# The Formation Stability of a Multi-Robotic Formation Control System

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## 1. Introduction

The recent trend in the design of a robot system towards to increase the capibility of a single robot for the increasing demand in different applications at home, industry or military(Murry 2007). However, it greatly increases not only the unit production cost but also the design complexity. One of the best solution for this problem is certainly identified as using many of the low cost robots in terms of the cooperative manner. Accordingly, the formation system may constitute a more effective achievement than a single robot in regard to the union of the different functions in the different Wheeled Mobile Robots(WMRs) of a team such that the cooperative system tends to the better signal resolution and the higher performance of the information system *etc*. Sometimes we could also suffer the task which is very diffcult to be achieved with a single robot, *i.e.*, move a large size object from one place to another one or deal with the highly flexible tasks.

The Multi-Robot Formation System (MRFS) is generally defined as a system contains a collection of the robotic subsystems which is able to be cooperation/competition with each other. The interconnected structure of the MRFS represents the physical or nonphysical interconnected relationship between any robots based on the specific robot in the formation team. Usually, the interconnected structure of the MRFS can be regarded as a topological structure with the communication system(Fax and Murray 2004; Olfati-Saber and Murray 2004). In other words, the topological structure is the foundation of the information system of the MRFS that is used to exchange their state information for the centralized coordinator(Matinez, Cortes et al. 2007). From geometrical persepective, the topological structure governs the shape stability or equivalently the graph rigidity(Lin, Francis et al. 2005) of the MRFS.

Reviewing with the information aspect, the interconnected structure is naturally constructed with respect to the physical limitation of the sensors or the communication power of the WMRs, *i.e.*, the resolution of the range sensor or the communication range *etc*. Sometimes, the interconnected structure may be assigned autonomously by the task manager who receives the global information of the systems and manages the performance of the task. The MRFS could surely be explained through the hierarchical system, ref. (Singh 1977). We emphasize that the MRFS is topologically constructed by the two hierarchy with low(subsystem) and high(interconnected system) level in this research. In most of the applications, the hierarchical system is diffeomorphic from low to high level or vice versa, see (Pappas, Lafferriere et al. 2000). However, in our research, it is complex beyond that: for the proposed hierarchical system, if we define an injective map: $f_M : \mathbb{R}^m \to \mathbb{R}^n$, it is said that $f_M$ is a smooth map or, furthermore we said that the interconnected system is locally submersion to the subsystems of the MRFS.

The field of the nonholonomic system has been extensively studied since it is founded from Euler and Lagrange's research result in the rolling without sliding mechanical system on a plane. Recently, the related control issues in the nonholonomic system are still an active research area either in the mathematics or in the engineering field, see (BLOCH, DRAKUNOV et al. 2000; Monforte 2002; Fernandez and Bloch 2008). Without loss of generality, the subject of the subsystem in this research can be practically restricted to a Wheeled Mobile Robot (WMR) so that the subsystem belongs to the nonholonomic system which statisfies both no sliding and no slipping constraints. More precisely, the motion of the nonholonomic system is physically subject to the nonholonomic constraints. Hence, we say that the MRFS is again the nonholonomic system so that the unreachable region locally varied with the instant of the velocity of the WMRs in a team. With this aspect, the research in (Brokett 1983) has indicated that there doesn't exist smooth feedback controller for a single WMR. Fortunately, the later works in (Koiller 1992; Murry and Sastry 1993) proposed that the periodic solution can be adopted for the control of the nonholonomic system. For instance, the asymptotically stable of the WMR is guaranteed.

Therefore, we consider the ways of relating the interconnected stability(or so-called the graph rigidity) to the stability of the interconnected nonholonomic subsystems. More prcisely, the task of the MRFS is generally classified as formation maintain and the formation switch. The task of the formation maintain is simply to maintain the formation shape when moves from the start point to the end point so that the control goal is to stabilize the formation variables and the subsystem variables simetinuously by the given trajectory, see (Chang and Fu 2008). Several elegant researches have been proposed in terms of the issue. The disctribute formation control architecture with respect to the consensus problem of the communication system is experimentally implemented and valited on the neighbour to neighbour information exchange in (Ren and Sorensen 2008). The control strategy with the input constraint in associated with the leader-follower control architecture of the MRFS has been obtained (Consolinia, Morbidib et al. 2008). A game theoretic modelling approach for the MRFS is provided by (Harmati and Skrzypczyk 2008). (Kaminka, Schechter-Glick et al. 2008) has been proposed the sensor based MRFS to achieve the minimal cost sensing system design. In addition, for overcomimg the trade off in the control goal between formation system and subsystems, a differential game approach has been used to model such the problem and a differential game based controller has been derived in (Kaminka, Schechter-Glick et al. 2008). Moreover, the construction for the MRFS can be additionally identified as virtual structure approach, leader-follower approach and the behaviour approach.

Also. the task of the formation switch is naturally defined that the swithing topology of the MRFS is physically performed while executing the task. Comparing to the proceed problem in the formation maintain, the switching stability has to be additionally cared particularlly in the nonholonomic system. Namely, it is lead to the topological structure switch of the formation system such that two main issues have to be concerned: the structure stability of the MRFS and the subsystem stability under the impluse response. Additionally, the impulse response of the nonholonomic system may lead to relase the nonholonomic constrains due to the desired state of the subsystem may locally stay on the unreachable region. The decentralized receding horizen controllers has been proposed in (Keviczky, Borrelli et al. 2008) that reside on each vehicle to achieve coordination among team members. The control of the MRFS which considered the changing formation has been studied in (Desai, Ostrowski et al. 2001). (Das, Fierro et al. 2002) has developed a framework for cooperative control of the MRFS that has been applied in the vision-based formation system. We do not have a study to the overall problem in the MRFS but we focus the problem of the formation stability analysis and the nonholonomic multi-robotic formation control design with respect to the capacity of the switching formation topology on-line which extends from the previous work(Chang and Fu 2008). This chapter is organized as follows: the general modelling of the MRFS is presented in Section 2. Additionally, the interconnected stability and formation control design is examined in Section 3. In Sections 4, the simulation are presented. Finally, in Section 5, the conclusions are made.

## 2. General Modelling of the MRFS

Consider a MRFS with the formation state : $z=\left\{z_{ij}\in\mathbb{R}^2 \middle| i\neq j;1\leq i,j\leq n\right\}$ . The MRFS is composed by $n$ WMRs whose state can be described as a vector matrix: $q=\begin{bmatrix}q_1 & \cdots & q_n\end{bmatrix}\in\mathbb{R}^{3n\times 2}$. More precisely, the $i^{th}$ WMR with the $j^{th}$ interconnection in a MRFS implies $z_{ij}\neq\phi$ if the $i^{th}$ WMR connects to the $j^{th}$ WMR. Suppose that the desired interconnected structure and a desired formation state $z_d=\left\{z_{dij}\in\mathbb{R}^2 \middle| i\neq j;1\leq i,j\leq n\right\}$ are given and a virtual formation center $q_c\in\mathbb{R}^2$ which locates inside the closed region of the formation shape is chosen. The kinematics of the WMR can be generally regarded as an driftless affine control system: $\dot{q}_i=\sum_{j=1}^{k}g_j\left(q_i\right)u_j$ with $u_j$ being the control; $k$ being a constant number of the control variables.

Now the virtual center of the MRFS moves along a desired trajectory $c_d\left(t\right)$ so that the MRFS is driven from an initial state $q_{c0}=c_d\left(t_0\right)$ to a final state $q_{cf}=c_d\left(t_f\right)$ where $t_0$ and $t_f$ denote the initial and final time respectively in addition to maintain the given formation shape derived from the desired formation state simultaneously. It is obviously that the set of the desired state $\left\{q_{d1},q_{d2},\cdots,q_{dn}\right\}$ of the WMRs in the formation team could be obtained with the well-known $c_d\left(t\right)$ and $z_d$ , *i.e.*, $q_{di}=c_d+f\left(z_d\right)$ where $f$ denotes a differentiable function. For all positive $\varepsilon_z,\varepsilon_q$ , there exists positive $\delta_z,\delta_\varepsilon$ such that $\sup B_{zi}\left(\varepsilon_z\right)=r_{zi}$ and $\sup B_{qi}\left(\varepsilon_q\right)=r_{qi}$ where $B_{zi}\left(\varepsilon_z\right)$ and $B_{qi}\left(\varepsilon_q\right)$ denote small enough balls ; $r_{zi}$ and $r_{qi}$ are the radius of the balls respectively;. Now we set $\varepsilon_z=\min\left(\delta_z,r_i\right)$, and the following definitions can be made:

**Definition 2.1(Interconnection stable)**: Consider a nonholonomic MRFS with its interconnected structure. Initially, we set $\sum_j\left\|z_{ij}\left(t_0\right)-z_{dij}\left(t_0\right)\right\|\leq\delta_z$ . If there exists the following condition: $\lim_{t\in[t_0,t_f]}\sum_j\left\|z_{ij}(t)-z_{dij}(t)\right\|\le\varepsilon_z$ for all $i$, the MRFS is said to be *interconnection stable*.

**Definition 2.2(Formation system stable)**: Let $z_{ij}$ be continuous in $t$. The equilibrium point $z_{ij}(0)=0$ and $q_i(0)=0$ information variable and individual variable respectively for all $i,j$ is

- *formation system stable*: Definition 2.1 holds and if there exists $\sum_j\left\|z_{ij}(t_0)-z_{dij}(t_0)\right\|\le\delta_z$ ; $\left\|q_i(t_0)-q_{di}(t_0)\right\|\le\delta_q$ then $\lim_{t\in[t_0,t_f]}\left\|q_i(t)-q_{di}(t)\right\|\le\varepsilon_q$, for all $i$;
- *asymptotically formation system stable*: Definition 2.1 holds and if there exists $\sum_j\left\|z_{ij}(t_0)-z_{dij}(t_0)\right\|\le\delta_z$ ; $\left\|q_i(t_0)-q_{di}(t_0)\right\|\le\delta_q$ then $\lim_{t\to\infty}\left\|q_i(t)-q_{di}(t)\right\|\to 0$, for all $i$;
- *formation system unstable*: if it is not formation system stable.

According to Definition 2.2, if the MRFS has the formation system stable, one of the necessary condition is that the interconnection sable has to be held. On the contrary, the interconnection stable cannot be the necessary condition for the formation system stable. In other words, the interconnection stability is clearly defined as the sufficient condition for achieving the formation stable. The formation system stability, no doubt, is thus based on the interconnection stable and the subsystem stable simeltineously. In addition, we have proved that if the Definition 2.2 is commitment, then the final state of the WMRs in the MRFS will be reached: $q_{df}=c_d(t_f)$, in section IV.

**Remark 2.3**: Considering the Definition 2.2, the following condition yields:

- if there exists $\lim_{t\in[t_0,t_f]}\left\|q_i(t)-q_{di}(t)\right\|\le\varepsilon_q$ then $\lim_{t\in[t_0,t_f]}\sum_j\left\|z_{ij}(t)-z_{dij}(t)\right\|\le\varepsilon_z$ ;
- if there exists $\lim_{t\to\infty}\left\|q_i(t)-q_{di}(t)\right\|\to 0$ then $\lim_{t\to\infty}\sum_j\left\|z_{ij}(t)-z_{dij}(t)\right\|\to 0$.

Thus, the formation system stable can be guaranteed by evaluating the convergence property of the individual states while performing the full state formation tracking.

As we know, the formation variables: the relative length and the relative heading angle, is abstracted from a collection of the states of nonholonomic WMRs. Also, the formation states can be written by general functions:

$$z_{ij}=\begin{bmatrix} l_{ij} \\ \varphi_{ij} \end{bmatrix}=\begin{bmatrix} f_{pij}\left(q_{pi},q_{pj}\right) \\ f_{ij}\left(q_{pi},q_{pj},q_{\theta i},q_{\theta j}\right) \end{bmatrix}\in Q$$

with $q_i=\begin{bmatrix} q_{pi} & q_{\theta i}\end{bmatrix}^T\in N_i$ and $q_j=\begin{bmatrix} q_{pj} & q_{\theta j}\end{bmatrix}^T\in N_j$ where $Q\in\mathbb{R}^m, N_i\in\mathbb{R}^k$ and $N_j\in\mathbb{R}^k$ denote the compact and differentiable manifolds.

Suppose the desired formation states are given and the formation system satisfies the condition of interconnection stable such that the solution of the individual states may not unique. For example, $\left(q_{pi},q_{pj}\right)=f_{pij}^{-1}\left(l_{ij}\right)$ and $\left(q_{pi},q_{pj},q_{\theta i},q_{\theta j}\right)=f_{ij}^{-1}\left(\varphi_{ij}\right)$, there are two equations but more than two unknown variables in both of the equations. Figure 1 shows the illustrated scenario with three WMRs in the MRFS.

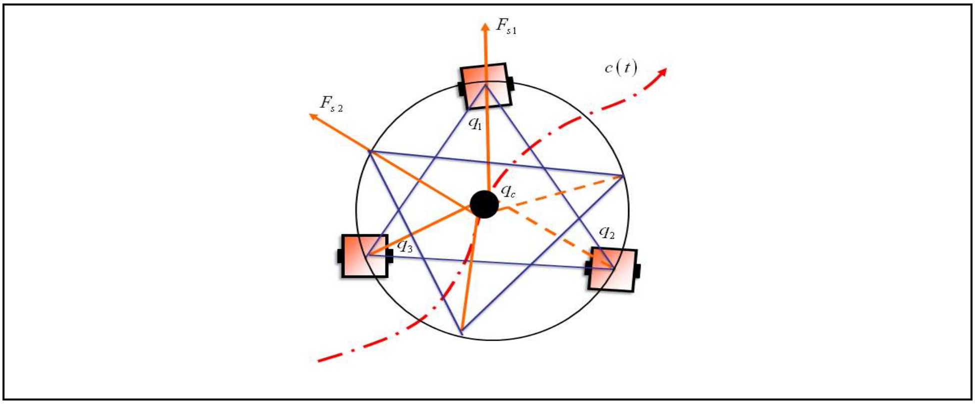


Fig. 1. A MRFS with three WMRs.

In Figure 1, the interconnected structures: $F_{s1}$ and $F_{s2}$, are both the solutions. If the additional nonholonomic constraints in each of the WMRs are called the nonholonomy, the design challenge of the MRFCS immediately arises that there may be infinite solutions or conversely no solutions. Thus we can conclude that the conditions of the solution depends on the nonholonomy. We can further explain that the nonholonomic constraint always forbids locally to reach some of the neighborhood of the WMR so that the nonholonomic system with redundent nonholonomy or holonomy equations(ususally the total equation number is over or equal to the dimension of the system) may not have physical solution.

Now we set oriented direction of the MRFS from $q_c$ to $q_1$ tangent to the desired path $c(t)$, see Figure 1. With respect to the interconnection stability and the subsystem stability, Definition 2.2 shall be further modified.

**Definition 2.4**: Let $z_{ij}$ be piecewise continuous in $t$. The equilibrium point $z_{ij}(0)=0$ and $q_i(0)=0$ in formation variable and individual variable respectively for all $i,j$ is

- *formation system stable*: Definition 2.1 holds and if there exist $\sum_j \left\| z_{ij}(t_0) - z_{dij}(t_0) \right\| \le \delta_z$ and $\left\| q_i(t_0) - q_{di}(t_0) \right\| \le \delta_q$ then $\lim_{t\in[t_0,t_f]} \sum_j \left\| z_{ij}(t) - z_{dij}(t) \right\| \le \varepsilon_z$ and $\lim_{t\in[t_0,t_f]} \left\| q_i(t) - q_{di}(t) \right\| \le \varepsilon$, for all $i$;
- *asymptotically formation system stable*: Definition 2.1 holds and if there exist $\sum_j \left\| z_{ij}(t_0) - z_{dij}(t_0) \right\| \le \delta_z$ and $\left\| q_i(t_0) - q_{di}(t_0) \right\| \le \delta_q$ then $\lim_{t\in[t_0,t_f]} \left\| q_i(t) - q_{di}(t) \right\| \le \varepsilon_q$ and $\lim_{t\to\infty} \left\| q_i(t) - q_{di}(t) \right\| \to 0$, for all $i$;
- *formation system unstable*: if it is not formation system stable.

No doubt, Definition 2.4 is more rigorous than Definiton 2.2 particularly it can be put on the condition after releasing the constraints on the formation state. So far, we got two unsolved problems in the design of the MRFS: first, the the uniqueness of the solution; second, the subsystem stability with respect to the interconnection stability.

For the first point, coneptually, the key step is how to select the adequate stable interconnected structure which corresponds to the number of the additional constraints.

Actually, this idea is simple but it is much complex than we expect in the design process resulted by the nonholonomic system of the WMR. As we know, the choice of the state of the MRFS can be either the relative length or the relative angle or even mix both of them and they are all capable to be the abstractive variables which are abstracted from the states of the nonholonomic subsystems. There also exists the nonlinear transfomation between the position and the oriented angle of the WMR so that, in the MRFS, the relative length couples the relative angle or vice versa. We, therefore, usually select one of them as the abstractive variables for simplifing the design complexity. With this aspect, if the minimal interconnected structure of the MRFS is performed, the process is the way regarded as to release some redundent abstracted equations. In this research, for this issue, we have proposed the minimal relization with respect to the stable interconnected structure in the controller design of the MRFS.

The second issue requires more detail study on the nonholonomic system. The nonholonomic constraints are assumed to be strictly satisfied in this research for applying the kinematics of the WMR. Hence, the output of the control velocity and the angular velocity is limited for avoiding to generate the large torque of the WMR. It immediately implies us that the unreachable region of the nonholonomic system is locally restricted by the limited torque. In real application of the MRFS, the desired state is usually given in the abstracted space. When we switch the interconnected topology, following the Remark 2.3, the nonholonomic subsystem may not be stable if $\lim_{t\to\infty}\sum_j \left\| z_{ij}(t) - z_{dij}(t) \right\| \to 0$. In this research, the Lyapunov based approach is proposed for dealing with this design issue.

## 3. Interconnected Stability and Formation Control Design

Formally, considering the nonholonomic constraints in a differential type WMR, the kinematics is able to be written by

$$\dot{q}_i = S_i u_i \tag{1}$$

where $q_i = \begin{bmatrix} q_{pi} & q_{\theta i} \end{bmatrix}^T \in \mathbb{R}^3$ denote the state of the WMR; $S_i = \begin{bmatrix} 0 & 0 & 1 \\ \cos q_{\theta i} & \sin q_{\theta i} & 0 \end{bmatrix}^T$ denotes the distribution; $u_i = \begin{bmatrix} v_i & w_i \end{bmatrix}^T \in \mathbb{R}^2$ denotes the control input. The formation state between two WMRs is distinctly defined as

$$z_{ij} = \begin{bmatrix} l_{ij} \\ \varphi_{ij} \end{bmatrix} \triangleq \begin{bmatrix} \left\| q_{pj} - q_{pi} \right\| \\ \left\| q_{\theta j} - q_{\theta i} \right\| \end{bmatrix} \tag{2}$$

In contrast to the relative formulation with two WMRs, the formation state to the $i^{th}$ WMR with respect to all $j^{th}$ connection without regarding with the interconnection structure is simply defined as the sum of the relative state:

$$z_i = \sum_j z_{ij} = \begin{bmatrix} \left\| q_{p1} - q_{pi} \right\| \\ \left\| q_{\theta 1} - q_{\theta i} \right\| \end{bmatrix} + \cdots + \begin{bmatrix} \left\| q_{pn} - q_{pi} \right\| \\ \left\| q_{\theta n} - q_{\theta i} \right\| \end{bmatrix} \tag{3}$$

and if $i = j$, $z_{ij} = 0$. Taking partial derivative to Eq. (3), we have the following equation:

$$\dot{z}_i = \begin{bmatrix} \sum_j \dot{l}_{ij} \\ \sum_j \dot{\varphi}_{ij} \end{bmatrix} = \sum_j \left( \frac{\partial z_i}{\partial q_i} + \frac{\partial z_i}{\partial q_j} \right) \tag{4}$$

For a MRFS, the neighbours of the $i^{th}$ WMR is noted as $q_j \sim q_i$ which corresponds to the interconnected structure and can be equivalently interpreted as an adjacency matrix. The adjacency matrix(Chung 1949) (or so-called interconnection matrix), $A_G$ , is represented as a binary matrix which is one-one maps from the interconnected structure to the elements of the matrix, *i.e.*, $q_j$ acts on $q_i$ if the element in $i^{th}$ row and $j^{th}$ column of the matrix equals "1", $A_G(i,j)=1$ but if $i=j$ , $A_G(i,j)=0$ . It is the fact that all of the connections of the $i^{th}$ WMR to the neighbour ones are a set: $a_{ij} = \left\{ A_G(i,j) \middle| 1 \le j \le n \right\}$ where $i$ and $j$ denotes the $i^{th}$ raw and $j^{th}$ column in the adjacency matrix. Therefore Eq. (4) could be naturally rewritten as

$$\dot{z}_i = \sum_j \frac{a_{ij}}{l_{ij}} \begin{bmatrix} q_{pij}^T I_2 \dot{q}_{pij} \\ q_{pij}^T J_2 \dot{q}_{pij} \end{bmatrix} = \sum_j \begin{bmatrix} q_{pij}^T \Omega_{ij}^I \dot{q}_{pij} \\ q_{pij}^T \Omega_{ij}^J \dot{q}_{pij} \end{bmatrix} \tag{5}$$

with $q_{ij} = \begin{bmatrix} q_{pij} & q_{\theta ij} \end{bmatrix}^T \in \mathbb{R}^3$ ; $\Omega_{ij}^I = \frac{a_{ij} I_2}{l_{ij}}$ ; $\Omega_{ij}^J = \frac{a_{ij} J_2}{l_{ij}}$ ; $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ; $J_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Now we summarize the result to the general formation dynamics form Eq. (1) and Eq. (5):

$$\left.\begin{aligned} \dot{z}_1 &= \sum_j a_{1j} \left( \frac{\partial z_1}{\partial q_i} + \frac{\partial z_1}{\partial q_j} \right) \\ &\vdots \\ \dot{z}_n &= \sum_j a_{nj} \left( \frac{\partial z_n}{\partial q_i} + \frac{\partial z_n}{\partial q_j} \right) \end{aligned}\right\} 2n \tag{6}$$

$$\left.\begin{aligned} \dot{q}_1 &= S_1 u_1 \\ &\vdots \\ \dot{q}_n &= S_n u_n \end{aligned}\right\} 3n \tag{7}$$

There are totally $5n$ equations in Eq. (6-7). Obviously, a number of $3n$ physical variables need to be solved so that we can freely choose $2n$ equations as a constraints, for example, minimizing Eq.(7) subject to Eq.(6) or minimizing the position subject to the heading angle of each WMRs and Eq.(6) and so forth. However, regarding with the interconnected structure, two problems yield: first, how to determine the minimal stable interconnected structure; second, how to guarantee the existence of the solution. For the first question, the following lemma will help us to make such a design:

**Lemma 3.1**: Considering the MRFS with a selective interconnection structure with totally $p$ connections, the stable minimal connection number of $p$ is $2n-3$ .

The proof follows the rigidity condition of the two dimensional graph, see (Laman 1970).

Now we begin with the second question for the existence of the MRFS. The existence of the solution is somehow linked to the subsystem stability if the designed nonholonomic control can derive the WMR to the admissible region within the control time. In other words, the existence of the solution is in the sense that there locally exist the reachable states of the

nonholonomic subsystem such that the WMR moves within the reachable region such that the sufficient condition of the subsystem stability is achieved.
Moreover, the coupling effect of the states in the WMR has to be considered. The state equation in Eq. (1) can be generally rewritten as

$$\begin{aligned} \dot{q}_{pi} &= f_{pi}\left(q_{\theta i}\right)v_i \\ \dot{q}_{\theta i} &= w_i \end{aligned} \tag{8}$$

where $f_{pi}:\mathbb{R}\rightarrow\mathbb{R}^2$ denotes a continuous and differentiable function; $v_i$ and $w_i$ denote the velocity and angular velocity respectively. Eq. (8) clearly represents the coupled effect between $q_{pi}$ and $q_{\theta i}$ in the nonholonomic system. It may be safety to assume that the velocity is a constant in the practical control design, the position and oriented angle can be derived by the assigned angular velocity simultaneously due to non-invloutive characteristic from Frobenious Thorem(Abraham and Marsden 1967). Conversely, if we set the angular velocity as a constant, the WMR is restricted to move along a line for the constrained oriented angle in the abstracted space. (BLOC and CROUC 1998) has indicated the general design rule of the nonholonomic control design which is stated in the following Remark:
**Remark 3.2**: Consider the nonholonomic system in Eq. (8). The system stability holds if the controller is designed for the WMR whose convergence rate of $q_{\theta i}$ is always faster than the one of $q_{pi}$.
Remark 3.2, for the MRFS, implies us that the subsystem stability is able to be designed by choosing the interconnected structure with respect to the relative length which is the function of $q_{pi}$. Through the way, another variable $q_{\theta i}$ is set free and is configurable. Therefore, the MRFS will be stable if the controller of the MRFS is carefully designed for satisfying Remark 3.2. Hence the formation dynamics for the $i^{th}$ WMR in Eq.(5) could be further reduced:

$$\dot{z}_i = \sum_j \frac{1}{l_{ij}} a_{ij}\left(q_{pij}^T I_2 \dot{q}_{pij}\right) = \sum_j q_{pij}^T \Omega_{ij}^I \dot{q}_{pij} \tag{9}$$

Rearranging the equation, the canonical form of the MRFS is further obtained with Eq. (6):

$$\begin{cases} \dot{z}_i = \sum_j q_{pij}^T \Omega_{ij}^I \left( f_{pj}\left(q_{\theta j}\right)v_j - f_{pi}\left(q_{\theta i}\right)v_i \right) \\ \dot{q}_{\theta 1} = w_1 \\ \vdots \\ \dot{q}_{\theta n} = w_n \end{cases} \tag{10}$$

**Corollary 3.3**: Consider the formation dynamics in Eq. (10), the state flow of the MRFS is equivalent to the state flow of the nonholonomic WMR. It can generally be written as the following formula:

$$
\begin{aligned}
\dot{z}_i &= \sum_j f_i^1\left(a_{ij}, z_{ij}, q_{pi}, q_{pj}, q_{\theta j}\right) + \left(\sum_j f_i^2\left(a_{ij}, z_{ij}, q_{pi}, q_{pj}, q_{\theta i}\right)\right) v_i \\
\dot{q}_{\theta 1} &= w_1 \\
&\vdots \\
\dot{q}_{\theta n} &= w_n
\end{aligned} \tag{11}
$$

Figure 2. shows the nonholonomic hierarchical structure in the nonholonomic formation dynamics in Eq. (11).

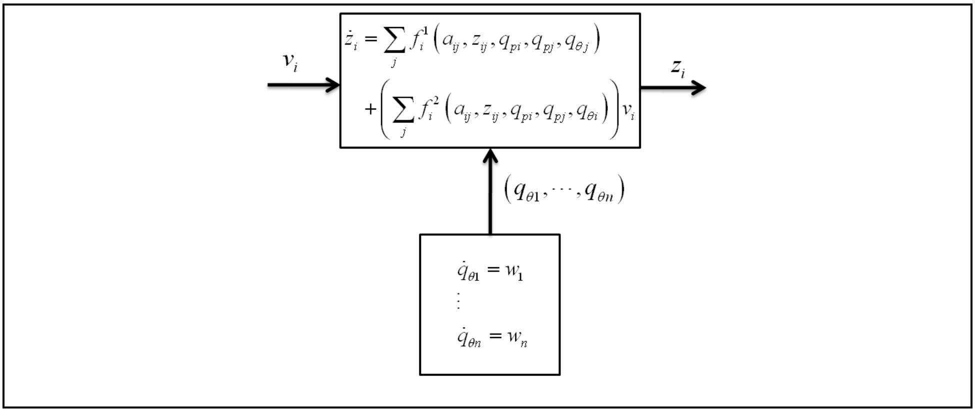


Fig. 2. the system structure of the nonholonomic formation dynamics.

**Remark 3.4**: Considering the MRFS, the interconnection matrix can be regarded as a linear operator of the formation dynamics.

For the Remark 3.4, an immediately result can be observed in Eq. (10). Hence, once the interconnected structure of the MRFS changes on-line so as to the interconnection matrix, the formation shape is able to be dynamically modified by applying the operator with the refreshed interconnection matrix. It is helpful in the implementation of the MRFS.

Now we shall prove the following statement: the interconnection stable is hold if and only if all of the eigenvalues of the interconnection matrix is positive. Purposely, the Lyapunov approach is adopted for minimizing the energy generated from the individual WMRs and the formation system. We select the Lyapunov function: $L_i = \frac{1}{2} a_{ii} q_i^T q_i$ , in each of the subsystem. This leads into the convergence rate of the heading angle of the WMR could be under our control. For helping the judgement, we also define the interconnection Lyapunov function: $L_{ij} = \sum_{j:j\neq i} \frac{1}{2} a_{ij} z_{ij}^T z_{ij}$ . Following these definitions, the formation Lyapunov function $L_i^F$ can be simply split into two parts: the individual Lyapunov function of the $i^{th}$ WMR and the interconnection Lyapunov functions of the $j^{th}$ WMR which acts on the $i^{th}$ WMR:

$$
L_i^F = L_i + \sum_j L_{ij} \tag{12}
$$

In Eq. (12), $L_i$ is generated from the $i^{th}$ subsystem and $\sum_j L_{ij}$ is produced by the interconnection of the MRFS for the $i^{th}$ subsystem. In the component form, it is able to be written as

$$L_i^F = \frac{1}{2}\begin{bmatrix} q_{pi} & q_{\theta i} \end{bmatrix} P_i \begin{bmatrix} q_{pi} \\ q_{\theta i} \end{bmatrix} + \frac{1}{2} A_{Gi} \begin{bmatrix} z_1^T z_1 & \cdots & z_n^T z_n \end{bmatrix}^T \tag{13}$$

where $P_i \in \mathbb{R}^{3\times 3}$ denotes the positive diagonal matrix of the $i^{th}$ WMR; $A_{Gi}$ denotes the $i^{th}$ raw of the interconnection matrix. Hence the necessary condition for the asymptotically formation stable is established via the following theorem:
**Theorem 3.5**: Considering the MRFS described in Eq. (11), the system, follows Definition 2.2, is said to be asymptotically interconnection stable.

**Proof**. Using Eq. (9), the time derivative of the Eq. (12) can be written as:

$$\sum_j \dot{L}_{ij} = \frac{1}{2}\left( \sum_j \dot{q}_{pij}^T \Omega_{ij}^I q_{pij} + q_{pij}^T \Omega_{ij}^I \dot{q}_{pij} \right) = \frac{1}{2}\left( \sum_j q_{pij}^T \left( F_i \Omega_{ij}^I + \Omega_{ij}^I F_i \right) q_{pij} \right) \tag{14}$$

where $F_i = \partial f_{pi} / \partial q_i$ denotes a linearized matrix from the nonlinear function $f_{pi}$ in Eq. (8). In order to state the stability condition on the MRFS, the Lyapunov function can be reproduced by Eq. (14) from single WMR to all WMRs in a formation team. Thus we reformulate the result in Eq. (14) in associated with a matrix formula:

$$F\Omega^I + \Omega^I F = -Q \tag{15}$$

where $Q_i$ are positive matrix. According to the Lyapunov stability theorem, if $\Omega^I$ and $Q_i$ are positive definite, then the MRFS in Eq. (11) is asymptotically stable. Q. E. D.
So far, the analysis result of the interconnection stability reveals us that the sufficient condition of the formation stable satisfies not only the existence of the positive definite interconnection matrix but also the subsystem stable by the Definition 2.4. Namely, if the formation stable holds, the necessary condition is that the interconnection matrix has to be positive definite. Note that the formation dynamics can be identified without driving the formation dynamics via Theorem 3.5. Practically, let us now consider the design of the control of the MRFS. The Lyapunov function in Eq. (12) can be further taken the partial derivative:

$$\begin{aligned} \dot{L}_i^F &= \frac{\partial L_i}{\partial q_i} + \sum_j \frac{\partial L_{ij}}{\partial q_j} \\ &= \sum_j f_i^1\left(a_{ij}, z_{ij}, q_{pi}, q_{pj}, q_{\theta j}\right) + \left( q_{pi} S_i + \sum_j f_i^2\left(a_{ij}, z_{ij}, q_{pi}, q_{pj}, q_{\theta i}\right) \right) v_i + q_{\theta i} w_i \end{aligned} \tag{16}$$

Therefore, the formation control can be chosen by the following Theorem:
**Theorem 3.6**: Considering the MRFS follows Eq. (11), if the velocity and angular velocity is chosen by:

$$v_i = \begin{cases} \left. \sum_j f_i^1\left(a_{ij}, z_{ij}, q_{pi}, q_{pj}, q_{\theta j}\right) - K_{pi} L_i^F \middle/ \left(q_{pi} S_i + \sum_j f_i^2\left(a_{ij}, z_{ij}, q_{pi}, q_{pj}, q_{\theta i}\right)\right) \right. ; \\ 0 \quad \text{if} \left(q_{pi} S_i + \sum_j f_i^2\left(a_{ij}, z_{ij}, q_{pi}, q_{pj}, q_{\theta i}\right)\right) = 0; \end{cases} \tag{17}$$

$$w_i = -K_{\theta i} q_{\theta i}.$$

then the MRFS is exponentially stable where $K_{pi} \geq K_{\theta i} \geq 0$ denote the constant real number.

**Proof**: After taking the controller in Eq. (17) into Eq. (16), the Lyapunov equation is obtained:

$$\dot{L}_i^F = -K_{pi} L_i^F - \left(K_{pi} - K_{\theta i}\right) q_{\theta i}^2 \leq -K_{pi} L_i^F \tag{18}$$

Consequently, the system is exponentially stable.

**Remark 3.**7 According to Theorem 3.6, the controller is capable of switching the interconnection structure in real-time by modifying the parameter: $a_{ij}$ .

Finally, the proposed formation stability theories and control design process in this section can be regarded as a useful tool.

## 4. Simulation

In this section, a simulation is performed for demonstrating the performance of the proposed nonholonomic multi-robotic formation control with respect to the formation stability. Figure 3 shows the simulation scenario with four WMRs in the MRFS. The team begins with the triangular shape and moves along a curve to the target with a square shape that shall change the interconnected structure on the middle way of the motion curve drawn as the solid line in Figure 3. Observing the interconnected structures, they satisfy the rigid condition which implies the interconnection stable of the MRFS in Lemma 3.1 so that the interconnection stability is promised by Definition 2.2.

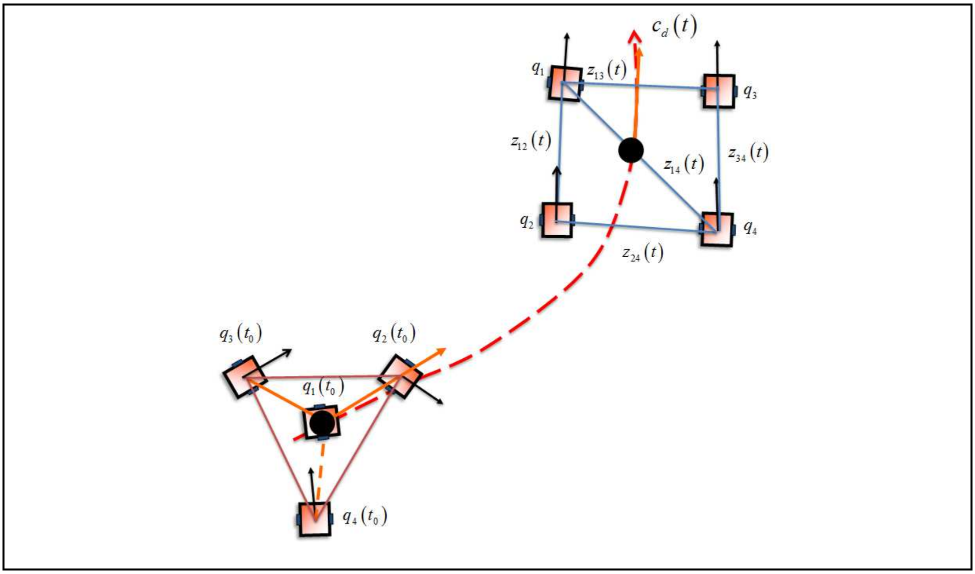


Fig. 3. the simulation scenario: from triangular to square structure of the MRFS.

In this simulation, we suppose that each of the WMRs is able to know the states from rest of the WMRs within the control time. Also, the physical configurations for the simulation are listed: the desired relative length is $l_{12} = l_{13} = l_{14} = 5(m)$; $l_{23} = l_{34} = l_{24} = 5\sqrt{3}(m)$ and the initial relative length is $l_{12} = l_{13} = l_{14} = 4(m)$ ; $l_{23} = l_{34} = l_{24} = 4\sqrt{3}(m)$ in the triangular shape and $l_{12} = l_{24} = l_{34} = l_{13} = 5(m)$ ; $l_{14} = 5\sqrt{2}(m)$ in the squire shape respectively. Considering the configuration of the single WMR, the initial oriented angles of the WMRs set to zero. The radius of the active wheels are $0.3(m)$ and the length of the axis of the active wheels is $0.5(m)$ . Practically, the control time is set to $0.01(\sec)$ in each of the WMRs.

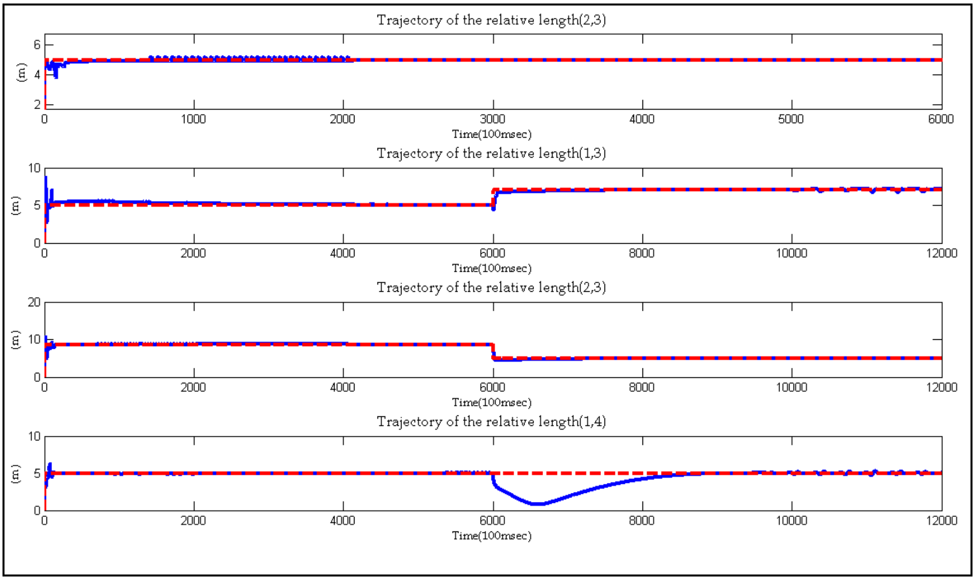


Fig. 4. The trajectory error of the relative length: $l_{23};l_{13};l_{23};l_{14}$ .

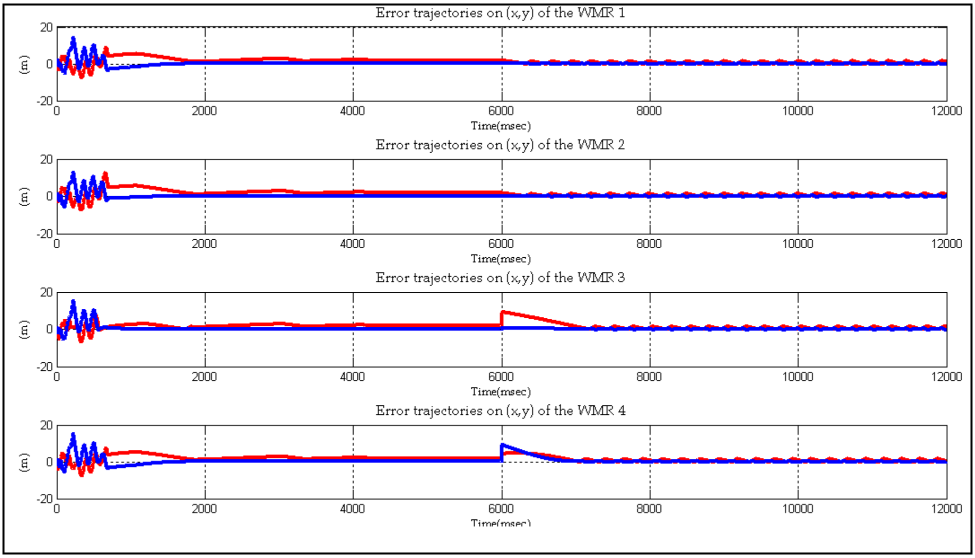


Fig. 5. The error trajectories on the X(red)-Y(blue) Plane from WMR 1-4.

The simulation results are drawn in Figure 4-5 where Figure 4 describes the relative lengths of the WMRs in the MRFS; Figure 5 draws the tracking error of the WMRs respectively. The diagrams indicate that the there exists impulse responses on each of the states of the subsystems when the interconnected structure is changed. In our proposed design, the subsystem stability can easily be handled.

## 5. Conclusion

The research reveal several important results: first, the formation stability could be hierarchically decoupled with the interconnection stability and the subsystem stability; second, the general framework of the MRFS with respect to the nonholonomic subsystems is obtained; third, the practical exponentially stable formation control is derived with respect to the minimal interconnection structure of the MRFS that can guarantee the subsystem stability. Clearly, our study provides a framework for designing and studying the modelling and the control problem in the nonholonomic MRFS. Finally, the simulation result shows the control performance so that the approach can be practically used in the switching interconnected structure of the MRFS on-line without adjusting any control parameters.

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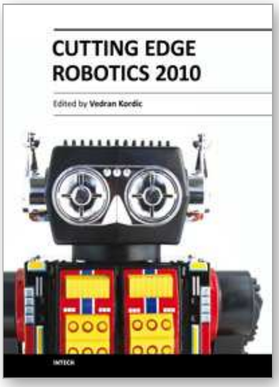


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Robotics research, especially mobile robotics is a young field. Its roots include many engineering and scientific disciplines from mechanical, electrical and electronics engineering to computer, cognitive and social sciences. Each of this parent fields is exciting in its own way and has its share in different books. This book is a result of inspirations and contributions from many researchers worldwide. It presents a collection of a wide range of research results in robotics scientific community. We hope you will enjoy reading the book as much as we have enjoyed bringing it together for you.