1. Introduction

A Shack-Hartmann sensor is one of the most important and popular wavefront sensors used in an adaptive optics system to measure the aberrations caused by either atmospheric turbulence, laser transmission, or the living eye [1-7]. Its design was based on an aperture array that was developed in 1900 by Johannes Franz Hartmann as a means to trace individual rays of light through the optical system of a large telescope, thereby testing the quality of the image.[8] In the late 1960s Roland Shack and Platt modified the Hartmann screen by replacing the apertures in an opaque screen by an array of lenslets [9-10]. The terminology as proposed by Shack and Platt was “Hartmann-screen”. The fundamental principle seems to be documented even before Huygens by the Jesuit philosopher, Christopher Scheiner [11].

The schematic of a Shack-Hartman wavefront sensor is shown in Figure 1. It consists of an array of lenses (called lenslets, see Figure 1) of the same focal length. Each is focused onto a photon sensor (typically a CCD array or quad-cell). The local tilt of the wavefront across each lens can then be calculated from the position of the focal spot on the sensor. Any phase aberration can be approximated to a set of discrete tilts. By sampling an array of lenslets, all of these tilts can be measured and the whole wavefront can be approximated. Since only tilts are measured, the Shack-Hartmann can not measure the discontinuous steps of wavefront.

Fig. 1. Schematic of a Shack-Hartmann wavefront sensor
Tyler and Fried have obtained the theory expression, which evaluates the angular position error when a quadrant detector is used in the SHWFS [12]. The formula they obtained, based on circular aperture diffraction, is shown in Eq. (1)

$$\sigma = \frac{3\pi}{16} \frac{1}{SNR} \frac{\lambda}{D}$$  \hspace{1cm} (1)

where SNR is defined as the ratio of the signal’s photoelectron counts to the noise’s fluctuation intensity within the detection area, $\lambda$ is the wavelength, and $D$ is the diameter of the aperture. Their analysis did not discuss the size of the incoming light spot on the detection area in detail. The formula was obtained based on a quadrant detector alone. Generally, the theory expression obtained by Tyler and Fried is not suitable for describing the angular position error when the scale of the discrete detector arrays is greater than $2 \times 2$ pixels.

Hardy has described formulas that can be used to evaluate the angular position error [13], under the conditions that the photon shot noise of signal is dominant. Although his formulas discussed the size of the diffraction-limited spot on the discrete detector arrays, it is reliable only under the approximation condition that $l/f > \lambda/D$ or $l/f < \lambda/D$ is satisfied, where $l$ is the length of a pixel, $f$ is the focal length, and $V_s$ is the count of signal photoelectrons. Eq. (2) shows the formulas based on square aperture diffraction.

$$\sigma = \begin{cases} 
0.277 \frac{\lambda}{D} / V_s^{1/2} & (\text{when } l/f < \lambda/D) \\
0.500 \frac{\lambda}{D} / V_s^{1/2} & (\text{when } l/f > \lambda/D)
\end{cases}$$  \hspace{1cm} (2)

Cao et al. have also analyzed the measurement error of a SHWFS. Their work emphasized the discrete sampling error of a CCD and obtained a formula which is used to describe the centroid position error induced by the readout noise of the CCD and the photon shot noise of the signal [14]. Their research results are only the approximation of some cases discussed in this article. Jiang et al. have partially analyzed the measurement error of SHWFS and performed their method by setting a fixed threshold to suppress the impact of the random noise [15].

In this chapter, the wavefront error of a Shack-Hartmann wavefront sensor was analyzed in detail based on the research results of angular position error and wavefront error [16-17], and the formula used to evaluate the wavefront error was derived, it concerns with the signal to noise ratio, number of photons and reconstruction matrix also.

### 2. The angular position error caused by random noise

The wavefront to be measured is segmented into many subwavefronts by lenslet arrays, and the light spots at the focal plane of the subapertures are detected by the CCD. Particularly, the analysis is based on the notion that the wavefront is essentially flat over each subaperture and $r_0 > D$ ($r_0$ is the coherent length of incoming wavefront). The centroid position can be calculated by Eq. (3) [18]. The detection area of the subaperture is $L_1 \times L_2$ pixels, and $x_{nm}$ and $y_{nm}$ are the $(n,m)$th pixel’s $X$ coordinate and $Y$ coordinates, respectively. $I_{nm}$ is the total intensity in the $(n,m)$th pixel, including the signal photons and all other noises.

$$\sigma_{nm} = \frac{2\pi}{16} \frac{1}{SNR} \frac{\lambda}{D}$$  \hspace{1cm} (3)
The formulas which evaluate the centroid error associated with the signal’s photon shot noise and the readout noise of the detector, respectively, have been derived by Cao et al. When the photon shot noise of the signal is considered alone, the centroid fluctuation error was obtained by introducing the Gauss width of the signal. When the readout noise of the detector is considered alone, the centroid fluctuation error was also obtained, and its results are shown in Eq. (4) and Eq. (5), respectively [14]. \(N_r\) is the rms error induced by the fluctuation of the readout noise in each pixel (with units of photoelectron counts). \(V_r\) is the sum of the readout noise’s photoelectron count among all of the pixels within the corresponding subaperture.

\[
\phi_{cs}^2 = \frac{G_s^2}{V_s}
\]

\[
\phi_{cr}^2 = \frac{N_r^2}{V_r^2} L_1 L_2 \frac{I_{1}^2 - 1}{12}
\]

\(\phi_{cs}^2\) is the variance of the centroid fluctuation in one direction (\(X\) or \(Y\)), induced by the photon shot noise of signal itself, and \(\phi_{cr}^2\) is the variance of the centroid fluctuation in one direction (\(X\) or \(Y\)), induced by the readout noise of the detector. \(\phi_{cs}^2\) and \(\phi_{cr}^2\) are both defined in pixel\(^2\) units. \(G_s\) is the equivalent Gauss width of the signal spot and is defined in pixel units by the expression:

\[
G_s = \eta \frac{f \lambda}{D}
\]

where \(\eta\) is the positive constant. \(\eta\) is 0.353 when the diffraction aperture is square and 0.431 when the diffraction aperture is circular.

Based on Eq. (3), the centroid position in the \(X\) direction can be expressed by Eq. (7). The detailed derivation process of this expression is shown in appendix 1.1.

\[
x_x = \frac{sbr}{1 + sbr} x_{cs} + \frac{1}{1 + sbr} x_{cb}
\]

where \(x_x\) is the calculated centroid position of the signal in the \(X\) direction, \(x_{cs}\) is the real centroid position of the signal in the \(X\) direction, and \(x_{cb}\) is the centroid position induced by the total noise, except for the signal in the \(X\) direction, where the total noise largely includes the readout noise of the detector and the heterogeneous light noise. \(sbr = \sum <S_{n,m}> / \sum <B_{n,m}>\), \(<S_{nm}>>\) is the collective average of the signal intensity in the \((n,m)\)th pixel, and \(<B_{nm}>>\) is the collective average of the total noise intensity in the \((n,m)\)th pixel (with units of ADU).
Based on the error transition principles, the rms error of centroid measurement induced by random noise in the X direction can be written in Eq. (8) as:

$$\phi_c = [(\frac{sbr}{1+sbr})^2 \phi_{cs}^2 + (\frac{1}{1+sbr})^2 \phi_{cb}^2]^{1/2}$$

(8)

where $\phi_{cs}$ is the rms error of centroid measurement in the X direction induced by the signal’s photon shot noise and $\phi_{cb}$ is the rms error of centroid measurement in the X direction induced by all the other noise. The signal is mostly comprised of heterogeneous light and readout noise.

If there were no heterogeneous light and readout noise in the detection area, the signal’s photon shot noise should be the unique noise resource which affects centroid measurement. Based on Eq. (4) and Eq. (6), when the discrete sampling error of the detector is ignored, the rms error of angular position in the X direction caused by the photon shot noise of the signal can be written as:

$$\sigma_1 = \left(\frac{C_s^2}{V_s}\right)^{1/2} = \eta \frac{\lambda}{D} V_s^{-1/2}$$

(9)

When the photon shot noise of the signal is small compared with the readout noise and the heterogeneous light noise, the heterogeneous light noise and the readout noise become the primary noise, which affects centroid calculation. When the heterogeneous light noise can be considered as a uniform noise, like the readout noise of the detector, it exists in each pixel and it has the same fluctuation characteristics among the pixels in the detection area. So, the noise in one pixel (including the heterogeneous light noise and the readout noise of the CCD) can be summed and described by $N_b$. $N_b$ is defined as the rms error of the heterogeneous light noise and the readout noise photoelectron count in one pixel, and it has the same fluctuation characteristics as the readout noise of the detector. $N_b$ has units of ADU. Subsequently, the rms error of centroid measurement in the X direction caused by the heterogeneous light noise and the readout noise of the CCD can be written as:

$$\phi_c = \frac{1}{1+sbr} \phi_{cb}$$

$$= (1 + \sum S_{nm} / \sum B_{nm})^{-1} [L_1 L_2 (L_1^2 - 1) / 12]^{1/2} N_b / V_b$$

(10)

$$= (\sum B_{nm} / N_b + C_{(\lambda,D,l,f)} \cdot snr)^{-1} [L_1 L_2 (L_1^2 - 1) / 12]^{1/2}$$

where $snr = \max(S_{nm}) / N_b$, $V_b = \sum B_{nm}$, and $\max(S_{nm})$ is the signal’s peak intensity. $snr$ is defined as the ratio of the signal’s peak intensity to the rms error induced by the background noise, $B_{nm}$ is the average intensity of noise in the $(n, m)^{th}$ pixel, which includes the heterogeneous light noise and the readout noise of the CCD. $C_{(\lambda,D,l,f)}$ is the light spot constant which is defined as the ratio of the total signal intensity to the signal’s peak intensity in the subaperture, and its value can be measured or calculated exactly by $C_{(\lambda,D,l,f)} = \sum S_{nm} / \max(S_{nm})$. 

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The intensity distribution of the signal’s light spot at the focal plane of the subaperture can be calculated by circular or square aperture diffraction approximations. On the other hand, the Gauss distribution can also be used to approximately describe the intensity distribution of the light spot. The analytic expressions of $C(\lambda, D, l, f)$ are described in Eq. (11) with different approximation conditions. The detailed derivation process is shown in appendix 2.1~2.3. The value of $C(\lambda, D, l, f)$ can be calculated by Eq. (11).

$$C(\lambda, D, l, f) = \begin{cases} 
1 / (1 - J_0^2(r_1) - J_1^2(r_1)) & \text{(circular aperture diffraction approximation)} \\
\frac{\pi}{2} / (-\sin^2 x_1 + \sum_{k=0}^{+\infty} \frac{(-1)^{k} \times (2x_1)^{2k+1}}{(2k+1)!(2k+1)!})^2 & \text{(square aperture diffraction approximation)} \\
1 / [1 - \exp(-n^2 \lambda^2 D^2)] & \text{(Gauss distribution approximation)} 
\end{cases}$$ (11) 

where $x_1 = \frac{\pi D}{\lambda} \sin(\theta) = \frac{\pi D}{\lambda} \sin(l/2f) = \frac{\pi Dl}{2\lambda f}$, $r_1 \approx \sqrt{\frac{4}{\pi}} x_1 = \sqrt{\frac{4}{\pi}} \frac{\pi Dl}{2\lambda f}$, $n$ is a positive constant, and $\theta$ is the diffraction angle.

When the direct current part of the noise (including the heterogeneous light noise and the readout noise of the CCD) is subtracted, it can be considered as white noise, and $\sum B_{nm} = 0$. Then, the standard deviation of the angular position error in the $X$ direction caused by noise can be described by Eq. (12):

$$\sigma_2 = \frac{[L_1 L_2 (L_1^2 - 1) / 12]^{1/2}}{C(\lambda, D, l, f)} \frac{1}{\text{snr} f}$$ (12) 

When $L_1 = L_2 = L$, the Eq. (12) can also be expressed by Eq. (13)

$$\sigma_2 = \frac{[L^2 (L^2 - 1) / 12]^{1/2}}{C(\lambda, D, l, f) \max(S_{nm}) / (L^2 N_b + V_s^{1/2}) f} \frac{1}{\text{snr} f}$$

$$= \frac{1}{\lambda / D \text{SNR} D}$$ (13) 

where $\text{SNR}$ has the same definition as in Eq. (1). $L^2 N_b$ is the sum of the rms error of total noise among all of the pixels within the detection area, and expresses the total intensity of noise fluctuation. $V_s^{1/2}$ expresses the photon shot noise induced only by the incoming signal.
ω is the position error constant, and it is weighted by the intensity of background noise and the intensity of the signal’s photon shot noise (defined in Eq. (14)):

$$\omega = [(L^2 - 1) / (12L^2)]^{1/2} \frac{1}{\lambda} \frac{L^2 N_b}{f} \frac{\lambda}{D} \frac{1}{(L^2 N_b + V_s^{1/2})}$$ (14)

Substituting Eq. (13) and Eq. (9) into Eq. (8), with the assumed condition that there are no correlations among the photon shot noise of the signal, the heterogeneous light noise, and the readout noise of CCD, then the total rms error of angular position in the X direction caused by random noise can be obtained:

$$\sigma = (\sigma_1^2 + \sigma_2^2)^{1/2}$$

$$\approx \omega \frac{1}{SNR} \frac{\lambda}{D} + \eta \frac{\lambda}{D} V_s^{-1/2}$$ (15)

Eq. (15) is the desired result which can be used to precisely describe the angular position error of a Shack-Hartmann wavefront sensor caused by random noise, and therefore, the centroid algorithm is used to calculate the spot position of the incoming light. Generally, when the ideal detector with very small readout noise is used and there is no background light noise (ω → 0), the photon shot noise of the signal becomes the theoretical limits imposed on the angular position measurement. Eq. (9) showed this expression. In practice, the theoretical limits may not be achieved for the hardware and environment limitations. When the photon shot noise is small enough compared with the heterogeneous light noise and readout noise, it could be ignored in Eq. (15), and Eq. (13) could be used to describe the angular position error caused by the random noise approximately. Commonly, it has enough accuracy. The position-error constant ω described in Eq. (14) is concerned with the scale of the discrete detector arrays in the detection area, the noise characteristics of the detector, and the system parameters. Clearly, the formula based on a quadrant detector obtained by Tyler and Fried is only a special case in this article. On the other hand, the formula obtained in Eq. (13) is suitable to evaluate the angular position error for both a circular and square aperture.

3. Wavefront measurement error caused by centroid position random error

In this chapter, Zernike modes are used as the basis for wavefront reconstruction. The wavefront measurement error can be written as [13, 19]

$$\Delta \phi = \phi - \phi'$$

$$= \sum_{j=1}^{P} (a_j - a_j') Z_j$$

$$= \sum_{j=1}^{P} \Delta a_j Z_j$$ (16)

where Δφ is the wavefront measurement error induced by centroid position random error, φ is the wavefront to be measured, φ' is the wavefront detected, P is the total number of Zernike modals, aj is the jth Zernike coefficient, and Z expresses the Zernike polynomial.
Then, the mean-square of wavefront measurement error can be written as shown in Eq. (17) [20]. The angle brackets denote a collective average.

\[
\sigma_\phi^2 = \langle \phi^2 \rangle - \langle \phi \rangle^2 = \sum_{j=1}^{p} \langle |\Delta a_j|^2 \rangle
\]  

(17)

Based on the principles of the Zernike modal wavefront reconstruction algorithm [20], the Zernike-coefficients vector of a wavefront can be obtained:

\[
A = E \cdot H
\]  

(18)

where \(E\) is the modal reconstruction matrix and \(H\) is the wavefront slope vector.

Therefore, the variance of the modal Zernike coefficients that describe the wavefront measurement error can be written as:

\[
\langle |\Delta a_j|^2 \rangle = \left| \sum_{k=1}^{2Q} e_{j,k} \cdot \Delta h_k \right|^2 
= \sum_{k=1}^{2Q} \sum_{l=1}^{2Q} e_{j,k}^* e_{j,l} \langle \Delta h_k \cdot \Delta h_l \rangle
\]  

(19)

where \(Q\) is the total number of subapertures, \(e_{j,k}\) is the element of \(E\), and \(\Delta h_k\) is the error of the \(k\)th slope element.

In order to simplify analysis, we assume that there are no correlations among the different slope vectors in the corresponding subapertures and the intensity of the signals are uniform and isotropic among the different subapertures. Subsequently, the following expression can be obtained:

\[
\langle \Delta h_k \cdot \Delta h_l \rangle = \sigma_c^2 \frac{\delta(j_k - j_l)}{f^2}
\]  

(20)

where \(\sigma_c^2\) is the variance of centroid position random error induced by random noise, \(f\) is the focal length of lenslets, \(\delta(x,y)\) is the Kronecker delta function [21], and \(j\) and \(k\) are the subapertures which are connected with the slope \(h_k\) and \(h_l\). Substituting Eq. (20) and Eq. (19) into Eq. (17), the mean-square of wavefront measurement error can be written as:

\[
\sigma_\phi^2 = \sum_{j=1}^{p} \langle |\Delta a_j|^2 \rangle 
= \sum_{j=1}^{p} \sigma_\phi^2 \cdot K(j,Q) 
= \sum_{j=1}^{p} \left( \frac{\sigma_c^2}{f^2} \cdot f_0^2 \right) \cdot K(j,Q)
\]  

(21)
where $\sigma_s$ is the wavefront average slope of the corresponding subaperture in the unit circle,

$$K(j,Q) = \sum_{k=1}^{Q} (e_{j,2k-1} + e_{j,2k})^2.$$  It is concerned with the subaperture segmentation number and the distribution of subapertures. $f_0$ describes the normalized relationship between the real wavefront slope vector and the normalized wavefront slope vector in the unit circle, and is defined by the expression:

$$f_0 = \frac{D}{2\lambda}$$  (22)

where D is the diameter of the aperture and $\lambda$ is the measuring wavelength.

Then, the root mean square value of wavefront measurement error caused by centroid position random error is obtained:

$$\sigma_\phi = \sigma_c \frac{D}{2\lambda f} \left[ \sum_{j=1}^{P} \sum_{k=1}^{Q} (e_{j,2k-1} + e_{j,2k})^2 \right]^{1/2} \tag{23}$$

Eq. (23) is the desired expression used to evaluate the wavefront measurement error associated with the centroid position random error. $\sigma_c$ is the standard deviation in pixels of centroid position random error caused by random noise. The formula described in Eq. (23) can help us to decide what the wavefront measurement error will be when the centroid position randomly fluctuates due to random noise, and it may be a factor that must be considered during the design of the SHWFS.

4. Wavefront measurement error analysis based on Zernike modal reconstruction

In a Shack-Hartmann wavefront sensor, the angular position can be calculated from the centroid position in each subaperture and is proportional to the centroid position. The relationship between centroid and angular position can be described by

$$\sigma = \frac{\sigma_c}{f}$$  (24)

In Eq. (15), the angular position error caused by random noise was obtained. In Eq. (23), the wavefront error caused by random centroid error was obtained. Therefore, the total wavefront measurement error can be described by Eq. (25):

$$\sigma_\phi = \sigma \cdot f \frac{D}{2\lambda f} \left[ \sum_{j=1}^{P} \sum_{k=1}^{Q} (e_{j,2k-1} + e_{j,2k})^2 \right]^{1/2}$$

$$\sigma_\phi = \frac{1}{2} \left( \frac{1}{\text{SNR}} + \eta \right) \frac{\sqrt{\text{SNR}}}{\sqrt{\text{SNR}}} \left[ \sum_{j=1}^{P} \sum_{k=1}^{Q} (e_{j,2k-1} + e_{j,2k})^2 \right]^{1/2} \tag{25}$$

where $\sigma_\phi$ is the total wavefront measurement error.
In this formula, we can determine the wavefront measurement error concerned with SNR (see the definition in Eq. (1)), aperture of lenslets (see the definition in Eq. (6)), counts of effective signal, and the reconstruction matrix parameters (see the definition in Eq. (19)).

5. Conclusions

In this chapter, the exact formula (Eq. (25)), which evaluates the Shack-Hartmann wavefront sensor’s measurement error associated with the signal to noise ratio of effective signal, was derived in detail. This study was performed based on a modal wavefront reconstruction with Zernike polynomials, and provided an exact and universal formula to describe the wavefront measurement error of a Shack-Hartmann wavefront sensor with discrete detector arrays. It is critical to an adaptive optics system when the Shack-Hartmann sensor is used as the wavefront sensor, and it provides a reference when designing a Shack-Hartmann wavefront sensor and calculating its reconstruction matrix.

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7. References

Advances in adaptive optics technology and applications move forward at a rapid pace. The basic idea of wavefront compensation in real-time has been around since the mid 1970s. The first widely used application of adaptive optics was for compensating atmospheric turbulence effects in astronomical imaging and laser beam propagation. While some topics have been researched and reported for years, even decades, new applications and advances in the supporting technologies occur almost daily. This book brings together 11 original chapters related to adaptive optics, written by an international group of invited authors. Topics include atmospheric turbulence characterization, astronomy with large telescopes, image post-processing, high power laser distortion compensation, adaptive optics and the human eye, wavefront sensors, and deformable mirrors.

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