockney & Eastwood, 1988; Omura & Matsumoto, 1994; Buneman, 1994) for further reading.

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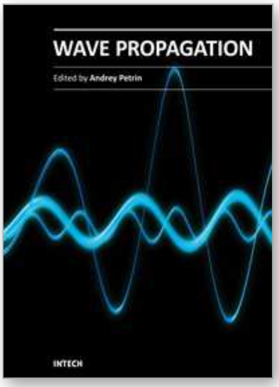

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The book collects original and innovative research studies of the experienced and actively working scientists in the field of wave propagation which produced new methods in this area of research and obtained new and important results. Every chapter of this book is the result of the authors achieved in the particular field of research. The themes of the studies vary from investigation on modern applications such as metamaterials, photonic crystals and nanofocusing of light to the traditional engineering applications of electrodynamics such as antennas, waveguides and radar investigations.