Using Transition Invariants for Reachability Analysis of Petri Nets

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1. Introduction

Petri nets are an important formal paradigm for modeling and analysis of discrete event systems. The related areas of application of Petri nets include deadlock avoidance and prevention, supervisory control, forbidden state detection, different aspects of flexible manufacturing systems, and many others (Zhou & DiCesare, 1993; Holloway et al., 1997; Boel et al., 1995). Quite often, given a discrete-event system, the designer is interested in determining whether the system can transit from an initial state to another, target state as a result of some operations from a predefined set. In terms of Petri nets, the answer to this question is obtained as a solution of a reachability problem.

The reachability problem in Petri nets is formulated as follows: for any Petri net $PN$, with an initial marking $M^0$, and for some other marking $M$, determine whether the relation $M \in R(PN, M^0)$ is true, where $R(PN, M^0)$ is the reachability set of $PN$ for its initial marking $M^0$ (Murata, 1989). The decidability of the reachability problem has been proved for a number of restricted classes of Petri nets, and there are efficient algorithms for such classes as acyclic Petri nets, marked graphs, and others (Kodama & Murata, 1988; Caprotti et al., 1995; Kostin, 1997).

It has been shown that the reachability problem is decidable for generalized Petri nets as well (Mayr, 1984). The fundamental contribution of the paper (Mayr, 1984) is in proving that the reachability problem for generalized Petri nets is decidable. However, being highly important theoretically, the practical use of the algorithm described in that paper is limited. Actually, the algorithm creates a series of so called regular constrained refined graphs, each of which is a generalization of the basic coverability tree. As the author admits, the first refined graph would enumerate the whole reachability set of the given Petri net.

In practice, two different approaches are used most often to determine the reachability of a marking in Petri nets. The first approach is based on the creation and investigation of a complete or reduced reachability graph. The main drawback of this approach is a state explosion problem. A closely related technique is the use of stubborn sets. The main purpose of the stubborn sets technique is to choose, for each marking of the net, a set of transitions to fire that is large enough to preserve some desired information about the Petri net, but is as small as possible to get a significant reduction of the resulting reachability graph (Varpaaniemi, 1998). Unfortunately, generation of minimal or reduced reachability graphs in finite state systems is known to be an NP-hard problem (Peled, 1993). If Petri net has no...
specific properties like a symmetry or reversibility, the corresponding reduced reachability graph will have almost the same size as that of the full reachability graph (Schmidt, 2000). The second approach is based on methods of linear algebra. Given a pure Petri net (i.e. a net without self-loops), with sets of transitions \( T \) and places \( P \), its structure is represented unambiguously by the incidence matrix

\[
D = [d(t_i, p_j)] = [d_{ij}], \quad i = 1, 2, \ldots, m = |T|, \quad j = 1, 2, \ldots, n = |P|,
\]

where \( d(t_i, p_j) = \text{Post}(p_j) - \text{Pre}(p_j, t_i) \). \( \text{Pre} \) and \( \text{Post} \) are the input and output functions of the Petri net, with \( \text{Pre}(p, t) = v \) if there is a directed arc from \( p \) to \( t \) with the weight \( v \), and \( \text{Post}(p, t) = v \) if there is an arc from \( t \) to \( p \) with the weight \( v \). Note that, in this matrix, rows correspond to transitions and columns correspond to places (Murata, 1989).

It is known that a necessary condition for reachability of marking \( M \) from some other marking \( M^0 \) is the existence of a nonnegative integer solution of the matrix equation

\[
M = M^0 + FD
\]

relative to \( F \), where \( F = [f_1, f_2, \ldots, f_m] \) is a nonnegative integer firing count vector. Note that in this chapter, all vectors are considered as row vectors. In particular, markings of PN will be expressed as \((1 \times n)\) vectors, so that we can write

\[
M^0 = [m_1^0, m_2^0, \ldots, m_n^0] \quad \text{and} \quad M = [m_1, m_2, \ldots, m_n],
\]

where the \( i \)th entry in vectors \( M^0 \) and \( M \) denotes the number of tokens in place \( p_i \in P \).

Equation (2), proposed in Murata (1977), is called the fundamental equation of Petri net. It is of the paramount importance for the investigation of the structural and behavioral properties of Petri nets with methods of linear algebra.

With the use of linear algebra, reachability analysis is usually carried out in two stages. At the first stage, by solving the equation (2) or its related integer programming form, firing count vectors are obtained. At the second stage, the computed firing count vectors are used in an attempt to determine legal firing sequences that transform initial marking \( M^0 \) into target marking \( M \).

Unfortunately, the existence of a nonnegative integer solution of equation (2) is not a sufficient condition for reachability of marking \( M \) from \( M^0 \) (Murata, 1989). That is, it is quite possible that, in a given Petri net, no legal firing sequences exist for the valid firing count vectors. In general, the equation (2) can have infinite number of nonnegative integer solutions. Some of these solutions can correspond to legal firing sequences, while others fail (Peterson, 1981). Thus, there is a challenging problem to select working firing count vectors.

In (Kostin, 2003), with the use of linear algebra, a method was proposed to restrict the number of firing count vectors to be tried for the determination of legal firing sequences, without the loss of reachability information. The method is applicable for reachability analysis of a particular class of place/transition Petri nets having no transition invariants, or T-invariants. Algebraically, T-invariants of a Petri net with incidence matrix \( D \) are nonnegative integer \((1 \times m)\) vectors \( F \) such that \( FD = 0 \) (Memmi & Roucairol, 1980).

According to the scheme proposed in (Kostin, 2003), given a Petri net with an initial and a target markings, a so called complemented Petri net is created that consists of the given Petri net and an additional, complementary transition with some input and output places of the original Petri net, which are uniquely determined by the initial and target markings. Then
the reachability problem is reduced to computation and investigation of T-invariants of the complemented Petri net. The main result of that paper is that legal firing sequences, if they exist, can be found using only those T-invariants of the complemented Petri net in which the complementary transition fires only once. It was shown that this set is finite. This chapter generalizes the approach described in (Kostin, 2003) for arbitrary place/transition nets, including Petri nets with T-invariants. The existence of T-invariants in the original Petri nets considerably complicates the reachability analysis. In contrast with the scheme in (Kostin, 2003), where the number of T-invariants of any complemented Petri net that are sufficient for performing the reachability analysis is proven to be finite, in the generalized scheme the set of T-invariants for investigation is theoretically infinite. Nevertheless, as will be shown in this chapter, it is always possible to effectively limit this set without the loss of reachability information and then to use T-invariants from this finite set for reachability analysis.

This chapter is an extended version of the author’s article published in Lecture Notes in Computer Science (Kostin, 2006). The use of the material of that article is done with kind permission of Springer Science and Business Media. The rest of the chapter is organized as follows. In Section 2, notation and basic statements used in the chapter are given. Section 3 explains how to compute so called minimal singular T-invariants of the complemented Petri net. In Section 4, a relation graph of T-invariants is introduced. Section 5 describes realization of T-invariants with borrowing of tokens. In Section 6, a scheme for linear combining of T-invariants is given. Section 7 illustrates the scheme by two examples. The most important points in sections are put down as proven statements. Some of the proofs are just skeletons or, for simple statements, omitted altogether.

2. Notation and basic statements

We adopt here the notation and basic statements from (Kostin, 2003). It is assumed without losing generality that Petri nets are pure, i.e. they have no self-loops. As was stated in the previous section, the structure of any pure Petri net is unambiguously represented by the incidence matrix (1).

Let $M^0$ be an initial marking and $M$ be some other marking of given Petri net $PN$. If we are interested in reachability of $M$ from $M^0$ then marking $M$ will be called the target marking. It is assumed, throughout the chapter, that $M^0 \neq M$.

If marking $M$ is reachable from marking $M^0$ in a Petri net $PN$, then there exists at least one sequence of markings $\mu = M^0 M^1 \ldots M^r$ with $M^r = M$, and a legal firing sequence $\tau = t_{i_1}t_{i_2} \ldots t_{i_r}$, with the two sequences related by the state equation

$$M^k = M^{k-1} + e[i_k]_m D,$$

$k = 1, 2, \ldots , r$. Here $e[i_k]_m$ is a $(1 \times m)$ control vector, in which $m - 1$ entries are zero and the $i$th entry is one, indicating that a transition $t_{i_k}$ fires at step $k$. Sequences $\mu$ and $\tau$ can be combined in one mixed sequence of interrelated markings and firing transitions that is called a reachability path from marking $M^0$ to marking $M^r$:

$$M^0 \xrightarrow{t_{i_1}} M^2 \xrightarrow{t_{i_2}} \ldots \xrightarrow{t_{i_r}} M^r.$$

Its determination is the main problem of reachability analysis. As was stated in Section 1, with linear algebra methods, this analysis is usually carried out in two stages. At the first
stage, it is important to limit the number of firing count vectors, without the loss of reachability information. In the proposed approach, this stage is done with the use of T-invariants of so called complemented Petri net which is a simple extension of the original net.

**Definition 1.** For any Petri net $PN$ with incidence matrix $D$ specified by (1), and initial and target markings $M^0$ and $M$ represented by vectors (3), there exists a unique complemented Petri net $PN_c$ that has the same set of places $P$ as $PN$, the set of transitions $T_c = T \cup \{t_{m+1}\}$, and is described structurally by the incidence matrix

$$D_c = \begin{bmatrix} D \\ \Delta M \end{bmatrix},$$

where $t_{m+1}$ is an additional, complementary transition, and $\Delta M = M^0 - M = [\Delta m_1, \Delta m_2, \ldots, \Delta m_n]$, with $\Delta m_i = m^0_i - m_i$, $i = 1, 2, \ldots, n$ (Kostin, 2003).

Using the right side of equation (2) with marking $M$ instead of $M^0$, control vector $e[m + 1]_{m+1}$ instead of $F$ and incidence matrix $D_c$ instead of $D$, one can obtain

$$M + e[m + 1]_{m+1} D_c = M + \Delta M = M^0 .$$

That is, a single firing of the complementary transition in marking $M$ of $PN_c$ results in marking $M^0$.

It is known that the reproducibility of a firing sequence in a Petri net indicates the existence of a T-invariant (Memmi & Roucairol, 1980). Thus the following statement holds.

**Statement 1.** Given a Petri net $PN$ with an initial marking $M^0$, a necessary condition for reachability of some other marking $M$ is the existence of a T-invariant of the complemented Petri net $PN_c$, with a single firing of the complementary transition.

Denote by $F_c = [f_1, f_2, \ldots, f_m, f_{m+1}]$ a firing count vector of the complemented Petri net $PN_c$. Now Statement 1 may be reformulated as follows: given a Petri net $PN$ with the incidence matrix $D$ and an initial marking $M^0$, a necessary (but generally not sufficient) condition for some other marking $M$ to be reachable from $M^0$ is the existence of an integer solution of the matrix equation

$$F_c D_c = 0$$

relative to $F_c$ such that $F_c \geq 0$ and $f_{m+1} = 1$. Here $D_c$ is the incidence matrix of $PN_c$ as defined by (5).

In sequel, each T-invariant of the complemented Petri net $PN_c$ having the last entry $f_{m+1} = 1$ will be called a singular complementary T-invariant.

The importance of Statement 1 is that the reachability analysis of the original Petri net $PN$ can be reduced to the computation and investigation of T-invariants of the complemented Petri net $PN_c$. One advantage of this reduction is the existence of efficient techniques for the calculation of T-invariants (Alaiwan, 1985; Krukeberg & Jaxy, 1987; Silva & Colom, 1991; Takano et al., 2001). Algorithms for the calculation of T-invariants are implemented in many Petri net software tools such as INA (Roch & Starke, 2001); GreatSPN (Chiola et al., 1995), TimeNET (German et al., 1995), and QPN (Bause & Kemper, 1994), to mention only a few.
It is known that, in any Petri net with T-invariants, there are \textit{minimal-support T-invariants} which can be used as generators of all T-invariants of the given net (Memmi & Roucairol, 1980; Murata, 1989). Let \( \Phi = \{ F_1, F_2, \ldots, F_s \} \) be the set of minimal-support T-invariants of some Petri net consisting of \( m = |T| \) transitions, where \( F_i = [f_{i1}, f_{i2}, \ldots, f_{im}] > 0 \), and \( s \) is the number of minimal-support T-invariants. We use here, for a vector \( X \), a denotation \( X \neq 0 \) if \( X \geq 0 \) and \( x_i \neq 0 \) for some \( i \)-th entry of \( X \). Each \( F_i \in \Phi \) specifies a nonempty subset of transitions \( \| F_i \| \subseteq T \) such that \( t_j \in \| F_i \| \) if and only if \( f_{ij} > 0 \), with \( \| F_i \| \neq \| F_k \| \) and \( \| F_k \| \neq \| F_i \| \) for every pair of distinct indices \( i, k = 1, 2, \ldots, s \). Here \( \| F_i \| \) represents the minimal support of T-invariant \( F_i \).

Statement 2. In any Petri net the number of minimal-support T-invariants is finite (Kostin, 2003).

Statement 3. For any Petri net \( PN \), its complemented net \( PN_c \) includes all T-invariants of \( PN \) (Kostin, 2003).

Statement 4. For every reachability path from an initial marking \( M^0 \) to a target marking \( M \) of a given Petri net \( PN \), there exists a T-invariant \( F = [f_1, f_2, \ldots, f_m, f_{m+1}] \) of the corresponding complemented Petri net \( PN_c \) with \( f_{m+1} = 1 \). That is, \( F \) is a singular complementary T-invariant of \( PN_c \).

Let

\[
M^0 \xrightarrow{t_1} M^1 \xrightarrow{t_2} \cdots \xrightarrow{t_k} M^k = M
\]  

be some reachability path from \( M^0 \) to \( M \) in given Petri net \( PN \), such that \( M^i \neq M \) and \( t_{i_j} \neq t_{m+1} \) for \( j = 1, 2, \ldots, k-1 \). Using this path, create an expanded reachability path

\[
M^0 \xrightarrow{t_1} M^1 \xrightarrow{t_2} \cdots \xrightarrow{t_k} M^k \xrightarrow{t_{k+1}} M^{k+1} = M^0.
\]  

Since \( M^k = M \), marking \( M^k \) can be transformed, according to (6), into marking \( M^0 \) by a single firing of the complementary transition \( t_{k+1} = t_{m+1} \). Consider now the firing count vector corresponding to the reachability path (9):

\[
F = [f_1, f_2, \ldots, f_m, f_{m+1}]
\]

where \( f_i \) is the number of times transition \( t_i \) appears in the sequence \( t_1t_2 \ldots t_{k+1} \), with \( f_{m+1} = 1 \). Since, in the reachability path (9), initial marking \( M^0 \) is transformed back into \( M^0 \), the corresponding firing count vector (10) is a T-invariant. Further, since the last entry in this vector \( f_{m+1} = 1 \), the vector is a singular complementary T-invariant of the complemented Petri net \( PN_c \).

Note that the reverse of Statement 4 is generally not true. That is, the existence of a singular complementary T-invariant does not guarantee that there exists a corresponding reachability path.

Corollary 1. For any Petri net, with given initial and target markings \( M^0 \) and \( M \) respectively, all existing reachability paths from \( M^0 \) to \( M \) are the paths that can be created on the set of singular complementary T-invariants. This corollary is a generalization of the corresponding result for T-invariant-less Petri nets obtained in (Kostin, 2003). It means that,
to perform reachability analysis of a Petri net, it is sufficient to search for reachability paths only on the set of singular complementary T-invariants. ♦

The set implied by Corollary 1 is infinite in general and includes singular minimal-support complementary T-invariants and all linear combinations of minimal-support T-invariants that yield the last entry \( f_{m+1} = 1 \). As will be shown, it is sufficient to consider in this set, without losing reachability information, only a finite subset.

Let

\[
\Phi_c = \{F_1, F_2, \ldots, F_w\}
\]

be a set of all minimal-support T-invariants of \( PN_c \), where

\[
F_j = [f_{j1}, f_{j2}, \ldots, f_{jm}, f_{jm+1}] \geq 0,
\]

with \( j = 1, 2, \ldots, w \). Notice that, according to the basic property of a T-invariant, each entry in vector \( F_j \) may be only a nonnegative integer (Memmi & Roucairol, 1980).

Now, depending on the value of the last entry, the minimal-support T-invariants of set \( \Phi_c \) can be classified into the following three disjoint groups:

\[
\{F_j \mid f_{jm+1} = 0, \ j \in I_w\}, \quad (13)
\]

\[
\{F_j \mid f_{jm+1} = 1, \ j \in I_w\}, \quad (14)
\]

\[
\{F_j \mid f_{jm+1} > 1, \ j \in I_w\}. \quad (15)
\]

where \( I_w = \{1, 2, \ldots, w\} \) is the indexing set of \( \Phi_c \). According to Statement 2, each of these groups is finite. Depending on the Petri net and its initial and target markings, some or even all these three groups can be empty.

Without the last, \((m+1)\)th entry, T-invariants of group (13), by Statement 3, are minimal-support T-invariants of the original Petri net \( PN \). We will call members of group (13) non-complementary minimal-support T-invariants of the complemented Petri net \( PN_c \). Group (14) consists of singular complementary T-invariants. Finally, members of group (15) are nonsingular complementary T-invariants in which the complementary transition fires more than once. Together, members of groups (14) and (15) are called minimal-support complementary T-invariants of \( PN_c \).

3. Computing minimal singular T-invariants of a complemented Petri net

By Corollary 1, the search for all reachability paths from initial marking \( M^0 \) to target marking \( M \) in a given Petri net can be carried out only on singular T-invariants of the corresponding complemented Petri net. These include, first of all, minimal-support T-invariants of group (14). However, these are not the only singular T-invariants of the complemented Petri net. Indeed, linear combinations of minimal-support T-invariants of groups (13), (14), and (15) can yield additional singular T-invariants. The number of such combinations is infinite in general. In this section, we will show that there exists a finite set of minimal singular T-invariants of the complemented Petri net. Then an approach to the computation of such a set will be described. In Section 6, it will be shown how the
computed minimal singular T-invariants can be combined with non-complementary T-invariants of group (13) to produce new, non-minimal singular T-invariants.

Consider a linearly-combined T-invariant

\[ F = [f_1, f_2, \ldots, f_m, f_{m+1}] = \sum_{j=1}^{w} k_j F_j \]  

with rational coefficients \( k_j \), where \( F_j \) are minimal-support T-invariants of groups (13), (14) and (15), and \( w \) is the number of elements in the three groups. In agreement with Corollary 1, we are looking only for those combined T-invariants \( F \) which yield \( f_{m+1} = 1 \). Thus, the following constraint must hold for each linear combination \( F \) in (16):

\[ f_{m+1} = \sum_{j=1}^{w} k_j f_{j,m+1} = 1. \]  

With \( k_j \geq 0 \), the product \( k_j F_j \) in (16) can be considered as a contribution of firings of transitions of T-invariant \( F_j \) to firings of transitions of the combined T-invariant \( F \). On the other hand, a negative coefficient \( k_j \) in (16) may be interpreted as a reverse, or backward firing of transitions, corresponding to T-invariant \( F_j \), and this is not legal in the normal semantics of Petri nets. Thus, for T-invariants of groups (14) and (15), taking into account (17), their coefficients \( k_j \) must be in the following range:

\[ 0 \leq k_j \leq 1. \]  

That is, for groups (14) and (15), in which \( f_{j,m+1} \geq 1 \), to satisfy (17) the following inequality must hold:

\[ \sum k_j \leq 1. \]  

However, coefficients \( k_j \) for T-invariants of group (13) in (16) may have arbitrary (non-negative) values without affecting the constraint (17). As a particular case, these T-invariants can be combined in (16) with coefficients \( k_j \leq 1 \). The case when T-invariants of group (13) can be included into combination (16) with arbitrary large coefficients is considered in Section 6.

The linearly-combined T-invariants (16), with the constraints (17), (18) and (19), are called minimal singular T-invariants of the complemented Petri net. As a subset, they include all minimal-support T-invariants of group (14).

Minimal singular T-invariants of the complemented Petri net can be computed in the following way. Rewrite (16) as a system of linear algebraic equations

\[ \Psi K^T = F^T, \]  

where \( \Psi \) is a matrix of size \((m + 1) \times w\) whose columns are transposed minimal-support T-invariants \( F_j \) from groups (13), (14) and (15), \( K = [k_1, k_2, \ldots, k_w] \), and \( F \) is vector (16), with \( f_{m+1} = 1 \).

In the system (20), not only coefficient vector \( K \), but also entries \( f_i \) of \( F \), for \( i = 1, 2, \ldots, m \), are not known. We will show, however, that the number of different integer-valued vectors \( F \) with \( f_{m+1} = 1 \) is finite. Then we will explain how to compute the valid vectors in (20). The word "valid" means here that, in addition to the requirement \( f_{m+1} = 1 \), all coefficients \( k_j \) in

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(16) satisfy the constrains (18) and (19). Taking into account (17) and (18), one can deduce that

\[ 0 \leq f_i \leq \max_j (f_{ji}), \quad j = 1, 2, \ldots, \ w; \quad i = 1, 2, \ldots, \ m, \]  

(21)

where entries \( f_i \) are integer-valued components of vector \( F \) in (16).

One can see now that the number of different integer-valued vectors \( F \) in the system (20) is

\[ N = \prod_{i=1}^{m} [ \max_j (f_{ji}) + 1]. \]  

(22)

This number includes one vector \( F \) with all zero entries except the last one, and all minimal-support \( T \)-invariants of group (14). Among the remaining vectors \( F \), there can be additional singular \( T \)-invariants. They can be computed in the following way.

Assume that, in the system (20), \( K \) is a vector of unknowns. Then \( \Psi \) can be considered as a coefficient matrix, so that the augmented matrix of the system (20) is \( U = \Psi \mid FT \). It is known that, by elementary row operations, each matrix can be transformed to an upper trapezoidal form (Goldberg, 1991). In particular, for the augmented matrix \( U \) the result of its transformation \( U^* \) can be written as follows:

\[
U^* = \begin{bmatrix}
* & * & * & * & \ldots & * & y_1 \\
0 & * & * & * & \ldots & * & y_2 \\
0 & 0 & * & * & \ldots & * & y_3 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & y_m \\
0 & 0 & 0 & 0 & \ldots & 0 & y_{m+1}
\end{bmatrix},
\]  

(23)

where the symbol \( * \) stands for some value (this value is not zero if the symbol is the first in the row), the symbol \( ° \) is a place holder, and \( y_i = y_i(f_1, f_2, \ldots, f_m, f_{m+1}) \) is some linear function of its arguments, \( i = 1, 2, \ldots, m+1 \). Each row in \( U^* \) consists of \( w + 1 \) elements.

For the system (20) to be consistent, the following equation must hold for each \( i \)th row of matrix \( U^* \) with all \( w \) leading elements equal to zero (Goldberg, 1991):

\[ y_i(f_1, f_2, \ldots, f_m, f_{m+1}) = 0. \]  

(24)

Collecting now all equations (24), we obtain a derived system of linear algebraic equations

\[
\begin{align*}
y_{j_1}(f_1, f_2, \ldots, f_m, f_{m+1}) &= 0 \\
y_{j_2}(f_1, f_2, \ldots, f_m, f_{m+1}) &= 0 \\
&\quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
y_{j_k}(f_1, f_2, \ldots, f_m, f_{m+1}) &= 0
\end{align*}
\]  

(25)

where \( k \leq m \) and \( f_{m+1} = 1 \).
Integer solutions of this system relative to $f_1, f_2, \ldots, f_m$ can be found using existing algorithms for integer systems of linear equations (Howell, 1971; Springer, 1986). With the constraints (21), the system has a finite number of solutions or no solutions at all. Note that, with nonempty group (14), for all its members $[f_{j1}, f_{j2}, \ldots, f_{jm}, 1]$, the system (25) has solutions at least for the trivial linear combinations

$$F = [f_1, f_2, \ldots, f_m, 1] = [f_{j1}, f_{j2}, \ldots, f_{jm}, 1],$$

(26)

since each vector (26) is the solution of (20), for which vector $K$ has some entry $k_j = 1$, with all other coefficient entries equal to zero.

To illustrate this method, consider a Petri net of 6 transitions and 6 places having the incidence matrix

$$D = \begin{bmatrix}
-1 & 1 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & 1 \\
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}$$

with the initial and target markings $M^0 = [2, 0, 0, 0, 0, 0]$ and $M = [0, 0, 0, 0, 0, 2]$, respectively. The corresponding complemented Petri net has two minimal-support T-invariants $F_1 = [0, 0, 2, 2, 2, 0, 1]$ and $F_2 = [2, 2, 0, 0, 0, 2, 1]$. Both are singular T-invariants (that is, they have $f_{m+1} = f_7 = 1$). We will try to determine whether there are some other minimal singular T-invariants. For this example, with $w = 2$, the augmented matrix of the system (20) and its upper trapezoidal form are

$$\begin{bmatrix}
0 & 2 & f_1 \\
0 & 2 & f_2 \\
2 & 0 & f_3 \\
2 & 0 & f_4 \\
2 & 0 & f_5 \\
0 & 2 & f_6 \\
1 & 1 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 1 \\
0 & 2 & f_1 \\
0 & 0 & f_1 - f_2 \\
0 & 0 & f_3 - f_4 \\
0 & 0 & f_4 - f_5 \\
0 & 0 & f_1 - f_6 \\
0 & 0 & f_1 + f_3 - 2
\end{bmatrix}.$$  

Thus, the system (25) is

$$f_1 - f_2 = 0$$
$$f_3 - f_4 = 0$$
$$f_4 - f_5 = 0$$
$$f_1 - f_6 = 0$$
$$f_1 + f_3 = 2$$
With the constraints $0 \leq f_1, f_2, f_3, f_4, f_5, f_6 \leq 2$, this system has the following three nonnegative integer solutions: $[0, 0, 2, 2, 2, 0, 1]$, $[2, 2, 0, 0, 0, 2, 1]$ and $[1, 1, 1, 1, 1, 1, 1]$. Clearly, the fist two solutions are minimal-support T-invariants $F_1$ and $F_2$, and the third solution is a minimal singular T-invariant that is the linear combination $F_3 = 0.5F_1 + 0.5F_2$. Neither $F_1$ nor $F_2$ are realizable in given initial marking. However, their linearly combined T-invariant $F_3$ is realizable. One legal firing sequence is $t_3 \ t_1 \ t_2 \ t_4 \ t_5 \ t_6 \ t_7$.

4. Relation graph of T-invariants

In general, each singular T-invariant should be tested for the creation of a reachability path (or a legal firing sequence) not only alone, but also in different linear combinations with non-complementary T-invariants (13), since these T-invariants can “help” the singular T-invariant to become realizable in given initial marking $M_0$ and to eventually provide a reachability path from $M_0$ to a target marking $M$. As will be shown in this section, in general not all non-complementary T-invariants can affect realization of the given singular T-invariant.

**Definition 2.** Let $F$ be a T-invariant of a Petri net, with the support $\| F \|$. Then

$$P(F) = \{ p_i \mid t_i \in \| F \|, d_{ij} \neq 0 \}$$

(27)

is a set of places of this Petri net affected by $F$ when it becomes realizable in some marking. Here, $d_{ij}$ is an element of the incidence matrix of the Petri net as specified by (1). ♦

**Statement 5.** Let $F_1$ and $F_2$ be some T-invariants of a Petri net, and let $P_1$ and $P_2$ be sets of places affected by $F_1$ and $F_2$ respectively. If $P_1 \cap P_2 = \emptyset$, then T-invariants $F_1$ and $F_2$ have no direct effect on the realizability of each other.

Assume that, contrary to the statement, $F_1$ can directly affect the realizability of $F_2$. This is possible only if $F_1$, during its realization, will change the number of tokens in some places affected by $F_2$. This can happen only if $P_1 \cap P_2 \neq \emptyset$. The contradiction proves the statement. ♦

Even if $P_1 \cap P_2 = \emptyset$, T-invariants $F_1$ and $F_2$ can indirectly affect the realizability of each other through other T-invariants having common affected places with $F_1$ and $F_2$.

**Corollary 2.** Let $F^{1}_{nc}, F^{2}_{nc}, \ldots, F^{k}_{nc}$ be some non-complementary T-invariants of a complemented Petri net, with sets of places $P^{1}_{nc}, P^{2}_{nc}, \ldots, P^{k}_{nc}$ affected by these T-invariants, respectively. Let further $F_c$ be a singular complementary T-invariant of this Petri net, with the set of affected places $P_c$. Denote by $P_{nc} = \bigcup_{i=1}^{k} P^{i}_{nc}$ a set of places of this Petri net affected by mentioned non-complementary T-invariants.

If $P_c \cap P_{nc} = \emptyset$, then realization of any linear combination of T-invariants $F^{1}_{nc}, F^{2}_{nc}, \ldots, F^{k}_{nc}$ has no effect on realization of $F_c$. Therefore these T-invariants may be excluded from consideration in the reachability analysis with T-invariant $F_c$ in given Petri net. ♦

To represent formally the effects of different T-invariants on each other in a Petri net, it is instructive to introduce into consideration a relation graph of T-invariants. Nodes in this graph are T-invariants. Two nodes corresponding to T-invariants $F_i$ and $F_j$ are connected by a non-oriented edge if $P(F_i) \cap P(F_j) \neq \emptyset$, and the corresponding T-invariants $F_i$ and $F_j$ are called directly connected T-invariants.
For a Petri net, such a graph generally consists of a number of connected components. A connected component may include complementary and non-complementary T-invariants, or only one type of T-invariants. We say that two T-invariants $F_i$ and $F_j$ can affect realizability of each other if they belong to the same connected component, even if $P(F_i) \cap P(F_j) = \emptyset$. On the other hand, if $F_i$ and $F_j$ belong to different connected components, they can not affect each other in no way, directly or indirectly.

The algorithm for determining all connected components of a graph is well known (Goodrich, 2002). In our problem, the algorithm will determine a connected component consisting of nodes representing a given singular T-invariant and non-complementary T-invariants. For this purpose, the algorithm will use the incidence matrix of the original Petri net and the array of T-invariants.

5. Realization of T-invariants with borrowing of tokens

In this section, the meaning of the help provided by one T-invariant to another one to become realizable is explained. Let $p$ be a place affected by two T-invariants $F_i$ and $F_j$ in a given Petri net. Assume that, in a given initial marking of the net, $F_i$ is realizable, but $F_j$ can become realizable if place $p$ accumulates $r_j$ tokens during realization of T-invariant $F_j$. Suppose further that, at some intermediate step during realization of $F_i$, $r_i$ tokens will be created in place $p$. If $r_i \geq r_j$ then, by temporary borrowing of $r_j$ tokens in place $p$, T-invariant $F_j$ becomes realizable and, at the end of its realization, will return the borrowed tokens to place $p$, so that T-invariant $F_i$ can complete its started realization.

With $r_i < r_j$, T-invariant $F_j$ cannot borrow the necessary number of tokens in place $p$. However, if T-invariant $F_j$, after creation of $r_i$ tokens in $p$ at some step of its first realization, can start a new realization before the completion of the first one, then additional $r_i$ tokens will be created in place $p$, so that this place will now accumulate $2r_i$ tokens. In general, if $F_i$ can start $z$ realizations before the completion of the previous ones, then place $p$ will accumulate $zr_i$ tokens. If, for some $z$, $zr_i \geq r_j$ then, after borrowing $r_j$ tokens in $p$, T-invariant $F_j$ becomes realizable. After the completion of its realization, all tokens borrowed by $F_j$ will be returned to place $p$, and T-invariant $F_i$ can complete all its started realizations.

![Fig. 1. Illustration of borrowing of tokens by a T-invariant.](www.intechopen.com)
becomes realizable only if it can borrow two tokens in place $p_2$, affected by the both T-invariants. These two tokens will be created here after T-invariant $F_1$ starts two realizations by firing transition $t_1$ two times. Afterwards, $F_2$ becomes realizable by borrowing two tokens in $p_2$. Then, after firing $t_3$ and $t_4$, the borrowed tokens reappear in $p_2$, and $F_1$ can complete its two started realizations. The corresponding sequence of transition firings for this example is $t_1t_1t_3t_4t_2t_2$.

To represent the relationship between connected T-invariants, when some non-realizable T-invariants can become realizable in given initial marking of a Petri net by borrowing tokens in places affected by other T-invariants, we will introduce a two-dimensional borrowing matrix $G$. In this matrix, rows correspond to T-invariants and columns correspond to places of the given Petri net. Formally, for a group of connected T-invariants,

$$ G = [g_{ij}], \quad i = 1, 2, \ldots, s; \quad j = 1, 2, \ldots, n, \quad (28) $$

where $s$ is the number of connected T-invariants in the group and $n$ is the number of places in the net. The elements of matrix $G$ are integers and have the following meaning. If $g_{ij} > 0$ then, for its realization, T-invariant $F_i$ needs to borrow $g_{ij}$ tokens in place $p_j$ affected by some other T-invariant of the considered group. If $g_{ij} < 0$ then T-invariant $F_i$, at some intermediate step of its single realization, creates $|g_{ij}|$ tokens in place $p_j$. Finally, $g_{ij} = 0$ means that $F_i$ does not affect place $p_j$.

As an example, matrix $G$ for minimal-support T-invariants of the Petri net shown in Fig. 1 is:

<table>
<thead>
<tr>
<th></th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$F_2$</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

One can see from this matrix that the number of tokens created in place $p_2$ during a single realization of $F_1$ is 1 and is not sufficient for $F_2$ to borrow two tokens. In this example borrowing is possible if T-invariant $F_1$ starts two interleaved realizations. The maximal number of realizations that can be started by $F_1$ depends on the initial marking of place $p_1$. In particular, if this place initially contains only one token, then $F_1$ is still realizable, but it will never create, during its realizations, more than one token in $p_2$.

For a group of connected T-invariants of a complemented Petri net, the borrowing matrix can be created with the use of the incidence matrix of the given original Petri net. Due to a relative simplicity of the underlying procedure and to space limitation, the details of this procedure are omitted.

### 6. Combining a singular complementary T-invariant with non-complementary T-invariants

Denote by $F_c$ a singular T-invariant of some complemented Petri net. It can be a member of group (14) or a minimal T-invariant calculated as was described in Section 3. Clearly, if group (13) is not empty, then the following linear combination

$$ F = F_c + \sum k_j F_{nc}^j, \quad (29) $$

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with coefficients $k_j \geq 0$, is also a singular T-invariant, if components of $F$ are nonnegative integers. Here $F_{nc}^{j}$ is a T-invariant of group (13). According to Corollary 2, it is sufficient to include in (29) only those T-invariants from (13) that belong to the same group of connected T-invariants together with $F_c$.

The expression (29) implies that the singular T-invariant $F_c$ in general should be tested for the determination of a reachability path not only alone, but also in different linear combinations with non-complementary T-invariants (13), since these T-invariants can “help” the non-realizable T-invariant $F_c$ to become realizable in a given initial marking $M^0$ and to eventually provide a reachability path from $M^0$ to a target marking $M$ of the given Petri net.

Without losing generality, we assume that coefficients $k_j$ in (29) are nonnegative integers. Indeed, if a singular T-invariant $F_c$ is realizable with some non-integer values of coefficients $k_j$ in (29), then it will remain realizable when these coefficient values are replaced by the nearest integer values not less than $k_j$. The case when $k_j \leq 1$ was considered in Section 3.

With integer coefficients $k_j > 1$, the product $k_j F_{nc}^{j}$ in (29) corresponds to a multiple realization of T-invariant $F_{nc}^{j}$. A multiple realization is a series of $k_j$ sequential or interleaved single realizations. Interleaved realizations of a T-invariant, if they are possible, can have a different effect on place marking in comparison with sequential realizations. Consider, for example, a simple Petri net consisting of two transitions $t_1, t_2$ and one place $p$ that is the output place for $t_1$ and the input place for $t_2$. This Petri net has a T-invariant $F = [1, 1]$ realizable in any initial marking of $p$. In particular, with the zero initial marking, place $p$ will never have more than one token if single realizations of $F$ are strictly sequential as in $t_1 t_2 t_1 t_2 t_1$. However, if single realizations of $F$ are interleaved, place $p$ can accumulate an arbitrary large number of tokens at some intermediate step.

In general, the number of valid combinations (29) is infinite. This section describes how to limit the values of coefficients $k_j$ in (29) without the loss of reachability information using the concept of structural boundedness of Petri nets.

It is known (Murata, 1989) that a Petri net is structurally bounded if and only if there exists a $(1 \times n)$ vector $Y = [y_1, y_2, ..., y_n]$ of positive integers, such that

$$D \cdot Y \leq 0,$$  \hspace{1cm} (30)

where $D$ is the $(m \times n)$ incidence matrix of the Petri net with $m$ transitions and $n$ places.

A Petri net is said to be not structurally bounded if and only if there exists a $(1 \times m)$ vector of (nonnegative) integers $X = [x_1, x_2, ..., x_m] \geq 0$, such that

$$D^T X^T = \Delta M^T$$  \hspace{1cm} (31)

for some $\Delta M \geq 0$, where $m$ is the number of transitions in the Petri net, and $\Delta M$ is a $(1 \times n)$ vector of marking increments as a result of firing of all transitions corresponding to vector $X$.  

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In a structurally unbounded Petri net, at least one place is structurally unbounded. A place \( p_i \) in such a Petri net is said to be structurally unbounded if and only if there exists a \((1 \times m)\) vector \( X > 0 \) of nonnegative integers, such that

\[
D^T X^T = \Delta M^T
\]

(32)

for some \( \Delta m_i > 0 \) in \( \Delta M = (\Delta m_1, \Delta m_2, ..., \Delta m_i, ..., \Delta m_n) \neq 0 \).

The structural unboundedness can be tested separately for each place \( p_i \) of the Petri net, by setting an appropriate integer \( \Delta m_i > 0 \) and \( \Delta m_j = 0 \) for all \( j \neq i \) in (32) and then trying to solve the system (32). The test may be done also simultaneously for a few desired places or even for all places of the net.

It is known that, according to Minkowski-Farkas' lemma (Kuhn & Tucker, 1956), one of the systems (30) or (31) has solutions. For our problem, we do not need to know all solutions of (30) or (31). Rather, it is sufficient to find only one, "minimal" solution of (30) or (31). The minimal solutions of (30) or (31) can be found as solutions of integer linear programming (ILP) problems. For the system (30), the corresponding ILP problem can be formulated as follows:

\[
\text{minimize } a = \sum_{i=1}^{n} y_i, \\
\text{subject to: } DY^T \leq 0, \quad y_i \geq 1, \quad i = 1, 2,...,n.
\]

(33)

For the system (31), the corresponding ILP problem is:

\[
\text{minimize } b = \sum_{i=1}^{m} x_i, \\
\text{subject to: } D^T X^T > 0, \quad \sum_{i=1}^{m} x_i \geq 1, \quad x_i \geq 0, \quad i = 1, 2,...,m.
\]

(34)

The property of structural boundedness can be considered also for subnets of a Petri net. We are interested in this property only for the subnets corresponding to non-complementary T-invariants \( F_{nc}^j \) in (29). For a non-complementary T-invariant \( F_{nc}^j \), the related subnet consists of transitions of the support \( \| F_{nc}^j \| \) and places \( P(F_{nc}^j) \) affected by \( F_{nc}^j \). The expressions (30) - (34) remain valid for the subnet corresponding to \( F_{nc}^j \) with the following restrictions: in the incidence matrix \( D \) rows are taken for transitions corresponding to nonzero entries in \( F_{nc}^j \), and columns are taken for places affected by \( F_{nc}^j \).

Let us consider initially the case when the subnet corresponding to \( F_{nc}^j \) is not structurally bounded and describe how to determine coefficients \( k_j \) for non-complementary T-invariants \( F_{nc}^j \) in the linear combination (29). If \( F_{nc}^j \) and \( F_c \) belong to different connected
components of the graph of relation of T-invariants then \( F_{nc}^j \) should be ignored at all, by setting \( k_j = 0 \) in (29).

If \( F_{nc}^j \) and \( F_c^j \) belong to the same connected component of the graph of relation of T-invariants then the subnet corresponding to \( F_{nc}^j \) will have common places with the subnets corresponding to \( F_c \) or other non-complementary T-invariants belonging to the same connected component. Thus, \( F_{nc}^j \) can affect realizability of \( F_c^j \), directly or indirectly, and therefore should be included in (29) with \( k_j > 0 \).

Suppose for definiteness that T-invariant \( F_{nc}^j \) has the support \( \{ t_1, t_2, \ldots, t_l \} \), \( l \leq m \), and the set of affected places

\[
\{ p_1, p_2, \ldots, p_q \}, \quad q \leq n, \tag{35}
\]

where \( m \) and \( n \) are the numbers of transitions and places in the original (non-complemented) Petri net. Assume that \( F_c \), to become realizable, needs to borrow \( n_i > 0, i = 1, 2, \ldots, h \), tokens at least in places

\[
\{ p_1, p_2, \ldots, p_h \}, \quad h \leq q, \tag{36}
\]

that belong to the set (35) and in which \( F_{nc}^j \) can create tokens during its realization. Then, to facilitate the realizability of \( F_c^j \), \( F_{nc}^j \) should be included in the linear combination (29) with a positive integer coefficient \( k_j \) determined by applying the following steps.

1. Try to solve an ILP problem:

\[
\text{minimize} \quad b = \sum_{i=1}^l x_i, \tag{37}
\]

subject to: \( D^T X^T \geq \Delta M^T \), \( \sum_{i=1}^l x_i \geq 1 \), \( x_i \geq 0 \),

where \( \Delta M = [\Delta m_1, \Delta m_2, \ldots, \Delta m_l, \Delta m_{l+1}, \ldots, \Delta m_q] = [n_1, n_2, \ldots, n_h, 0, \ldots, 0] \) is a vector of the desired numbers of tokens which are expected to be created in places (36) as a result of one or more realizations of \( F_{nc}^j \), \( l \) is the number of transitions in the subnet corresponding to \( F_{nc}^j \), and \( q \) is the number of places affected by \( F_{nc}^j \). In the matrix multiplication, only those rows and columns of \( D \) are used which correspond to the support of \( F_{nc}^j \) and to places affected by \( F_{nc}^j \).

2. If, for the specified vector \( \Delta M \), the problem (37) has a solution \( X^* = [x_1^*, x_2^*, \ldots, x_l^*] \), then components of \( X^* \) represent the total numbers of firings of respective transitions sufficient to accumulate the desired number of tokens in places of set (36) in a few realizations of \( F_{nc}^j \), and ratio \( \frac{x_i^*}{f_i^j} \) the number of realizations of \( F_{nc}^j \) to provide the necessary number of firings of transition \( t_i \), \( i = 1, 2, \ldots, l \). In this case,
3. If, on the other hand, the problem (37) has no feasible solution then it means that at least one of places in set (36) \( p_i \) is structurally bounded and can not accumulate the desired number of tokens \( \Delta m_i \) in multiple realizations of \( F_{nc}^j \). In this case, using (32), determine all structurally unbounded places in set (36). Since, as is assumed, the subnet for \( F_{nc}^j \) is not structurally bounded, there is at least one structurally unbounded place in this subnet.

4. Solve the ILP problem (37) simultaneously for all structurally unbounded places found at the previous step, to obtain a solution vector \( X^* \). That is, in solving (37), vector \( \Delta M \) should have nonzero entries \( \Delta m_i = n_i \) only for structurally unbounded places. According to Minkowski-Farkas’ lemma, this solution always exists. Then coefficient \( k_j \) is determined by the use of expression (38).

In case, when the subnet for \( F_{nc}^j \) is found to be structurally bounded, then the number of tokens in each of its places is bounded. However, this bound generally depends on realizations of other, connected T-invariants and is not known in advance. For such a subnet, coefficient \( k_j \) can be evaluated with the use of the borrowing matrix (28) computed for \( F_c \) and all its connected non-complementary T-invariants, including \( F_{nc}^j \). Let, in this matrix, \( c \) and \( j \) be indexes of rows corresponding to \( F_{nc}^j \) and \( F_c \) respectively. Then it is sufficient to include \( F_{nc}^j \) in the linear combination (29) with coefficient \( k_j \) computed with the use of the expression

\[
k_j = \sum \left[ \frac{g_{ci}}{|g_{ji}|} \right], \quad (39)
\]

where \( g_{ci} \) and \( g_{ji} \) are entries in the borrowing matrix, and the sum is computed for all pairs \( g_{ci} > 0 \) and \( g_{ji} < 0 \). Indeed, with this coefficient, the sufficient number of interleaved realizations of \( F_{nc}^j \) are allowed to accumulate the required numbers of tokens in places which are common for \( F_c \) and \( F_{nc}^j \) and in which T-invariant \( F_c \) can borrow them during its realization.

However, the possibility of realizations of \( F_{nc}^j \) depends on marking of places in its subnet. For example, in the Petri net of Fig. 1, T-invariant \( F_2 \) can become realizable only with the help of T-invariant \( F_1 \) for which the corresponding subnet is structurally bounded. The borrowing matrix for this example has only one pair of non-zero entries \( g_{12} = -1 \) (for \( F_1 \)) and \( g_{22} = 2 \) (for \( F_2 \)). Thus, using (39), one can obtain \( k_1 = 2 \). That is, two interleaved realizations of \( F_1 \) are sufficient to create two tokens in place \( p_2 \) to make \( F_2 \) realizable. But this is possible only if place \( p_1 \) holds initially at least two tokens. If this place holds one token, \( F_1 \) is
sequentially realizable but it can never create two tokens in \( p_2 \) to facilitate the realizability of \( F_2 \).

In general, coefficient \( k_j \) calculated as was described for the two cases can result in a larger number of realizations of T-invariant \( F_{nc}^j \) than is actually necessary. The reason is that other T-invariants in (29) can also create tokens in places (36) and contribute to the realizability of \( F_c \).

After computing coefficients \( k_j \) in (29), an appropriate method can be applied to find a reachability path (or a legal firing sequence) for the combined T-invariant \( F \) if such a path exists. The task here is the following. For a Petri net with given initial and target markings \( M^0 \) and \( M \) and a combined T-invariant \( F \), find a legal firing transition sequence. To find a legal firing transition sequence, or reachability path as defined in (4), known computational techniques can be used (Kostin, 2003; Taoka et al., 2003; Watanabe, 2000; Huang & Murata, 1998).

7. Examples

This section illustrates the proposed reachability analysis scheme by two examples. The examples were tested in Windows XP OS with a prototype C program that implemented
almost all steps of the scheme, with the major exception of the sub-algorithm for solving an ILP problem. To solve this problem, the interactive system QS was used (Chang & Sullivan, 1996). For the first example, Fig. 2 shows a Petri net consisting of \( m = 10 \) transitions and \( n = 9 \) places, with its incidence matrix (recall that rows correspond to transitions), and the initial and target markings \( M^0 = [2, 0, 0, 0, 0, 0, 0, 0, 0] \) and \( M = [2, 0, 0, 0, 0, 0, 0, 0, 1] \), respectively. To get the complemented Petri net, the algorithm appends a row \( \Delta M = M^0 - M = [0, 0, 0, 0, 0, 0, 0, 0, -1] \) to the original incidence matrix. Minimal-support T-invariants of the corresponding complemented Petri net are two non-complementary T-invariants \( F_1 = [0, 0, 1, 1, 0, 1, 0, 0, 0] \) and \( F_2 = [1, 1, 0, 0, 1, 1, 0, 0, 0] \), and one singular complementary T-invariant \( F_3 = [0, 0, 0, 0, 0, 0, 1, 1, 1] \), with the sets of affected places \( \{p_1, p_3, p_4, p_5\} \), \( \{p_1, p_2, p_3, p_5, p_6\} \) and \( \{p_6, p_7, p_8, p_9\} \), respectively. Thus, all these T-invariants are connected and should be considered together. The borrowing matrix \( G \) for this example contains the following data:

\[
D = \begin{bmatrix}
-1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 \end{bmatrix}
\]

Fig. 2. Petri net of Example 1 and its incidence matrix.

Thus, each of these T-invariants can become realizable if it borrows tokens in some of common affected places. Specifically, \( F_1 \) needs to borrow two tokens in place \( p_5 \), \( F_2 \) needs to borrow one token in place \( p_5 \), and \( F_3 \) borrows two tokens in place \( p_6 \). Note that a token borrowed by \( F_2 \) in place \( p_3 \) can be produced by \( F_1 \) in a single realization. In its turn, \( F_2 \) is capable, in a single realization, to lend one token to \( F_1 \), instead of necessary two tokens. Therefore, \( F_1 \) and \( F_2 \) can help each other to become realizable. Together, they are capable to produce 2 tokens in place \( p_5 \) to be borrowed by \( F_3 \).

The desired number of tokens in \( p_5 \) can be accumulated if the subnet corresponding to \( F_2 \) is not structurally bounded. To check this, the ILP problem (33) for \( F_2 \) is solved, in the following form:

\[
\begin{array}{cccccccc}
\text{ } & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 \\
F_1 & -1 & 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\
F_2 & -1 & -1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\
F_3 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & -1 \\
\end{array}
\]
Using Transition Invariants for Reachability Analysis of Petri Nets

\[
\text{minimize } a = y_1 + y_2 + y_3 + y_5 + y_6, \\
\text{subject to: } -y_1 + y_2 - y_3 \leq 0, -y_2 + y_3 + y_5 + y_6 \leq 0, -y_5 \leq 0, y_1 - y_6 \leq 0, y_1, y_2, y_3, y_5, y_6 \geq 1.
\]

This ILP problem has no feasible solution. Thus, the subnet corresponding to \(F_2\) is not structurally bounded, so that at least one of its affected places is not structurally bounded.

We are interested in accumulating two tokens in \(p_5\), so that \(\Delta M = [0, 0, 2, 0]\). Therefore, now the ILP problem (37) should be attempted, in the following form:

\[
\text{minimize } b = x_1 + x_2 + x_6 + x_7, \\
\text{subject to: } -x_1 + x_7 \geq 0, x_1 - x_2 \geq 0, x_1 + x_2 \geq 0, x_2 - x_7 \geq 0, x_1 + x_2 + x_6 + x_7 \geq 1.
\]

This ILP problem has the optimal (minimal) solution

\[X^* = [x_1^*, x_2^*, x_6^*, x_7^*] = [2, 2, 2, 0].\]

Now, using (38), one can find that

\[k_2 = \max\left(\frac{x_i^*}{f_{2i}} \mid i = 1, 2, 6, 7\right) = 2.\]

Since T-invariant \(F_2\) borrows only one token in place \(p_3\) and this token can be created during a single realization of \(F_1\), it is sufficient to have \(k_1 = 1\). Thus, the combined T-invariant \(29\), with \(F_c = F_3\), is \(F = F_1 + 2F_2 + F_3 = [2, 2, 1, 1, 1, 2, 3, 1, 1, 1, 1]\). For this T-invariant, a legal firing sequence can be found consisting of 15 firing transitions \(t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}t_{11}t_{12}t_{13}t_{14}\) and transforming \(M^0\) into \(M\). This is the shortest sequence although there exist other sequences of the same length. Using the computed sequence, the corresponding reachability path (4) from \(M^0\) to \(M\) can be easily found.
Thus, all these T-invariants are connected. Linear combinations of \(\\{\text{complementary T-invariants}\}\) according to Section 3, yield four additional minimal singular \(p_0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\), connected non-complementary T-invariants, the borrowing matrix

\[
D = \begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2
\end{bmatrix}.
\]

Fig. 3. Petri net of Example 2 and its incidence matrix.

Fig. 3 shows the second example of a Petri net, consisting of \(m = 13\) transitions and \(n = 9\) places. With the initial and target markings

\[
M_0 = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \quad \text{and} \quad M = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1],
\]

there are seven minimal-support T-invariants in the corresponding complemented Petri net: six non-complementary T-invariants \(F_1 = [1, 1, 1, 2, 2, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0], F_2 = [2, 2, 2, 1, 1, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0], F_3 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0], F_4 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], F_5 = [0, 0, 0, 0, 3, 3, 6, 0, 1, 1, 0, 2, 0, 0, 0, 0], F_6 = [2, 2, 2, 1, 1, 0, 0, 1, 1, 2, 0, 0, 0, 0, 0, 0], \) and one singular complementary T-invariant \(F_7 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1], \) with the sets of affected places \(\{p_1, p_2, p_3, p_4, p_5\}, \{p_1, p_2, p_3, p_4, p_5\}, \{p_3, p_4, p_5\}, \{p_3, p_4, p_5\}, \{p_3, p_4, p_5\}, \{p_3, p_4, p_5\}, \) respectively.

Thus, all these T-invariants are connected. Linear combinations of \(F_7\) with non-complementary T-invariants, according to Section 3, yield four additional minimal singular T-invariants \(F_8 = [1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1], F_9 = [2, 2, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 1, 1, 1, 1], F_{10} = [0, 0, 0, 2, 4, 1, 1, 1, 0, 2, 1, 1, 1, 1] \) and \(F_{11} = [0, 0, 0, 1, 1, 2, 2, 1, 1, 0, 2, 1, 1, 1, 1]. \)

For reachability analysis, consider the singular complementary T-invariant \(F_7\). For \(F_7\) and its connected non-complementary T-invariants, the borrowing matrix \(G\) contains the following data:

\[
\begin{array}{cccccccc}
\text{ } & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 \\
F_1 & -1 & -1 & -2 & -1 & -2 & 0 & 0 & 0 & 0 \\
F_2 & -1 & -1 & -2 & -1 & -2 & 0 & 0 & 0 & 0 \\
F_3 & 0 & 0 & 3 & 0 & 0 & -1 & -2 & 0 & 0 \\
F_4 & 0 & 0 & 6 & 0 & 0 & -1 & -2 & 0 & 0 \\
F_5 & 0 & 0 & 6 & -1 & -2 & -1 & -2 & 0 & 0 \\
F_6 & -1 & -1 & -2 & -1 & -2 & -1 & -2 & 0 & 0 \\
F_7 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & -1
\end{array}
\]
Thus, T-invariant $F_7$ can become realizable if only it borrows tokens. Specifically, $F_7$ needs to borrow one token in place $p_5$ and two tokens in place $p_7$. The necessary number of tokens in the both places can be produced by realizable T-invariant $F_6$ alone. Indeed, $F_6$ creates, in a single realization, two tokens in place $p_5$ and two tokens in place $p_7$. However, at this point we cannot say that there exist a state of the Petri net in which places $p_5$ and $p_7$ hold at least one and two tokens, respectively. To learn this possibility, it is necessary initially to test the structural boundedness of the subnet corresponding to $F_6$, by attempting to solve the ILP problem (33), in the following form:

$$\text{minimize } a = y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7,$$

subject to:

$$y_1 - y_5 \leq 0, -y_1 + y_2 + y_3 \leq 0, -y_2 + y_3 \leq 0, -y_3 + y_4 + y_5 \leq 0, -y_4 + y_5 \leq 0,$$

$$-3y_3 + y_6 + y_7 \leq 0, -y_6 + y_7 \leq 0, -y_7 \leq 0, y_1, y_2, y_3, y_4, y_5, y_6, y_7 \geq 1.$$ 

This ILP problem has no feasible solution. Thus, the subnet corresponding to $F_6$ is not structurally bounded, so that at least one of its affected places is not structurally bounded. We are interested in having at least one token in $p_5$ and at least two tokens in $p_7$, so that $\Delta M = [0, 0, 0, 0, 1, 0, 2]$. Therefore, now it is necessary to try to solve the ILP problem (37), in the following form:

$$\text{minimize } b = x_1 + x_2 + x_3 + x_4 + x_5 + x_8 + x_9 + x_{10},$$

subject to:

$$x_1 - x_2 \geq 0, x_2 - x_3 \geq 0, x_2 + x_3 - x_4 - 3x_8 \geq 0, x_4 - x_5 \geq 0, -x_1 + x_4 + x_5 \geq 1,$$

$$x_8 - x_9 \geq 0, x_8 + x_9 - x_{10} \geq 2, x_1 + x_2 + x_3 + x_4 + x_5 + x_8 + x_9 + x_{10} \geq 1.$$ 

This ILP problem has the optimal (minimal) solution

$$X^* = [x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_8^*, x_9^*, x_{10}^*] = [3, 3, 2, 2, 1, 1, 0].$$

Now, using (38), we can find that

$$k_6 = \max\left(\frac{x_i^*}{f_{6i}}\right) \mid i = 1, 2, \ldots, 5, 8, 9, 10 = 2.$$ 

Thus, the combined complementary T-invariant (29), with $F_c = F_7$, is $F = 2F_6 + F_7 = [4, 4, 4, 2, 2, 0, 0, 2, 2, 4, 0, 1, 1, 1]$. For $F$, a legal firing sequence can be found consisting of 26 transition firings and transforming $M^0$ into $M$. This is not the shortest sequence. The shortest sequence exists for the decremented value of coefficient $k_6 = 1$ and consists of 14 transition firings $t_2t_3t_4t_1t_2t_3t_5t_6t_7t_8t_9t_10t_11t_{10}$.

Although non-complementary T-invariants $F_1$ and $F_2$ are realizable as well, they are not appropriate to be combined with $F_7$ to create a realizable combined T-invariant since they cannot produce tokens in place $p_7$. The necessary number of tokens could be produced in $p_7$ also by $F_3$ but it needs itself to borrow six tokens in place $p_3$.

In this way, one can proceed with the remaining singular T-invariants $F_8$, $F_9$, $F_{10}$, and $F_{11}$. Calculating coefficient $k_6 = 2$ and combining each of these T-invariants with $F_6$, it will be possible to successfully find the corresponding legal firing sequences and, if necessary, reachability paths. In all cases, coefficient $k_6$ can be decremented to one, to get the shortest legal firing sequence.
This example shows that, in general, it is not necessary to compute coefficients $k_j$ for all T-invariants $F_{nc}^j$ in (29). The reachability test can be done as soon as coefficient $k_j$ is computed for the first $F_{nc}^j$. If this test fails, then coefficient $k_j$ is computed for the next $F_{nc}^j$, until the reachability test is successful or all connected non-complementary T-invariants in (29) are considered.

8. Conclusion

A new approach to reachability analysis in general Petri nets is proposed, formally described, and illustrated by examples tested with a prototype program. For a given original Petri net, the reachability analysis is reduced to the computation and investigation of T-invariants of the complemented Petri net consisting of the original Petri net and an additional, complementary transition with input and output arcs depending on the given initial and target markings. It is shown that, without the loss of reachability information, one can carry out reachability analysis using only a finite number of T-invariants.

We did not address, in this chapter, complexity aspects of the proposed approach to reachability analysis. Complexity of some problems of Petri nets, including the reachability problem, was investigated elsewhere (Jones et al., 1977). Most of the running time in the proposed reachability analysis scheme will be spent in computing minimal-support T-invariants and their linear combinations, solving ILP problems, and trying to find legal firing sequences for the computed T-invariants. This can be done with the use of existing methods (Watanabe, 2000; Yamauchi & Watanabe, 1998; Huang & Murata, 1998).

9. References


Although many other models of concurrent and distributed systems have been developed since the introduction in 1964 Petri nets are still an essential model for concurrent systems with respect to both the theory and the applications. The main attraction of Petri nets is the way in which the basic aspects of concurrent systems are captured both conceptually and mathematically. The intuitively appealing graphical notation makes Petri nets the model of choice in many applications. The natural way in which Petri nets allow one to formally capture many of the basic notions and issues of concurrent systems has contributed greatly to the development of a rich theory of concurrent systems based on Petri nets. This book brings together reputable researchers from all over the world in order to provide a comprehensive coverage of advanced and modern topics not yet reflected by other books. The book consists of 23 chapters written by 53 authors from 12 different countries.

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