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Factorization of overdetermined boundary value problems

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1. Introduction

The purpose of this chapter is to present the application of the factorization method of linear elliptic boundary value problems to overdetermined problems. The factorization method of boundary value problems is inspired from the computation of the optimal feedback control in linear quadratic optimal control problems. This computation uses the invariant embedding technique of R. Bellman (1): the initial problem is embedded in a family of similar problems starting from the current time with the current position. This allows to express the optimal control as a linear function of the current state through a gain that is built using the solution of a Riccati equation. The idea of boundary value problem factorization is similar with a space-wise invariant embedding. The method has been presented and justified in (5) in the simple situation of a Poisson equation in a cylinder. In this case the family of spatial subdomains is simply a family of subcylinders. The method can be generalized to other elliptic operators than the laplacian (7) and more general spatial embeddings (6). The output of the method is to furnish an equivalent formulation of the boundary value problem as the product of two uncoupled Cauchy initial value problems that are to be solved successively in a spatial direction in opposite ways. These problems need the knowledge of a family of operators that satisfy a Riccati equation and that relate on the boundaries of the subdomains the Dirichlet and Neumann boundary conditions. This factorization can be viewed as an infinite dimensional generalization of the block Gauss *LU* factorization. It inherits the same properties: once the factorization is done (i.e. the Riccati equation has been solved), solving the same problem with new data needs only the integration of the two Cauchy problems.
Here we consider the situation where one wants to simulate a phenomenon described by a model using an elliptic operator with physical boundary conditions but using also an additional information that may come from boundary measurements. In general this extra information is not compatible with the model and one explains it as a small disturbance of the data of the model that is to be minimized. That is to say that we want to solve the model satisfying all the boundary conditions but the equation is to be solved in the least mean square sense. Then the factorization method is applied to the normal equations for this least square problem. It will need now the solution of two equations for operators: one is the same Riccati equation as for the well-posed problem and the second is a linear Lyapunov equation. It preserves the

property of reducing the solution of the problem for extra sets of data or measurements to two Cauchy problems.
This chapter is organized in the following way: section 2 states the well-posed and overdetermined problems to be solved. Section 3 gives a reformulation of the well-posed problem as a control problem which gives a clue to the factorization method. In section 5 the normal equations for the problem with additional boundary conditions are derived. In section 4 the factorization method for the well-posed elliptic problem is recalled. Section 6 gives the main result of the chapter with the derivation of the factorization of the normal equation of the overdetermined problem. Section 7 presents mathematical properties of the operators P and Q and of the equations they satisfy.

2. Position of the problem

Let Ω be the cylinder $\Omega =]0,1[\times\mathcal{O}$, $x' = (x,y) \in \mathbb{R}^n$, where x is the coordinate along the axis of the cylinder and $\mathcal{O}$, a bounded open set in $\mathbb{R}^{n-1}$, is the section of the cylinder. Let $\Sigma =]0,1[\times\partial\mathcal{O}$ be the lateral boundary, $\Gamma_0 = \{0\} \times \mathcal{O}$ and $\Gamma_1 = \{1\} \times \mathcal{O}$ be the faces of the cylinder.
We consider the following Poisson equation with mixed boundary conditions

$$(\mathcal{P}_0) \begin{cases} -\Delta z = -\dfrac{\partial^2 z}{\partial x^2} - \Delta_y z = f \quad \text{in } \Omega, \\ z|_\Sigma = 0, \\ -\dfrac{\partial z}{\partial x}|_{\Gamma_0} = z_0, \quad z|_{\Gamma_1} = z_1. \end{cases} \tag{1}$$

If $f \in L^2(\Omega)$, $z_0 \in (H_{00}^{1/2}(\mathcal{O}))'$ and $z_1 \in H_{00}^{1/2}(\mathcal{O})$, problem $(\mathcal{P}_0)$ has an unique solution in

$$\mathcal{Z} = \{z \in H^1(\Omega) : \Delta z \in L^2(\Omega), z|_\Sigma = 0\}.$$

We also assume that we want to simulate a system satisfying the previous equation but we want also to use an extra information we have on the "real" system which is a measurement of the flux on the boundary Γ_1. The problem is now overdetermined so, we will impose to satisfy both Dirichlet and Neumann boundary conditions on Γ_1, the state equation being satisfied only in the mean square sense. That will define problem $(\mathcal{P}_1)$ that we shall make precise in section 5.

3. Associated control problem

In this section, for the sake of simplicity, we consider $z_0 = 0$. We define an optimal control problem that we will show to be equivalent to $(\mathcal{P}_0)$. The control variable is v and the state z verifies equation (2) below. Let $\mathcal{U} = L^2(\mathcal{O})$ be the space of controls. For each $v \in \mathcal{U}$, we represent by $z(v)$ the solution of the problem:

$$\begin{cases} \dfrac{\partial z}{\partial x} = v \quad \text{in } \Omega, \\ z(1) = z_1. \end{cases} \tag{2}$$

We consider the following set of admissible controls:

$$\mathcal{U}_{ad} = \{v \in \mathcal{U} : z(v) \in X_{z_1}\}$$

where

$$X_{z_1} = \{h \in L^2(0,1;H_0^1(\mathcal{O})) \cap H^1(0,1;L^2(\mathcal{O})) : h(1) = z_1\}.$$

The cost function is

$$J(v) = \|z(v) - z_d\|^2_{L^2(0,1;H_0^1(\mathcal{O}))} + \|v\|^2_{L^2(\Omega)} = \int_0^1 \|\nabla_y z(v) - \nabla_y z_d\|^2_{L^2(\mathcal{O})} dx +$$
$$+ \int_0^1 \int_{\mathcal{O}} v^2 dxdy, \quad v \in \mathcal{U}_{ad}.$$

The desired state z_d is defined in each section by the solution of

$$\begin{cases} -\Delta_y \varphi(x) = f(x) & \text{in } \mathcal{O}, \\ \varphi|_{\partial\mathcal{O}} = 0, \end{cases} \tag{3}$$

where $\varphi \in L^2(0,1;H_0^1\mathcal{O})$. Consequently, we have

$$z_d = (-\Delta_y)^{-1} f \in L^2(0,1;H_0^1(\mathcal{O})).$$

Now we look for $u \in \mathcal{U}_{ad}$, such that

$$J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v).$$

Taking into account that $\mathcal{U}_{ad}$ is not a closed subset in $L^2(\Omega)$, we cannot apply the usual techniques to solve the problem, even it is not clear under that form that this problem has a solution. Nevertheless we can rewrite it as an equivalent minimization problem with respect to the state

$$\mathcal{U}_{ad} = \left\{ \frac{\partial h}{\partial x} : h \in X_{z_1} \right\}$$

and, consequently

$$J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v) = \inf_{h \in X_{z_1}} \bar{J}(h) = \bar{J}(z)$$

where $\frac{\partial z}{\partial x} = u$, and

$$\bar{J}(h) = \|h - z_d\|^2_{L^2(0,1;H_0^1(\mathcal{O}))} + \|\frac{\partial h}{\partial x}\|^2_{L^2(\Omega)} = \int_0^1 \|\nabla_y h - \nabla_y z_d\|^2_{L^2(\mathcal{O})} dx +$$
$$+ \int_0^1 \int_{\mathcal{O}} \left| \frac{\partial h}{\partial x} \right|^2 dxdy.$$

We remark that X_{z_1} is a closed convex subset in the Hilbert space

$$X = L^2(0,1;H_0^1(\mathcal{O})) \cap H^1(0,1;L^2(\mathcal{O}))$$

and $(\bar{J}(h))^{\frac{1}{2}}$ is a norm equivalent to the norm in X. Then by Theorem 1.3, chapter I, of (8), there exists a unique $z \in X_{z_1}$, such that:

$$\bar{J}(z) = \inf_{h \in X_{z_1}} \bar{J}(h)$$

which is uniquely determined by the condition

$$\bar{J}'(z)(h-z) \geq 0, \forall h \in X_{z_1}.$$

But X_0 is a subspace, and so the last condition is equivalent to

$$\bar{J}'(z)(h) = 0, \forall h \in X_0. \tag{4}$$

Now we have

$$\bar{J}'(z)(h) = 0 \Leftrightarrow \lim_{\theta \to 0^+} \frac{1}{\theta}[\bar{J}(z+\theta h) - \bar{J}(z)] = 0 \Leftrightarrow \int_0^1 \int_{\mathcal{O}} \nabla_y (z - z_d).\nabla_y h dxdy +$$
$$+ \int_0^1 \int_{\mathcal{O}} \frac{\partial z}{\partial x} \frac{\partial h}{\partial x} dxdy = 0, \forall h \in X_0$$

which implies that

$$\int_0^1 \left\langle -\Delta_y (z - z_d), h \right\rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})} dx + \int_0^1 \int_{\mathcal{O}} \frac{\partial z}{\partial x} \frac{\partial h}{\partial x} dxdy = 0, \forall h \in X_0.$$

Then, taking into account that $z_d = (-\Delta_y)^{-1} f$, we obtain

$$\int_0^1 \left\langle -\Delta_y (z) - f, h \right\rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})} dx + \int_0^1 \int_{\mathcal{O}} \frac{\partial z}{\partial x} \frac{\partial h}{\partial x} dxdy = 0, \forall h \in X_0.$$

If we consider $h \in \mathcal{D}(\Omega)$, then

$$\left\langle -\Delta_y z - \frac{\partial^2 z}{\partial x^2} - f, h \right\rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = 0, \forall h \in \mathcal{D}(\Omega)$$

so, we may conclude that $-\Delta z = f$ in the sense of distributions. But $f \in L^2(\Omega)$, and so we deduce that $z \in Y$, where

$$Y = \left\{ v \in X_{z_1} : \Delta v \in L^2(\Omega) \right\}.$$

We now introduce the adjoint state:

$$\begin{cases} \dfrac{\partial p}{\partial x} = -\Delta_y z - f & \text{in } \Omega, \\ p(0) = 0. \end{cases}$$

We know that $-\Delta_y z - f \in L^2(0,1; H^{-1}(\mathcal{O}))$. For each $h \in X_0$

$$\int_0^1 \left\langle -\Delta_y z - f, h \right\rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})} dx = \int_0^1 \left\langle \frac{\partial p}{\partial x}, h \right\rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})} dx =$$
$$= -\int_0^1 \int_{\mathcal{O}} p \frac{\partial h}{\partial x} dxdy$$

and so $p \in L^2(\Omega)$. Using the optimality condition (4), we obtain:

$$\int_0^1 \int_{\mathcal{O}} \left(-p + \frac{\partial z}{\partial x} \right) \frac{\partial h}{\partial x} dxdy = 0, \forall h \in X_0,$$

which implies

$$-p+\frac{\partial z}{\partial x}\in H^{1}(0,1;L^{2}(\mathcal{O}))\subset C([0,1]\,;L^{2}(\mathcal{O}))$$

and

$$\frac{\partial}{\partial x}(-p+\frac{\partial z}{\partial x})=0.$$

Then there exists $c(y)\in L^{2}(\mathcal{O})$, such that:

$$(-p+\frac{\partial z}{\partial x})|_{\Gamma_s}=c(y),\forall s\in[0,1]\,.$$

On the other hand, integrating by parts, we obtain:

$$\int_{\mathcal{O}}c(y)\,h|_{\Gamma_0}(y)dy=0,\forall h\in X_0,$$

and consequently $c(y)=0$. It follows that $-p+\dfrac{\partial z}{\partial x}=0$.
We have thus shown that problem

$$(\mathcal{P}_{1,z_1})\begin{cases}\dfrac{\partial z}{\partial x}=p\quad\text{in }\Omega,\quad z\,(1)=z_1,\\[2mm] \dfrac{\partial p}{\partial x}=-\Delta_y z-f\quad\text{in }\Omega,\quad p\,(0)=0,\end{cases}\tag{5}$$

admits a unique solution $\{z,p\}\in H_0^1(\Omega)\times L^2(\Omega)$, where z is the solution of $(\mathcal{P}_0)$.
We can represent the optimality system (5) in matrix form as follows:

$$\mathcal{A}\begin{pmatrix}p\\ \\ z\end{pmatrix}=\begin{pmatrix}0\\ \\ f\end{pmatrix},\ z(1)=z_1,\ p(0)=0,\tag{6}$$

with

$$\mathcal{A}=\begin{pmatrix}-I & \frac{\partial}{\partial x}\\ -\frac{\partial}{\partial x} & -\Delta_y\end{pmatrix}.$$

4. Factorization of problem $(\mathcal{P}_0)$ by invariant embedding

Following R. Bellman (1), we embed problem $(\mathcal{P}_{1,z_1})$ in the family of similar problems defined on $\Omega_s=]0,s[\times\mathcal{O},0<s\leq 1$:

$$(\mathcal{P}_{s,h})\begin{cases}\dfrac{\partial\varphi}{\partial x}-\psi=0\quad\text{in }\Omega_s,\quad\varphi(s)=h,\\[2mm] \varphi|_{\Sigma}=0,\\[2mm] -\dfrac{\partial\psi}{\partial x}-\Delta_y\varphi=f\quad\text{in }\Omega_s,\quad\psi(0)=-z_0,\end{cases}\tag{7}$$

where h is given in $H_{00}^{1/2}(\mathcal{O})$. When $s = 1$ and $h = z_1$ we obtain problem $(\mathcal{P}_{1,z_1})$. Due to the linearity of the problem, the solution $\{\varphi_{s,h}, \psi_{s,h}\}$ of $(\mathcal{P}_{s,h})$ verifies

$$\psi_{s,h}(s) = P(s)h + r(s), \tag{8}$$

where $P(s)$ and $r(s)$ are defined as follows:
1) We solve

$$\begin{cases} \dfrac{\partial \beta}{\partial x} - \gamma = 0 \;\text{ in } \Omega_s, \quad \beta(s) = h, \\ \beta|_\Sigma = 0, \\ -\dfrac{\partial \gamma}{\partial x} - \Delta_y \beta = 0 \;\text{ in } \Omega_s, \quad \gamma(0) = 0. \end{cases} \tag{9}$$

This defines $P(s)$ as:

$$P(s)h = \gamma(s).$$

We remark that $P(s)$ is the Dirichlet-to-Neumann operator on Γ_s relative to the domain Ω_s.
2) We solve

$$\begin{cases} \dfrac{\partial \eta}{\partial x} - \xi = 0 \;\text{ in } \Omega_s, \quad \eta(s) = 0, \\ \eta|_\Sigma = 0, \\ -\dfrac{\partial \xi}{\partial x} - \Delta_y \eta = f \;\text{ in } \Omega_s, \quad \xi(0) = -z_0. \end{cases} \tag{10}$$

The remainder $r(s)$ is defined by:

$$r(s) = \xi(s).$$

Furthermore, the solution $\{z, p\}$ of $(\mathcal{P}_{1,z_1})$ restricted to $]0,s[$ satisfies $(\mathcal{P}_{s,z|_{\Gamma_s}})$, for $s \in]0,1[$, and so one has the relation

$$p(x) = P(x)z(x) + r(x), \forall x \in]0,1[. \tag{11}$$

>From (11) and the boundary conditions at $x = 0$, we easily deduce that

$$P(0) = 0, \;\; r(0) = -z_0.$$

Formally, taking the derivative with respect to x on both sides of equation (11), we obtain:

$$\frac{\partial p}{\partial x}(x) = \frac{dP}{dx}(x)z(x) + P(x)\frac{\partial z}{\partial x}(x) + \frac{dr}{dx}(x)$$

and, substituting from (5) and (11) we conclude that:

$$\begin{aligned} &-\Delta_y z - f = \frac{dP}{dx}(x)z(x) + P(x)(P(x)z(x) + r(x)) + \frac{dr}{dx} \Leftrightarrow \\ &(\frac{dP}{dx} + P^2 + \Delta_y)z + \frac{dr}{dx} + Pr + f = 0. \end{aligned} \tag{12}$$

Then, taking into account that $z(x) = h$ is arbitrary, we obtain the following decoupled system:

$$\frac{dP}{dx} + P^2 + \Delta_y = 0, \quad P(0) = 0, \tag{13}$$

$$\frac{\partial r}{\partial x} + Pr = -f, \quad r(0) = -z_0, \tag{14}$$

$$\frac{\partial z}{\partial x} - Pz = r, \quad z(1) = z_1, \tag{15}$$

where P and r are integrated from 0 to 1, and finally z is integrated backwards from 1 to 0. We remark that P is an operator on functions defined on $\mathcal{O}$ verifying a Riccati equation.
We have factorized problem $(\mathcal{P}_0)$ as:

$$\text{“} -\Delta\text{”} = -\left(\frac{d}{dx} + P\right)\left(\frac{d}{dx} - P\right).$$

This decoupling of the optimality system (5) may be seen as a generalized block LU factorization. In fact, for this particular problem, we may write

$$\mathcal{A} = \begin{pmatrix} I & 0 \\ -P & -\frac{d}{dx} - P \end{pmatrix} \begin{pmatrix} -I & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & -P \\ 0 & \frac{d}{dx} - P \end{pmatrix}.$$

We will see in section 7 that P is self adjoint. So, the first and third matrices are adjoint of one another and are, respectively, lower triangular and upper triangular.

5. Normal equations for the overdetermined problem

>From now on, we suppose $z_0 \in H^{1/2}_{00}(\mathcal{O})$, $z_1 \in H^{3/2}_{0}(\mathcal{O})$, $z_2 \in H^{1/2}_{00}(\mathcal{O})$, $f \in H^{5/2}(\Omega)$ and $(\Delta f)|_\Sigma = 0$.
Assuming we have an extra information, given by a Neumann boundary condition at point 1, we consider the overdetermined system

$$\mathcal{A}\begin{pmatrix} p \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \; z(1) = z_1, \; p(0) = -z_0, \; \frac{\partial z}{\partial x}(1) = z_2. \tag{16}$$

If the data are not compatible with (5), this system should be satisfied in the least square sense. We introduce a perturbation,

$$\mathcal{A}\begin{pmatrix} p \\ z \end{pmatrix} = \begin{pmatrix} \delta g \\ f + \delta f \end{pmatrix}, \; z(1) = z_1, \; p(0) = -z_0, \; \frac{\partial z}{\partial x}(1) = z_2. \tag{17}$$

We want to minimize the norm of the perturbation,

$$J(\delta f, \delta g) = \frac{1}{2}\int_0^1 \left(\|\delta f\|^2_{L^2(\mathcal{O})} + \|\delta g\|^2_{L^2(\mathcal{O})}\right) dx, \tag{18}$$

subject to the constraint given by (17). This defines problem $(\mathcal{P}_1)$.
We remark that, like in section 3, this is an ill-posed problem. We could solve it by regularization, taking $\frac{\partial \delta g}{\partial x} \in L^2(\Omega)$, $\delta g(0) = \delta g(1) = 0$ and considering the problem of the minimization of the functional

$$J_\varepsilon(\delta f, \delta g) = J(\delta f, \delta g) + \frac{\varepsilon}{2} \int_0^1 \left\| \frac{\partial \delta g}{\partial x} \right\|^2_{L^2(\mathcal{O})} dx, \tag{19}$$

subject to the constraint given by (17), which is a well-posed problem.
However, like in section 3, the final optimality problem is well-posed. >From now on we consider the final problem and take the corresponding Lagrangian.
Taking, for convenience, the Lagrange multiplier of the second equation of (17) as $\bar{z} - f$,

$$\begin{aligned} L(\delta f, \delta g, z, p, \bar{z}, \bar{p}) &= J(\delta f, \delta g) + \int_0^1 \left(\bar{p}, \frac{\partial z}{\partial x} - p - \delta g\right)_{L^2(\mathcal{O})} dx + \\ &+ \int_0^1 \left(\bar{z} - f, -\frac{\partial p}{\partial x} - \Delta_y z - f - \delta f\right)_{L^2(\mathcal{O})} dx + \left(\mu, \frac{\partial z}{\partial x}(1) - z_2\right)_{L^2(\mathcal{O})}. \end{aligned} \tag{20}$$

Taking into account that $\frac{\partial z}{\partial x}(1) = p(1) + \delta g(1)$, we obtain

$$\left(\frac{\partial L}{\partial z}, \varphi\right) = \int_0^1 (\bar{z} - f, -\Delta_y \varphi)_{L^2(\mathcal{O})} dx + \int_0^1 \left(\bar{p}, \frac{\partial \varphi}{\partial x}\right)_{L^2(\mathcal{O})} dx, \ \forall \varphi \in \mathcal{Y},$$

where

$$\mathcal{Y} = \left\{ \varphi \in \mathcal{Z} : \frac{\partial \varphi}{\partial x}(0) = 0, \ \varphi(1) = 0 \right\}$$

and, integrating by parts, we derive

$$\left(\frac{\partial L}{\partial z}, \varphi\right) = \int_0^1 (-\Delta_y(\bar{z} - f), \varphi)_{L^2(\mathcal{O})} dx - (\bar{p}(0), \varphi(0)) + \int_0^1 \left(-\frac{\partial \bar{p}}{\partial x}, \varphi\right)_{L^2(\mathcal{O})} dx.$$

Now, if $\bar{p}(0) = 0$, and because all the functions are null on Σ, we conclude that:

$$\frac{\partial L}{\partial z} = 0 \Leftrightarrow -\frac{\partial \bar{p}}{\partial x} - \Delta_y \bar{z} = -\Delta_y f.$$

On the other hand

$$\begin{aligned} &\left(\frac{\partial L}{\partial p}, \psi\right) = \int_0^1 \left(\bar{z} - f, -\frac{\partial \psi}{\partial x}\right)_{L^2(\mathcal{O})} dx + \int_0^1 (\bar{p}, -\psi)_{L^2(\mathcal{O})} dx + \\ &(\mu, \psi(1))_{L^2(\mathcal{O})} = \int_0^1 \left(\frac{\partial(\bar{z} - f)}{\partial x}, \psi\right)_{L^2(\mathcal{O})} dx + (\bar{z}(0) - f(0), \psi(0)) - \\ &- (\bar{z}(1) - f(1), \psi(1)) + \int_0^1 (-\bar{p}, \psi)_{L^2(\mathcal{O})} dx + (\mu, \psi(1)) \end{aligned}$$

and, if $\psi(0) = 0$ and $\bar{z}(1) - f(1) = \mu$ arbitrary, then

$$\frac{\partial L}{\partial p} = 0 \Leftrightarrow \frac{\partial \bar{z}}{\partial x} - \bar{p} = \frac{\partial f}{\partial x}.$$

We have thus obtained:

$$\begin{cases} \dfrac{\partial \bar{z}}{\partial x} - \bar{p} = f_1 := \dfrac{\partial f}{\partial x}, \quad \bar{z}\,(1) \text{ arbitrary}, \\ -\dfrac{\partial \bar{p}}{\partial x} - \Delta_y \bar{z} = f_2 := -\Delta_y f, \quad \bar{p}\,(0) = 0. \end{cases} \tag{21}$$

We finally evaluate the optimal values for δf and δg. We have:

$$\left(\frac{\partial L}{\partial(\delta f)}, \gamma \right) = \int_0^1 (\delta f, \gamma)_{L^2(\mathcal{O})} dx + \int_0^1 (\bar{z} - f, -\gamma)_{L^2(\mathcal{O})} dx, \;\; \forall \gamma \in L^2(\Omega)$$

and for all $\xi \in L^2(\Omega)$ such that $\frac{\partial \xi}{\partial x} \in L^2(\Omega)$,

$$\left(\frac{\partial L}{\partial(\delta g)}, \xi \right) = \int_0^1 (\delta g, \xi)_{L^2(\mathcal{O})} dx + \int_0^1 (\bar{p}, -\xi)_{L^2(\mathcal{O})} dx.$$

At the minimum, we must have

$$\frac{\partial L}{\partial(\delta f)} = 0 \Leftrightarrow \delta f = \bar{z} - f$$

and

$$\frac{\partial L}{\partial(\delta g)} = 0 \Leftrightarrow \delta g = \bar{p}.$$

In conclusion, we obtain

$$\mathcal{A} \begin{pmatrix} p \\ z \end{pmatrix} = \begin{pmatrix} \delta g \\ f + \delta f \end{pmatrix} = \begin{pmatrix} \bar{p} \\ \bar{z} \end{pmatrix}, \tag{22}$$

and the normal equation is given by

$$\mathcal{A}^2 \begin{pmatrix} p \\ z \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \; p(0) = -z_0, \; z(1) = z_1, \; \frac{\partial z}{\partial x}(1) = z_2, \; -\frac{\partial z}{\partial x}(0) = z_0. \tag{23}$$

>From (23), we have

$$-\Delta \bar{z} = -\frac{\partial^2 \bar{z}}{\partial x^2} - \Delta_y \bar{z} = \frac{\partial^2 f}{\partial x^2} - \Delta_y f = -\Delta f \tag{24}$$

and, from (23) and (24),

$$\begin{aligned} -\Delta f = -\Delta \bar{z} &= -\Delta \left(-\frac{\partial p}{\partial x} - \Delta_y z \right) = -\Delta \left(\frac{\partial \bar{p}}{\partial x} - \frac{\partial^2 z}{\partial x^2} - \Delta_y z \right) \\ &= -\Delta \left(-\Delta_y \bar{z} + \Delta_y f - \Delta z \right) = \Delta^2 z + \Delta_y \left(\Delta \bar{z} - \Delta f \right) = \Delta^2 z \end{aligned}$$

We now notice that

$$\frac{\partial^2 \bar{p}}{\partial x^2} = \frac{\partial}{\partial x} \left(\Delta_y f - \Delta_y \bar{z} \right) = \Delta_y \left(\frac{\partial f}{\partial x} - \frac{\partial \bar{z}}{\partial x} \right) = -\Delta_y \bar{p}$$

and, remarking that $\bar{p}(0) = 0$, we derive $-\Delta_y \bar{p}(0) = 0$ which implies that

$$\frac{\partial(\Delta z)}{\partial x}(0) = \frac{\partial^2 p}{\partial x^2}(0) - \frac{\partial \bar{z}}{\partial x}(0) = -\Delta_y \bar{p}(0) - \bar{p}(0) - \frac{\partial f}{\partial x}(0) = -\frac{\partial f}{\partial x}(0)$$

Now we can write the normal equation as

$$(\mathcal{P}_2) \quad \begin{cases} \Delta^2 z = -\Delta f, \quad \text{in } \Omega, \\ z|_\Sigma = 0, \quad \Delta z|_\Sigma = 0, \\ -\dfrac{\partial z}{\partial x}(0) = z_0, \quad \dfrac{\partial \Delta z}{\partial x}(0) = -\dfrac{\partial f}{\partial x}(0), \\ z(1) = z_1, \ \dfrac{\partial z}{\partial x}(1) = z_2. \end{cases} \tag{25}$$

6. Factorization of the normal equation by invariant embedding

In order to factorize problem (25) we consider an invariant embedding using the family of problems $(\mathcal{P}_{s,h,k})$ defined in $\Omega_s =]0,s[\times\mathcal{O}$, for each $h \in H^{\frac{3}{2}}(\mathcal{O})$ and each $k \in (H_{00}^{\frac{1}{2}}(\mathcal{O}))'$. These problems can be factorized in two second order boundary value problems. Afterwards we will show the relation between $(\mathcal{P}_{s,h,k})$ for $s = 1$ and problem (25).

$$(\mathcal{P}_{s,h,k}) \quad \begin{cases} \Delta^2 z = -\Delta f, \quad \text{in } \Omega_s, \\ z|_\Sigma = 0, \quad \Delta z|_\Sigma = 0, \\ -\dfrac{\partial z}{\partial x}(0) = z_0, \quad \dfrac{\partial \Delta z}{\partial x}(0) = -\dfrac{\partial f}{\partial x}(0), \\ z|_{\Gamma_s} = h, \ \Delta z|_{\Gamma_s} = k. \end{cases} \tag{26}$$

Due to the linearity of the problem, for each $s \in]0,1]$, h, k, the solution of $(\mathcal{P}_{s,h,k})$ verifies:

$$\frac{\partial z}{\partial x}(s) = P(s)h + Q(s)k + \tilde{r}(s). \tag{27}$$

In fact, let us consider the problem

$$\begin{cases} \Delta^2 \gamma_1 = 0, \quad \text{in } \Omega_s, \\ \gamma_1|_\Sigma = 0, \quad \Delta\gamma_1|_\Sigma = 0, \\ \dfrac{\partial \gamma_1}{\partial x}(0) = 0, \quad \dfrac{\partial \Delta\gamma_1}{\partial x}(0) = 0, \\ \gamma_1|_{\Gamma_s} = h, \ \Delta\gamma_1|_{\Gamma_s} = 0. \end{cases} \tag{28}$$

This problem reduces to:

$$\begin{cases} \Delta \gamma_1 = 0, \quad \text{in } \Omega_s, \\ \gamma_1|_\Sigma = 0, \\ \dfrac{\partial \gamma_1}{\partial x}(0) = 0, \quad \gamma_1|_{\Gamma_s} = h. \end{cases} \tag{29}$$

Setting $P_1(s)h = \dfrac{\partial \gamma_1}{\partial x}(s)$, from (9) we may conclude that $P_1 = P$. On the other hand, given

$$\begin{cases} \Delta^2 \gamma_2 = 0, \quad \text{in } \Omega_s, \\ \gamma_2|_\Sigma = 0, \quad \Delta\gamma_2|_\Sigma = 0, \\ \dfrac{\partial \gamma_2}{\partial x}(0) = 0, \quad \dfrac{\partial \Delta\gamma_2}{\partial x}(0) = 0, \\ \gamma_2|_{\Gamma_s} = 0, \ \Delta\gamma_2|_{\Gamma_s} = k, \end{cases} \tag{30}$$

we define:

$$Q(s)k = \frac{\partial \gamma_2}{\partial x}(s).$$

Problem (30) can be decomposed in two second order boundary value problems. Finally, we solve:

$$\begin{cases} \Delta^2 \beta = -\Delta f, \quad \text{in } \Omega_s, \\ \beta|_\Sigma = \Delta\beta|_\Sigma = 0, \\ -\dfrac{\partial \beta}{\partial x}(0) = z_0, \quad \dfrac{\partial \Delta\beta}{\partial x}(0) = -\dfrac{\partial f}{\partial x}(0), \\ \beta|_{\Gamma_s} = \Delta\beta|_{\Gamma_s} = 0 \end{cases} \tag{31}$$

and set:

$$\tilde{r}(s) = \frac{\partial \beta}{\partial x}(s).$$

Then, the solution of the normal equation restricted to $]0,s[$, verifies $(\mathcal{P}_{s,z|_{\Gamma_s},\Delta z|_{\Gamma_s}})$, for $s \in]0,1[$. So, one has the relation

$$\frac{\partial z}{\partial x}|_{\Gamma_s} = P(s)z|_{\Gamma_s} + Q(s)\Delta z|_{\Gamma_s} + \tilde{r}(s). \tag{32}$$

>From (32), it is easy to see that $Q(0) = 0$ and $\tilde{r}(0) = -z_0$. On the other hand, we may consider the following second order problem on Δz as a subproblem of problem (26)

$$\begin{cases} \Delta(\Delta z) = -\Delta f, \quad \text{in } \Omega_s, \\ \Delta z|_\Sigma = 0, \\ \dfrac{\partial \Delta z}{\partial x}(0) = -\dfrac{\partial f}{\partial x}(0), \\ \Delta z|_{\Gamma_1} = c, \end{cases} \tag{33}$$

where c is to be determined later, in order to be compatible with the other data. >From (14) and (15), it admits the following factorization:

$$\begin{cases} \dfrac{\partial t}{\partial x} + Pt = -\Delta f, \quad t(0) = -\dfrac{\partial f}{\partial x}(0), \\ -\dfrac{\partial \Delta z}{\partial x} + P\Delta z = -t, \quad \Delta z(1) = c. \end{cases} \tag{34}$$

Formally, taking the derivative with respect to x on both sides of (32), we obtain:

$$\frac{\partial^2 z}{\partial x^2}(x) = \frac{dP}{dx}(x)z(x) + P(x)\frac{\partial z}{\partial x}(x) + \frac{dQ}{dx}(x)\Delta z(x) + Q(x)\frac{\partial \Delta z}{\partial x}(x) + \frac{d\tilde{r}}{dx}(x)$$

and, substituting from (32) and (34), we obtain:

$$\Delta z - \Delta_y z = \frac{dP}{dx} z + P(Pz + Q\Delta z + \tilde{r}) + \frac{dQ}{dx}\Delta z + Q(P\Delta z + t) + \frac{d\tilde{r}}{dx} \tag{35}$$

or which is equivalent

$$(\frac{dP}{dx} + P^2 + \Delta_y) + (\frac{dQ}{dx} + PQ + QP - I)\Delta z + \frac{d\tilde{r}}{dx} + P\tilde{r} + Qt = 0. \tag{36}$$

Now, taking into account that $z|_{\Gamma_s} = h$ and $\Delta z|_{\Gamma_s} = k$ are arbitrary, we derive

$$\frac{dP}{dx} + P^2 + \Delta_y = 0, \quad P(0) = 0, \tag{37}$$

$$\frac{dQ}{dx} + PQ + QP = I, \quad Q(0) = 0, \tag{38}$$

$$\frac{\partial t}{\partial x} + Pt = -\Delta f, \quad t(0) = -\frac{\partial f}{\partial x}(0), \tag{39}$$

$$\frac{\partial \tilde{r}}{\partial x} + P\tilde{r} = -Qt, \quad \tilde{r}(0) = -z_0, \tag{40}$$

$$\frac{\partial \Delta z}{\partial x} - P\Delta z = t, \quad \Delta z(1) = c, \tag{41}$$

$$\frac{\partial z}{\partial x} - Pz = Q\Delta z + \tilde{r}, \quad z(1) = z_1. \tag{42}$$

It is easy to see, from the definition, that $Q(1)$ is a bijective operator from $(H_{00}^{\frac{1}{2}}(\mathcal{O}))'$ to $H^{\frac{1}{2}}(\mathcal{O})$, so we can define $(Q(1))^{-1}$. >From (27) and the regularity assumptions made at beginning of this section, we can define:

$$c = (Q(1))^{-1}(z_2 - P(1)\, z_1 - \tilde{r}(1)). \tag{43}$$

Once again we can remark the interest of the factorized form if the same problem has to be solved many times for various sets of data (z_1, z_2). Once the problem has been factorized, that is P and Q have been computed, and t and $\tilde{r}$ are known, the solution for a data set (z_1, z_2) is obtained by solving (43) and then the Cauchy initial value problems (41), (42) backwards in x. We have factorized problem $(\mathcal{P}_2)$. We may write

$$\mathcal{A}^2 = \begin{pmatrix} -\frac{d}{dx} - P & 0 \\ -Q & -\frac{d}{dx} - P \end{pmatrix} \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{d}{dx} - P & -Q \\ 0 & \frac{d}{dx} - P \end{pmatrix}.$$

We will see in section 7 that P and Q are self adjoint. So, the first and third matrices are adjoint of one another and are, respectively, lower triangular and upper triangular.

7. Some properties of P and Q

The Riccati equation (37) was studied in (3), using a Yosida regularization.
1− For each $s \in [0,1]$, $P(s) \in \mathcal{L}(H_0^1(\mathcal{O}), L^2(\mathcal{O}))$.
2− For each $s \in [0,1]$, $P(s)$ is a self-adjoint and positive operator. In fact, the property is obviously true when $s = 0$. On the other hand, let $s \in]0,1]$, $h_1, h_2 \in L^2(\mathcal{O})$, and $\{\beta_1, \gamma_1\}$,$\{\beta_2, \gamma_2\}$

the corresponding solutions of (9) to h_1 and h_2. From the definition of P we may conclude that $P(s)h_i = \dfrac{\partial \beta_i}{\partial x}|_{\Gamma_s}$, where β_i is the solution of:

$$\begin{cases} -\Delta\beta_i = 0 \quad \text{in } \Omega_s, \\ \beta_i|_\Sigma = 0, \\ -\dfrac{\partial \beta_i}{\partial x}|_{\Gamma_0} = 0, \quad \beta_i|_{\Gamma_s} = h_i. \end{cases}$$

We then have that:

$$0 = \int_{\Omega_s} (-\Delta\beta_1)\beta_2 dxdy = \int_{\Omega_s} \nabla\beta_1 \nabla\beta_2 dxdy - \int_{\partial\Omega_s} \frac{\partial \beta_1}{\partial x}\beta_2 d\sigma$$

and, taking into account that $\beta_2|_\Sigma = 0, \dfrac{\partial \beta_1}{\partial x}|_{\Gamma_0} = 0$ and $\beta_2|_{\Gamma_s} = h_2$, we conclude that:

$$(P(s)h_1, h_2) = \int_{\Gamma_s} \frac{\partial \beta_1}{\partial x}(s)\beta_2(s) d\sigma = \int_{\Omega_s} \nabla\beta_1 \nabla\beta_2 dxdy$$

which shows that $P(s)$ is a self-adjoint and positive operator.
3−

$$\|P(s)h\|_{L^2(\mathcal{O})} \leq \|h\|_{H_0^1(\mathcal{O})}, \forall h \in H_0^1(\mathcal{O}), \forall s \in [0,1].$$

4− For each $s \in [0,1]$, $Q(s)$ is an operator from $(H_{00}^{1/2}(\mathcal{O}))'$ into $H_{00}^{1/2}(\mathcal{O})$ and from $L^2(\mathcal{O})$ into $H_0^1(\mathcal{O})$.
5− For each $s \in [0,1]$, $Q(s)$ is a linear, self-adjoint, non negative operator in $L^2(\mathcal{O})$, and it is positive if $s \neq 0$. In fact, the result is obviously verified if $s = 0$. On the other hand, if $s \in]0,1]$, $k_i \in L^2(\mathcal{O})$, and γ_i are the solutions of the problems:

$$\begin{cases} \Delta^2\gamma_i = 0, \quad \text{in } \Omega_s, \\ \gamma_i|_\Sigma = 0, \quad \Delta\gamma_i|_\Sigma = 0, \\ \dfrac{\partial \gamma_i}{\partial x}(0) = 0, \quad \dfrac{\partial \Delta\gamma_i}{\partial x}(0) = 0, \\ \gamma_i|_{\Gamma_s} = 0, \ \Delta\gamma_i|_{\Gamma_s} = k_i, \quad i = 1,2, \end{cases} \tag{44}$$

then, by Green's formula, noticing that $\gamma_1|_\Sigma = \gamma_1|_{\Gamma_s} = 0$ and $\frac{\partial \Delta\gamma_2}{\partial x}(0) = 0$, we have:

$$0 = \int_{\Omega_s} \gamma_1 \Delta^2 \gamma_2 dxdy = -\int_{\Omega_s} \nabla\gamma_1 \nabla(\Delta\gamma_2) dxdy$$

and, again by Green's formula, remarking that $\Delta\gamma_2|_\Sigma = 0$ and $\frac{\partial \gamma_1}{\partial x}(0) = 0$, we obtain

$$(Q(s)k_1, k_2) = \int_{\Gamma_s} \frac{\partial \gamma_1}{\partial x}(s)\Delta\gamma_2(s) d\sigma = \int_{\Omega_s} \Delta\gamma_1 \Delta\gamma_2 dxdy$$

which shows that $Q(s)$ is a self-adjoint non negative operator in $L^2(\mathcal{O})$. On the other hand

$$\langle Q(s)k, k\rangle = 0 \Leftrightarrow \int_{\Omega_s} (\Delta\gamma)^2 dxdy = 0 \Rightarrow \Delta\gamma = 0 \text{ in } \Omega_s \Rightarrow k = \Delta\gamma|_{\Gamma_s} = 0$$

and so $Q(s)$ is positive for $s \in]0,1]$.
6− For each $x \in [0,1]$, $-P(x)$ is the infinitesimal generator of a strongly continuous semigroup of contractions in $L^2(\mathcal{O})$.
In fact we know that, for each $x \in [0,1]$, $P(x)$ is an unbounded and self-adjoint operator from $L^2(\mathcal{O})$ into $L^2(\mathcal{O})$ with domain $H_0^1(\mathcal{O})$. By (4), proposition II.16, page 28, $-P(x)$ is a closed operator. On the other hand

$$(-P(x)h, h) \leq 0, \forall h \in H_0^1(\mathcal{O})$$

so, $-P(x)$ is a dissipative operator. Finally, by (9), Corollary 4.4, page 15, $-P(x)$ is the infinitesimal generator of a strongly continuous semigroup of contractions in $L^2(\mathcal{O})$, $\{\exp(-tP(x)\}_{t\geq 0}$.

It is easy to see that the family $\{-P(x)\}_{x\in[0,1]}$ verifies the conditions of Theorem 3.1, with the slight modification of remark 3.2, of (9). This implies that there exists a unique evolution operator $U(x,s)$ in $L^2(\mathcal{O})$, that is, a two parameter family of bounded linear operators in $L^2(\mathcal{O})$, $U(x,s)$, $0 \leq s < x \leq 1$, verifying $U(x,x) = I$, $U(x,r)U(r,s) = U(x,s), 0 \leq s \leq r \leq x \leq 1$, and $(x,s) \longmapsto U(x,s)$ is strongly continuous for $0 \leq s \leq x \leq 1$. Moreover, $\|U(x,s)\|_{\mathcal{L}(L^2(\mathcal{O}))} \leq 1$ and

$$\frac{\partial}{\partial s} U(x,s)h = U(x,s)P(s)h, \quad \forall h \in H_0^1(\mathcal{O}), \text{ a.e. in } 0 \leq s \leq x \leq 1.$$

Formally, from equation (38), we have:

$$\frac{\partial}{\partial s}(U(x,s)Q(s)U^*(x,s)) = U(x,s)U^*(x,s).$$

Integrating from 0 to x, and remarking that $Q(0) = 0$,

$$Q(x) = \int_0^x U(x,s)U^*(x,s)\, ds.$$

We define a mild solution of (38) by

$$(Q(x)h, \bar{h}) = \int_0^x (U^*(x,s)h, U^*(x,s)\bar{h})\, ds, \quad \forall h, \bar{h} \in H_0^1(\mathcal{O}).$$

By the preceeding remarks, equation (38) has a unique mild solution.

Again formally, from equation (39), we have

$$\frac{\partial}{\partial s}(U(x,s)t(s)) = U(x,s)\frac{\partial t}{\partial s} + U(x,s)P(s)t = -U(x,s)\Delta f,$$

so we define a mild solution of (39) by

$$t(x) = -U(x,0)\frac{\partial f}{\partial x}(0) - \int_0^x U(x,s)\Delta f\; ds.$$

For equations (40), (41) and (42) we proceed in a similar way, noting that for (41) and (42) the integral is taken between x and 1.

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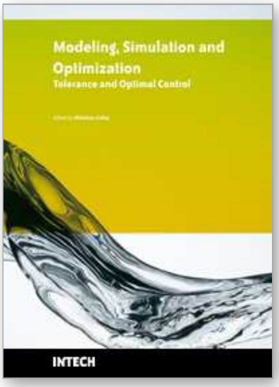

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Parametric representation of shapes, mechanical components modeling with 3D visualization techniques using object oriented programming, the well known golden ratio application on vertical and horizontal displacement investigations of the ground surface, spatial modeling and simulating of dynamic continuous fluid flow process, simulation model for waste-water treatment, an interaction of tilt and illumination conditions at flight simulation and errors in taxiing performance, plant layout optimal plot plan, atmospheric modeling for weather prediction, a stochastic search method that explores the solutions for hill climbing process, cellular automata simulations, thyristor switching characteristics simulation, and simulation framework toward bandwidth quantization and measurement, are all topics with appropriate results from different research backgrounds focused on tolerance analysis and optimal control provided in this book.