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# The Wigner-Heisenberg Algebra in Quantum Mechanics

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Additional information is available at the end of the chapter

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## 1. Introduction

While the supersymmetry in quantum mechanics (SUSY QM) algebra has thus received much operator applications for potential problems [1–4], another algebra, the general Wigner-Heisenberg(WH) oscillator algebra [5–8], which already possesses an in built structure which generalises the usual oscillators ladder operators, has not, however, in our opinion, received its due attention in the literature as regards its potential for being developed as an effective operator technique for the spectral resolution of oscillator-related potentials. The purpose of the chapter is to bridge this gulf.

The WH algebraic technique which was super-realized for quantum oscillators [9–11], is related to the paraboson relations and a graded Lie algebra structure analogous to Witten's SUSY QM algebra was realized in which only annihilation operators participate, all expressed in terms of the Wigner annihilation operator of a related super Wigner oscillator system [12]. In this reference, the coherent states are investigate via WH algebra for bound states, which are defined as the eigenstates of the lowering operator, according to the Barut-Girardello approach [13]. Recently, the problem of the construction of coherent states for systems with continuous spectra has been investigated from two viewpoints by Bragov *et al.* [17]. They adopt the approach of Malkin-Manko [18] to systems with continuous spectra that are not oscillator-like systems. On the other hand, they generalize, modify and apply the approach followed in [19] to the same kind of systems.

To illustrate the formalism we consider here simpler types of such potentials only, of the full 3D isotropic harmonic oscillator problem (for a particle of spin  $\frac{1}{2}$ ) [9] and non-relativistic Coulomb problem for the electron [16].

The WH algebra has been considered for the three-dimensional non-canonical oscillator to generate a representation of the orthosymplectic Lie superalgebra  $osp(3/2)$ , and recently

Paley have investigated the 3D Wigner oscillator under a discrete non-commutative context [20]. Let us now point out the following (anti-)commutation relations ( $[A, B]_+ \equiv AB + BA$  and  $[A, B]_- \equiv AB - BA$ ).

Also, the relevance of WH algebra to quantization in fractional dimension has been also discussed [21] and the properties of Weyl-ordered polynomials in operators  $P$  and  $Q$ , in fractional-dimensional quantum mechanics have been developed [22].

The Kustaanheimo-Stiefel mapping [25] yields the Schrödinger equation for the hydrogen atom that has been exactly solved and well-studied in the literature. (See for example, Chen [26], Cornish [27], Chen and Kibler [28], D’Hoker and Vinet [29].)

Kosteletsky, Nieto and Truax [30] have studied in a detailed manner the relation of the supersymmetric (SUSY) Coulombian problem [31–33] in D-dimensions with that of SUSY isotropic oscillators in D-dimensions in the radial version.

The vastly simplified algebraic treatment within the framework of the WH algebra of some other oscillator-related potentials like those of certain generalised SUSY oscillator Hamiltonian models of the type of Celka and Hussin which generalise the earlier potentials of U<sub>i</sub> and Balantekin have been applied by Jayaraman and Rodrigues [10]. Also, the connection of the WH algebra with the Lie superalgebra  $sl(1|n)$  has been studied in a detailed manner [34].

Also, some super-conformal models are sigma models that describe the propagation of a non-relativistic spinning particle in a curved background [35]. It was conjectured by Gibbons and Townsend that large  $n$  limit of an  $N = 4$  superconformal extension of the  $n$  particle Calogero model [36] might provide a description of the extreme Reissner-Nordström black hole near the horizon [37]. The superconformal mechanics, black holes and non-linear realizations have also been investigated by Azcárraga *et al.* [38].

## 2. The abstract WH algebra and its super-realisation

Six decades ago Wigner [5] posed an interesting question as if from the equations of motion determine the quantum mechanical commutation relations and found as an answer a generalised quantum commutation rule for the one-dimensional harmonic oscillator. Starting with the Schrödinger equation  $H|\psi_n\rangle = E_n|\psi_n\rangle$ , where the Hamiltonian operator becomes

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{x}^2) = \frac{1}{2}[\hat{a}^-, \hat{a}^+]_+ = \frac{1}{2}(\hat{a}^- \hat{a}^+ + \hat{a}^+ \hat{a}^-) \tag{1}$$

(we employ the convention of units such that  $\hbar = m = \omega = 1$ ) where the abstract Wigner Hamiltonian  $\hat{H}$  is expressed in the symmetrised bilinear form in the mutually adjoint abstract operators  $a^\pm$  defined by

$$\hat{a}^\pm = \frac{1}{\sqrt{2}}(\pm i\hat{p} - \hat{x}) \quad (\hat{a}^+)^\dagger = \hat{a}^- \tag{2}$$

Wigner showed that the Heisenberg equations of motion

$$[\hat{H}, \hat{a}^\pm]_- = \pm \hat{a}^\pm \tag{3}$$

obtained by also combining the requirement that  $x$  satisfies the equation of motion of classical form. The form of this general quantum rule can be given by

$$[\hat{a}^-, \hat{a}^+]_- = 1 + c\hat{R} \rightarrow [\hat{x}, \hat{p}]_- = i(1 + c\hat{R}) \tag{4}$$

where  $c$  is a real constant that is related to the ground-state energy  $E^{(0)}$  of  $\hat{H}$  and  $\hat{R}$  is an abstract operator, Hermitian and unitary, also possessing the properties

$$\hat{R} = \hat{R}^\dagger = \hat{R}^{-1} \rightarrow \hat{R}^2 = 1, \quad [\hat{R}, \hat{a}^+]_+ = 0 \rightarrow [\hat{R}, \hat{H}]_- = 0. \tag{5}$$

It follows from equations (1) and (4) that

$$\hat{H} = \begin{cases} \hat{a}^+ \hat{a}^- + \frac{1}{2}(1 + c\hat{R}) \\ \hat{a}^- \hat{a}^+ - \frac{1}{2}(1 + c\hat{R}) \end{cases} \tag{6}$$

Abstractly  $\hat{R}$  is the Klein operator  $\pm \exp[i\pi(\hat{H} - E_0)]$  while in Schrödinger coordinate representation, first investigated by Yang,  $\hat{R}$  is realised by  $\pm P$  where  $P$  is the parity operator:

$$P|x \rangle = \pm |x \rangle, \quad P^{-1} = P, \quad P^2 = 1, \quad PxP^{-1} = -x. \tag{7}$$

The basic (anti-)commutation relation (1) and (3) together with their derived relation (4) will be referred to here as constituting the WH algebra which is in fact a parabose algebra for one degree of freedom. We shall assume here in after, without loss of generality, that  $c$  is positive, i.e.  $c = |c| > 0$ . Thus, in coordinate representation the generalized quantization à la Wigner requires that

$$\hat{x} = x, \quad \infty - < x < \infty, \quad \hat{p} = -i \frac{d}{dx} + \frac{ic}{2x} P, \quad R = P. \tag{8}$$

Indeed, following Yang representation [6] we obtain the coordinate representation for the ladder operators as given by

$$\hat{a}^\pm \rightarrow a_{\frac{c}{2}}^\pm = \frac{1}{\sqrt{2}} \left( \pm \frac{d}{dx} \mp \frac{c}{2x} P - x \right). \tag{9}$$

Yang's wave mechanical description was further investigated in [7, 8].

The present author have applied a super-realization so that  $R = \Sigma_3$  to illustrate the first application of our operator method to the cases of the Hamiltonian of an isotonic oscillator (harmonic plus a centripetal barrier) system. To obtain a super-realisation of the WH algebra, we introduce, in addition to the usual bosonic coordinates  $(x, -i\frac{d}{dx})$ , the fermionic ones  $b^\mp (= (b^\pm)^\dagger)$  that commute with the bosonic set and are represented in terms of the usual Pauli matrices  $\Sigma_i, (i = 1, 2, 3)$ . Indeed, expressing  $a^\pm(\frac{c}{2})$  in the following respective factorised forms:

$$a^+(\frac{c}{2}) = \frac{1}{\sqrt{2}}\Sigma_1 x^{(1/2)c\Sigma_3} \exp(\frac{1}{2}x^2) (\frac{d}{dx}) \exp(-\frac{1}{2}x^2) x^{-(1/2)c\Sigma_3} \quad (10)$$

$$a^-(\frac{c}{2}) = \frac{1}{\sqrt{2}}\Sigma_1 x^{(1/2)c\Sigma_3} \exp(-\frac{1}{2}x^2) (\frac{-d}{dx}) \exp(\frac{1}{2}x^2) x^{-(1/2)c\Sigma_3}, \quad (11)$$

where these ladder operators satisfy the algebra of Wigner-Heisenberg. From (1), (11) and (10) the Wigner Hamiltonian becomes

$$\begin{aligned} H(\frac{c}{2}) &= \frac{1}{2} [a^+(\frac{c}{2}), a^-(\frac{c}{2})]_+ \\ &= \begin{pmatrix} H_-(\frac{c}{2}-1) & 0 \\ 0 & H_+(\frac{c}{2}-1) = H_-(\frac{c}{2}) \end{pmatrix}, \end{aligned} \quad (12)$$

where the even and odd sector Hamiltonians are respectively given by

$$H_-(\frac{c}{2}-1) = \frac{1}{2} \left\{ -\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}(\frac{c}{2})(\frac{c}{2}-1) \right\} \quad (13)$$

and

$$H_+(\frac{c}{2}-1) = \frac{1}{2} \left\{ -\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}(\frac{c}{2})(\frac{c}{2}+1) \right\} = H_-(\frac{c}{2}). \quad (14)$$

The time-independent Schrödinger equation for these Hamiltonians of an isotonic oscillator (harmonic plus a centripetal barrier) system becomes the following eigenvalue equation:

$$H_\pm(\frac{c}{2}-1) |m, \frac{c}{2}-1\rangle = E_\pm(\frac{c}{2}-1) |m, \frac{c}{2}-1\rangle. \quad (15)$$

Thus, from the annihilation condition  $a^-|0\rangle = 0$ , the ground-state energy is given by

$$E^{(0)}\left(\frac{c}{2}\right) = \frac{1}{2}(1+c) > \frac{1}{2}, \quad c > 0. \tag{16}$$

At this stage an independent verification of the existence or not of a zero ground-state energy for  $H\left(\frac{c}{2}\right)$  suggested by its positive semi-definite form may be in order.

The question we formulate now is the following: What is the behaviour of the ladder operators on the autokets of the Wigner oscillator quantum states? To answer this question is obtained via WH algebra, note that the Wigner oscillator ladder operators on autokets of these quantum states are given by

$$\begin{aligned} a_{\frac{c}{2}}^- |2m, \frac{c}{2}\rangle &= \sqrt{2m} |2m-1, \frac{c}{2}\rangle \\ a_{\frac{c}{2}}^- |2m+1, \frac{c}{2}\rangle &= \sqrt{2(m+E^{(0)})} |2m, \frac{c}{2}\rangle \\ a_{\frac{c}{2}}^+ |2m, \frac{c}{2}\rangle &= \sqrt{2(m+E^{(0)})} |2m+1, \frac{c}{2}\rangle \\ a_{\frac{c}{2}}^+ |2m+1, \frac{c}{2}\rangle &= \sqrt{2(m+1)} |2m+2, \frac{c}{2}\rangle. \end{aligned} \tag{17}$$

Now, from the role of  $a^+\left(\frac{c}{2}\right)$  as the energy step-up operator (the upper sign choice) the excited-state energy eigenfunctions and the complete energy spectrum of  $H\left(\frac{c}{2}\right)$  are respectively given by  $\psi^{(n)}\left(\frac{c}{2}\right) \propto [a^+\left(\frac{c}{2}\right)]^n \psi^{(0)}\left(\frac{c}{2}\right)$  and  $E^{(n)}\left(\frac{c}{2}\right) = E^{(0)} + n, \quad n = 0, 1, 2, \dots$ .

It is known that the operators  $\pm \frac{i}{2} (a^\pm\left(\frac{c}{2}\right))^2$  and  $\frac{1}{2}H\left(\frac{c}{2}\right)$  can be chosen as a basis for a realization of the  $so(2, 1) \sim su(1, 1) \sim sl(2, \mathbf{R})$  Lie algebra. When projected the  $-\frac{1}{2} (a^\pm)^2$  operators in the even sector with  $\frac{1}{2}(\mathbf{1} + \Sigma_3)$ , viz.,

$$\frac{1}{2}(\mathbf{1} + \Sigma_3)B^- = \frac{1}{2}(\mathbf{1} + \Sigma_3) (a^-)^2 = \begin{pmatrix} B^- & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\frac{1}{2}(\mathbf{1} + \Sigma_3)B^+ = \frac{1}{2}(\mathbf{1} + \Sigma_3) (a^+)^2 = \begin{pmatrix} B^+ & 0 \\ 0 & 0 \end{pmatrix}$$

we obtain

$$B^-\left(\frac{c}{2} - 1\right) = \frac{1}{2} \left\{ \frac{d^2}{dx^2} + 2x \frac{d}{dx} + x^2 - \frac{\left(\frac{c}{2} - 1\right)\frac{c}{2}}{x^2} + 1 \right\} \tag{18}$$

and

$$B^+(\frac{c}{2} - 1) = (B^-)^\dagger = \frac{1}{2} \left\{ \frac{d^2}{dx^2} - 2x \frac{d}{dx} + x^2 - \frac{(\frac{c}{2} - 1)\frac{c}{2}}{x^2} - 1 \right\}. \tag{19}$$

Thus, the Lie algebra becomes

$$[K_0, K_-]_- = -K_-, \quad [K_0, K_+]_- = +K_+, \quad [K_-, K_+]_- = 2K_0, \tag{20}$$

where  $K_0 = \frac{1}{2}H_-$ ,  $K_- = -\frac{1}{2}B^-$  and  $K_+ = \frac{1}{2}B^+$  generate once again the  $su(1, 1)$  Lie algebra.

Therefore, these ladder operators obey the following commutation relations:

$$\begin{aligned} [B^-(\frac{c}{2} - 1), B^+(\frac{c}{2} - 1)]_- &= 4H_-(\frac{c}{2} - 1) \\ [H_-, B^\pm(\frac{c}{2} - 1)]_- &= \pm 2B^\pm(\frac{c}{2} - 1). \end{aligned} \tag{21}$$

Hence, the quadratic operators  $B^\pm(\frac{c}{2} - 1)$  acting on the orthonormal basis of eigenstates of  $H_-(\frac{c}{2} - 1)$ ,  $\{| m, \frac{c}{2} - 1 \rangle\}$  where  $m = 0, 1, 2, \dots$  have the effect of raising or lowering the quanta by two units so that we can write

$$B^-(\frac{c}{2} - 1) | m, \frac{c}{2} - 1 \rangle = \sqrt{2m(2m + c + 1)} | m - 1, \frac{c}{2} - 1 \rangle \tag{22}$$

and

$$B^+(\frac{c}{2} - 1) | m, \frac{c}{2} - 1 \rangle = \sqrt{2(m + 1)(2m + c + 1)} | m + 1, \frac{c}{2} - 1 \rangle \tag{23}$$

giving

$$| m, \frac{c}{2} - 1 \rangle = 2^{-m} \left\{ \frac{\Gamma(\frac{c+1}{2})}{m! \Gamma(\frac{c+1}{2} + m)} \right\}^{1/2} \left\{ B^+(\frac{c}{2} - 1) \right\}^m | 0, \frac{c}{2} - 1 \rangle, \tag{24}$$

where  $\Gamma(x)$  is the ordinary Gamma Function. Note that  $B^\pm(\frac{c}{2} - 1) | m, \frac{c}{2} - 1 \rangle$  are associated with the energy eigenvalues  $E_-^{(m\pm 1)} = \frac{c+1}{2} + 2(m \pm 1)$ ,  $m = 0, 1, 2, \dots$

Let us to conclude this section presenting the following comments: one can generate the called canonical coherent states, which are defined as the eigenstates of the lowering operator  $B^-(\frac{c}{2} - 1)$  of the bosonic sector, according to the Barut-Girardello approach [12, 13] and generalized coherent states according to Perelomov [14, 15]. Results of our investigations on these coherent states will be reported separately.

### 3. The 3D Wigner and SUSY systems

As is well-known, the quantum mechanical (QM)  $N = 2$  supersymmetry (SUSY) algebra of Witten [1–4]

$$\begin{aligned} H_{ss} &= [Q_-, Q_+]_+ \\ (Q_-)^2 &= (Q_+)^2 = 0 \quad \rightarrow \quad [H_{ss}, Q_{\mp}] = 0, \end{aligned} \quad (25)$$

involves bosonic and fermionic sector Hamiltonians of the SUSY Hamiltonian  $H_{ss}$  (the even element), which get intertwined through the nilpotent charge operators  $Q_- = (Q_+)^{\dagger}$  (the odd elements).

The connection between the 3D Wigner Hamiltonian  $H(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$  and a 3D SUSY isotropic harmonic oscillator for spin  $\frac{1}{2}$  is given by anti-commutation relation (25) and the mutually adjoint charge operators  $Q_{\mp}$  in terms of the Wigner system ladder operators:

$$\begin{aligned} Q_- &= \frac{1}{2}(1 - \Sigma_3)a^-(\vec{\sigma} \cdot \vec{L} + \mathbf{1}) \\ Q_+ &= Q_-^{\dagger} = \frac{1}{2}(1 + \Sigma_3)a^+(\vec{\sigma} \cdot \vec{L} + \mathbf{1}), \quad (Q_-)^2 = (Q_+)^2 = 0. \end{aligned} \quad (26)$$

In this Section, we consider the 3D isotropic spin- $\frac{1}{2}$  oscillator Hamiltonian in the bosonic sector of a Wigner system

$$H_-(\vec{\sigma} \cdot \vec{L}) = \frac{1}{2}(p^2 + r^2) = \frac{1}{2}\left(-\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}(\vec{\sigma} \cdot \vec{L})(\vec{\sigma} \cdot \vec{L} + \mathbf{1}) + r^2\right) \quad (0 < r < \infty) \quad (27)$$

for a non-relativistic 3D isotropic oscillator with spin- $\frac{1}{2}$  represented here by  $\frac{1}{2}\sigma$ . With the use of the following familiar spin- $\frac{1}{2}$  equalities:

$$\vec{\sigma} \cdot \vec{p} = \sigma_r p_r + \frac{i}{r}\sigma_r(\vec{\sigma} \cdot \vec{L} + \mathbf{1}), \quad p_r = -i\left(\frac{1}{r} + \frac{\partial}{\partial r}\right) = \frac{1}{r}\left(-i\frac{\partial}{\partial r}\right)r = p_r^{\dagger} \quad (28)$$

$$\sigma_r = \frac{1}{r}\vec{\sigma} \cdot \vec{r}, \quad \sigma_r^2 = 1, \quad [\sigma_r, \vec{\sigma} \cdot \vec{L} + \mathbf{1}]_+ = 0, \quad L^2 = \vec{\sigma} \cdot \vec{L}(\vec{\sigma} \cdot \vec{L} + \mathbf{1}). \quad (29)$$

The 3D fermionic sector Hamiltonian becomes

$$H_+(\vec{\sigma} \cdot \vec{L}) = H_-(\vec{\sigma} \cdot \vec{L} + \mathbf{1}) = \frac{1}{2} \left( -\frac{\partial^2}{\partial r^2} - \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} (\vec{\sigma} \cdot \vec{L} + \mathbf{1})(\vec{\sigma} \cdot \vec{L} + \mathbf{2}) + r^2 \right). \quad (30)$$

In this case the connection between  $H_{ss}$  and  $H_W$  is given by

$$H_{ss} = H_W - \frac{1}{2} \Sigma_3 \{ 1 + 2(\vec{\sigma} \cdot \vec{L} + \mathbf{1}) \Sigma_3 \}, \quad (31)$$

where  $H_W = \text{diag}(\tilde{H}_-(\vec{\sigma} \cdot \vec{L}), \tilde{H}_+(\vec{\sigma} \cdot \vec{L}))$  is the diagonal 3D Wigner Hamiltonian given by

$$H_W = H_W(\vec{\sigma} \cdot \vec{L} + \mathbf{1}) = \begin{pmatrix} H_-(\vec{\sigma} \cdot \vec{L}) & 0 \\ 0 & H_-(\vec{\sigma} \cdot \vec{L} + \mathbf{1}) \end{pmatrix}. \quad (32)$$

Indeed from the role of  $a^+$  as the energy step-up operator, the complete excited state wave functions  $\Psi_w^n$  are readily given by the step up operation with  $a^+$ :

$$\psi_W^{(n)} \propto (a^+)^n \psi_{W,+}^{(0)}(r, \theta, \varphi) = (a^+)^n \tilde{R}_{1,+}^{(0)}(r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} y_+(\theta, \varphi). \quad (33)$$

On the eigenspaces of the operator  $(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$ , the 3D Wigner algebra gets reduced to a 1D from with  $(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$  replaced by its eigenvalue  $\mp(\ell + 1)$ ,  $\ell = 0, 1, 2, \dots$ , where  $\ell$  is the orbital angular momentum quantum number. The eigenfunctions of  $(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$  for the eigenvalues  $(\ell + 1)$  and  $-(\ell + 1)$  are respectively given by the well known spin-spherical harmonic  $y_{\mp}$ . Thus, from the super-realized first order ladder operators given by

$$a^{\pm}(\ell + 1) = \frac{1}{\sqrt{2}} \left\{ \pm \frac{d}{dr} \pm \frac{(\ell + 1)}{r} \Sigma_3 - r \right\} \Sigma_1, \quad (34)$$

where  $\frac{\ell}{2} = \ell + 1$ , the Wigner Hamiltonian becomes

$$\begin{aligned} H(\ell + 1) &= \frac{1}{2} [a^+(\ell + 1), a^-(\ell + 1)]_+ \\ &= \begin{pmatrix} H_-(\ell) & 0 \\ 0 & H_+(\ell) = H_-(\ell + 1) \end{pmatrix}, \end{aligned} \quad (35)$$

where in the representation of the radial part wave functions,  $\chi(r) = rR(r)$ , the even and odd sector Hamiltonians are respectively given by



$$H_-(\ell) = \frac{1}{2} \left\{ -\frac{d^2}{dr^2} + r^2 + \frac{1}{r^2} \ell(\ell + 1) \right\} \quad (36)$$

and

$$H_+(\ell) = \frac{1}{2} \left\{ -\frac{d^2}{dr^2} + r^2 + \frac{1}{r^2} (\ell + 1)(\ell + 2) \right\} = H_-(\ell + 1). \quad (37)$$

In this representation the eigenvalue equation becomes

$$H_{\pm}(\ell)\chi(r) = E_{\pm}(\ell)\chi(r), \quad \chi(r) = rR(r). \quad (38)$$

The WH algebra ladder relations are readily obtained as

$$[H(\ell + 1), a^{\pm}(\ell + 1)]_{-} = \pm a^{\pm}(\ell + 1). \quad (39)$$

Equations (35) and (39) together with the commutation relation

$$[a^{-}(\ell + 1), a^{+}(\ell + 1)]_{-} = 1 + 2(\ell + 1)\Sigma_3 \quad (40)$$

constitute the WH algebra.

The Wigner eigenfunctions that generate the eigenspace associated with even(odd)  $\sigma_3$ -parity for even(odd) quanta  $n = 2m(n = 2m + 1)$  are given by

$$|n = 2m, \ell + 1\rangle = \begin{pmatrix} |m, \ell\rangle \\ 0 \end{pmatrix}, \quad |n = 2m + 1, \ell\rangle = \begin{pmatrix} 0 \\ |m, \ell\rangle \end{pmatrix} \quad (41)$$

and satisfy the following eigenvalue equation

$$H(\ell + 1) |n, \ell + 1\rangle = E^{(n)} |n, \ell + 1\rangle, \quad (42)$$

the non-degenerate energy eigenvalues are obtained by the application of the raising operator on the ground eigenstate and are given by

$$\psi_W^{(n)}(r) \propto (a^{+})^n \psi_{W,+}^{(0)}(r) = (a^{+})^n \chi_{1,+}^{(0)}(r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (43)$$

and

$$E^{(n)} = \ell + \frac{3}{2} + n, \quad n = 0, 1, 2, \dots \tag{44}$$

The ground state energy eigenvalue is determined by the annihilation condition which reads as:

$$a^- \psi_{\mathbf{W},+}^{(0)} = 0, \quad (\vec{\sigma} \cdot \vec{L} + \mathbf{1}) \rightarrow \ell + 1; \tag{45}$$

which, after operation on the fermion spinors and the spin-angular part, turns into

$$\begin{pmatrix} \exp(\frac{r^2}{2})r^{-\ell-1}\chi_{1,+}^{(0)}(r) \\ \exp(\frac{r^2}{2})r^{-\ell+1}\chi_{2,+}^{(0)}(r) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{46}$$

Retaining only the non-singular and normalizable  $R_{1,+}^{(0)}(r)$ , we simply take the singular solution  $R_{2,+}^{(0)}(r)$ , which is physically non-existing, as identically zero. Hence the Wigner’s eigenfunction of the ground state becomes

$$\psi_{\mathbf{W},+}^{(0)} = \begin{pmatrix} \chi_{1,+}^{(0)}(r)y_+ \\ 0 \end{pmatrix}, \quad \chi_{1,+}^{(0)}(r) \propto r^{\ell+1}\exp(-\frac{r^2}{2}), \tag{47}$$

where  $0 < r < \infty$ . For the radial oscillator the energy eigenvectors satisfy the following eigenvalue equations

$$H_-(\ell) | m, \ell \rangle = E_-^{(m)} | m, \ell \rangle, \tag{48}$$

where the eigenvalues are exactly constructed via WH algebra ladder relations and are given by

$$E_-^{(m)} = \ell + \frac{3}{2} + 2m, \quad m = 0, 1, 2, \dots \tag{49}$$

We stress that similar results can be adequately extended for any physical D-dimensional radial oscillator system by the Hermitian replacement of  $-i\left(\frac{d}{dr} + \frac{1}{r}\right) \rightarrow -i\left(\frac{d}{dr} + \frac{D-1}{2r}\right)$  and the Wigner deformation parameter  $\ell + 1 \rightarrow \ell_D + \frac{1}{2}(D - 1)$  where  $\ell_D(\ell_D = 0, 1, 2, \dots)$  is the D-dimensional oscillator angular momentum.

#### 4. The constrained Super Wigner Oscillator in 4D and the hydrogen atom

In this section, the complete spectrum for the hydrogen atom is found with considerable simplicity. Indeed, the solutions of the time-independent Schrödinger equation for the hydrogen atom were mapped onto the super Wigner harmonic oscillator in 4D by using the Kustaanheimo-Stiefel transformation. The Kustaanheimo-Stiefel mapping yields the Schrödinger equation for the hydrogen atom that has been exactly solved and well-studied in the literature. (See for example, [16].)

Kostelecky, Nieto and Truax have studied in a detailed manner the relation of the SUSY Coulombian problem in D-dimensions with that of SUSY isotropic oscillators in D-dimensions in the radial version. (See also Lahiri *et. al.* [2].)

The bosonic sector of the above eigenvalue equation can immediately be identified with the eigenvalue equation for the Hamiltonian of the 3D Hydrogen-like atom expressed in the equivalent form given by [16]

The usual isotropic oscillator in 4D has the following eigenvalue equation for it's Hamiltonian  $H_{osc}^B$ , described by (employing natural system of units  $\hbar = m = 1$ ) time-independent Schrödinger equation

$$H_{osc}^B \Psi_{osc}^B(y) = E_{osc}^B \Psi_{osc}^B(y), \tag{50}$$

with

$$H_{osc}^B = -\frac{1}{2}\nabla_4^2 + \frac{1}{2}s^2, \quad s^2 = \sum_{i=1}^4 y_i^2, \tag{51}$$

$$\nabla_4^2 = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2} + \frac{\partial^2}{\partial y_4^2} = \sum_{i=1}^4 \frac{\partial^2}{\partial y_i^2}, \tag{52}$$

where the superscript  $B$  in  $H_{osc}^B$  is in anticipation of the Hamiltonian, with constraint to be defined, being implemented in the bosonic sector of the super 4D Wigner system with unitary frequency. Changing to spherical coordinates in 4-space dimensions, allowing a factorization of the energy eigenfunctions as a product of a radial eigenfunction and spin-spherical harmonic.

In (52), the coordinates  $y_i(i = 1, 2, 3, 4)$  in spherical coordinates in 4D are defined by [26, 29]

$$\begin{aligned} y_1 &= s \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{\varphi - \omega}{2}\right) \\ y_2 &= s \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\varphi - \omega}{2}\right) \\ y_3 &= s \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\varphi + \omega}{2}\right) \\ y_4 &= s \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\varphi + \omega}{2}\right), \end{aligned} \tag{53}$$

where  $0 \leq \theta \leq \pi$ ,  $0 \leq \varphi \leq 2\pi$  and  $0 \leq \omega \leq 4\pi$ .

The mapping of the coordinates  $y_i(i = 1, 2, 3, 4)$  in 4D with the Cartesian coordinates  $\rho_i(i = 1, 2, 3)$  in 3D is given by the Kustaanheimo-Stiefel transformation

$$\rho_i = \sum_{a,b=1}^2 z_a^* \Gamma_{ab}^i z_b, \quad (i = 1, 2, 3) \tag{54}$$

$$z_1 = y_1 + iy_2, \quad z_2 = y_3 + iy_4, \tag{55}$$

where the  $\Gamma_{ab}^i$  are the elements of the usual Pauli matrices. If one defines  $z_1$  and  $z_2$  as in Eq. (55),  $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$  is a two dimensional spinor of  $SU(2)$  transforming as  $Z \rightarrow Z' = UZ$  with  $U$  a two-by-two matrix of  $SU(2)$  and of course  $Z^\dagger Z$  is invariant. So the transformation (54) is very spinorial. Also, using the standard Euler angles parametrizing  $SU(2)$  as in transformations (53) and (55) one obtains

$$\begin{aligned} z_1 &= s \cos\left(\frac{\theta}{2}\right) e^{\frac{i}{2}(\varphi-\omega)} \\ z_2 &= s \sin\left(\frac{\theta}{2}\right) e^{\frac{i}{2}(\varphi+\omega)}. \end{aligned} \tag{56}$$

Note that the angles in these equations are divided by two. However, in 3D, the angles are not divided by two, viz.,  $\rho_3 = \rho \cos^2(\frac{\theta}{2}) - \rho \sin^2(\frac{\theta}{2}) = \rho \cos\theta$ . Indeed, from (54) and (56), we obtain

$$\rho_1 = \rho \sin\theta \cos\varphi, \quad \rho_2 = \rho \sin\theta \sin\varphi, \quad \rho_3 = \rho \cos\theta \tag{57}$$

and also that

$$\begin{aligned} \rho &= \left\{ \rho_1^2 + \rho_2^2 + \rho_3^2 \right\}^{\frac{1}{2}} = \left\{ (\rho_1 + i\rho_2)(\rho_1 - i\rho_2) + \rho_3^2 \right\}^{\frac{1}{2}} \\ &= \left\{ (2z_1^* z_2)(2z_1 z_2^*) + (z_1^* z_1 - z_2^* z_2)^2 \right\}^{\frac{1}{2}} \\ &= (z_1 z_1^* + z_2 z_2^*) = \sum_{i=1}^4 y_i^2 = s^2. \end{aligned} \tag{58}$$

The complex form of the Kustaanheimo-Stiefel transformation was given by Cornish [27].

Thus, the expression for  $H_{osc}^B$  in (51) can be written in the form

$$H_{osc}^B = -\frac{1}{2} \left( \frac{\partial^2}{\partial s^2} + \frac{3}{s} \frac{\partial}{\partial s} \right) - \frac{2}{s^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\sin^2 \theta} \left( 2 \cos \theta \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \omega} \right) \frac{\partial}{\partial \omega} \right] + \frac{1}{2} s^2. \quad (59)$$

We obtain a constraint by projection (or "dimensional reduction") from four to three dimensional. Note that  $\psi_{osc}^B$  is independent of  $\omega$  provides the constraint condition

$$\frac{\partial}{\partial \omega} \Psi_{osc}^B(s, \theta, \varphi) = 0, \quad (60)$$

imposed on  $H_{osc}^B$ , the expression for this restricted Hamiltonian, which we continue to call as  $H_{osc}^B$ , becomes

$$H_{osc}^B = -\frac{1}{2} \left( \frac{\partial^2}{\partial s^2} + \frac{3}{s} \frac{\partial}{\partial s} \right) - \frac{2}{s^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] + \frac{1}{2} s^2. \quad (61)$$

Identifying the expression in bracket in (61) with  $L^2$ , the square of the orbital angular momentum operator in 3D, since we always have

$$L^2 = (\vec{\sigma} \cdot \vec{L})(\vec{\sigma} \cdot \vec{L} + 1), \quad (62)$$

which is valid for any system, where  $\sigma_i (i = 1, 2, 3)$  are the Pauli matrices representing the spin  $\frac{1}{2}$  degrees of freedom, we obtain for  $H_{osc}^B$  the final expression

$$H_{osc}^B = \frac{1}{2} \left[ - \left( \frac{\partial^2}{\partial s^2} + \frac{3}{s} \frac{\partial}{\partial s} \right) + \frac{4}{s^2} (\vec{\sigma} \cdot \vec{L})(\vec{\sigma} \cdot \vec{L} + 1) + s^2 \right]. \quad (63)$$

Now, associating  $H_{osc}^B$  with the bosonic sector of the super Wigner system,  $H_W$ , subject to the same constraint as in (60), and following the analogy with the Section II of construction of super Wigner systems, we first must solve the Schrödinger equation

$$H_W \Psi_W(s, \theta, \varphi) = E_W \Psi_W(s, \theta, \varphi), \quad (64)$$

where the explicit form of  $H_W$  is given by

$$H_W(2\vec{\sigma} \cdot \vec{L} + \frac{3}{2}) = \begin{pmatrix} -\frac{1}{2}(\frac{\partial}{\partial s} + \frac{3}{2s})^2 + \frac{1}{2}s^2 + \frac{(2\vec{\sigma} \cdot \vec{L} + \frac{1}{2})(2\vec{\sigma} \cdot \vec{L} + \frac{3}{2})}{2s^2} & 0 \\ 0 & -\frac{1}{2}(\frac{\partial}{\partial s} + \frac{3}{2s})^2 + \frac{1}{2}s^2 + \frac{(2\vec{\sigma} \cdot \vec{L} + \frac{3}{2})(2\vec{\sigma} \cdot \vec{L} + \frac{5}{2})}{2s^2} \end{pmatrix}. \quad (65)$$

Using the operator technique in references [9, 10], we begin with the following super-realized mutually adjoint operators

$$a_W^\pm \equiv a^\pm(2\vec{\sigma} \cdot \vec{L} + \frac{3}{2}) = \frac{1}{\sqrt{2}} \left[ \pm \left( \frac{\partial}{\partial s} + \frac{3}{2s} \right) \Sigma_1 \mp \frac{1}{s} (2\vec{\sigma} \cdot \vec{L} + \frac{3}{2}) \Sigma_1 \Sigma_3 - \Sigma_1 s \right], \quad (66)$$

where  $\Sigma_i (i = 1, 2, 3)$  constitute a set of Pauli matrices that provide the fermionic coordinates commuting with the similar Pauli set  $\sigma_i (i = 1, 2, 3)$  already introduced representing the spin  $\frac{1}{2}$  degrees of freedom.

It is checked, after some algebra, that  $a^+$  and  $a^-$  of (66) are indeed the raising and lowering operators for the spectra of the super Wigner Hamiltonian  $H_W$  and they satisfy the following (anti-)commutation relations of the WH algebra:

$$\begin{aligned} H_W &= \frac{1}{2} [a_W^-, a_W^+]_+ \\ &= a_W^+ a_W^- + \frac{1}{2} \left[ 1 + 2(2\vec{\sigma} \cdot \vec{L} + \frac{3}{2}) \Sigma_3 \right] \\ &= a_W^- a_W^+ - \frac{1}{2} \left[ 1 + 2(2\vec{\sigma} \cdot \vec{L} + \frac{3}{2}) \Sigma_3 \right] \end{aligned} \quad (67)$$

$$[H_W, a_W^\pm]_- = \pm a_W^\pm \quad (68)$$

$$[a_W^-, a_W^+]_- = 1 + 2(2\vec{\sigma} \cdot \vec{L} + \frac{3}{2}) \Sigma_3, \quad (69)$$

$$[\Sigma_3, a_W^\pm]_+ = 0 \Rightarrow [\Sigma_3, H_W]_- = 0. \quad (70)$$

Since the operator  $(2\vec{\sigma} \cdot \vec{L} + \frac{3}{2})$  commutes with the basic elements  $a^\pm, \Sigma_3$  and  $H_W$  of the WH algebra (67), (68) and (69) it can be replaced by its eigenvalues  $(2\ell + \frac{3}{2})$  and  $-(2\ell + \frac{5}{2})$  while acting on the respective eigenspace in the form

$$\Psi_{\text{osc}}(s, \theta, \varphi) = \begin{pmatrix} \Psi_{\text{osc}}^{\text{B}}(s, \theta, \varphi) \\ \Psi_{\text{osc}}^{\text{F}}(s, \theta, \varphi) \end{pmatrix} = \begin{pmatrix} R_{\text{osc}}^{\text{B}}(s) \\ R_{\text{osc}}^{\text{F}}(s) \end{pmatrix} y_{\pm}(\theta, \varphi) \quad (71)$$

in the notation where  $y_{\pm}(\theta, \varphi)$  are the spin-spherical harmonics [43],

$$\begin{aligned} y_+(\theta, \varphi) &= y_{\ell+\frac{1}{2}; j=\ell+\frac{1}{2}, m_j}(\theta, \varphi) \\ y_-(\theta, \varphi) &= y_{\ell+1+\frac{1}{2}; j=(\ell+1)-\frac{1}{2}, m_j}(\theta, \varphi) \end{aligned} \quad (72)$$

so that, we obtain:  $(\vec{\sigma} \cdot \vec{L} + 1)y_{\pm} = \pm(\ell + 1)y_{\pm}$ ,  $(2\vec{\sigma} \cdot \vec{L} + \frac{3}{2})y_+ = (2\ell + \frac{3}{2})y_+$  and  $(2\vec{\sigma} \cdot \vec{L} + \frac{3}{2})y_- = -[2(\ell + 1) + \frac{1}{2}]y_-$ . Note that on these subspaces the 3D WH algebra is reduced to a formal 1D radial form with  $H_W(2\vec{\sigma} \cdot \vec{L} + \frac{3}{2})$  acquiring respectively the forms  $H_W(2\ell + \frac{3}{2})$  and

$$H_W \left( -2\ell - \frac{5}{2} \right) = \Sigma_1 H_W \left( 2\ell + \frac{3}{2} \right) \Sigma_1. \quad (73)$$

Thus, the positive finite form of  $H_W$  in (67) together with the ladder relations (68) and the form (69) leads to the direct determination of the state energies and the corresponding Wigner ground state wave functions by the simple application of the annihilation conditions

$$a^-(2\ell + \frac{3}{2}) \begin{pmatrix} R_{\text{osc}}^{\text{B}(0)}(s) \\ R_{\text{osc}}^{\text{F}(0)}(s) \end{pmatrix} = 0. \quad (74)$$

Then, the complete energy spectrum for  $H_W$  and the whole set of energy eigenfunctions  $\Psi_{\text{osc}}^{(n)}(s, \theta, \varphi) (n = 2m, 2m + 1, m = 0, 1, 2, \dots)$  follows from the step up operation provided by  $a^+(2\ell + \frac{3}{2})$  acting on the ground state, which are also simultaneous eigenfunctions of the fermion number operator  $N = \frac{1}{2}(1 - \Sigma_3)$ . We obtain for the bosonic sector Hamiltonian  $H_{\text{osc}}^{\text{B}}$  with fermion number  $n_f = 0$  and even orbital angular momentum  $\ell_4 = 2\ell, (\ell = 0, 1, 2, \dots)$ , the complete energy spectrum and eigenfunctions given by

$$\left[ E_{\text{osc}}^{\text{B}} \right]_{\ell_4=2\ell}^{(m)} = 2\ell + 2 + 2m, \quad (m = 0, 1, 2, \dots), \quad (75)$$

$$\left[ \Psi_{osc}^B(s, \theta, \varphi) \right]_{\ell_4=2\ell}^{(m)} \propto s^{2\ell} \exp\left(-\frac{1}{2}s^2\right) L_m^{(2\ell+1)}(s^2) \begin{cases} y_+(\theta, \varphi) \\ y_-(\theta, \varphi) \end{cases} \quad (76)$$

where  $L_m^\alpha(s^2)$  are generalized Laguerre polynomials [9]. Now, to relate the mapping of the 4D super Wigner system with the corresponding system in 3D, we make use of the substitution of  $s^2 = \rho$ , Eq. (60) and the following substitutions

$$\frac{\partial}{\partial s} = 2\sqrt{\rho} \frac{\partial}{\partial \rho}, \quad \frac{\partial^2}{\partial s^2} = 4\rho \frac{\partial^2}{\partial \rho^2} + 2\frac{\partial}{\partial \rho}, \quad (77)$$

in (65) and divide the eigenvalue equation for  $H_W$  in (64) by  $4s^2 = 4\rho$ , obtaining

$$\begin{pmatrix} -\frac{1}{2} \left( \frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial}{\partial \rho} \right) - \frac{1}{2} \left[ -\frac{1}{4} - \frac{\vec{\sigma} \cdot \vec{L} (\vec{\sigma} \cdot \vec{L} + 1)}{\rho^2} \right] & 0 \\ 0 & -\frac{1}{2} \left( \frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial}{\partial \rho} \right) - \frac{1}{2} \left[ -\frac{1}{4} - \frac{(\vec{\sigma} \cdot \vec{L} + \frac{1}{2})(\vec{\sigma} \cdot \vec{L} + \frac{3}{2})}{\rho^2} \right] \end{pmatrix} \begin{pmatrix} \Psi^B \\ \Psi^F \end{pmatrix} = \frac{1}{4\rho} E_W \begin{pmatrix} \Psi^B \\ \Psi^F \end{pmatrix}. \quad (78)$$

The bosonic sector of the above eigenvalue equation can immediately be identified with the eigenvalue equation for the Hamiltonian of the 3D Hydrogen-like atom expressed in the equivalent form given by

$$\left\{ -\frac{1}{2} \left( \frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial}{\partial \rho} \right) - \frac{1}{2} \left[ -\frac{1}{4} - \frac{\vec{\sigma} \cdot \vec{L} (\vec{\sigma} \cdot \vec{L} + 1)}{\rho^2} \right] \right\} \psi(\rho, \theta, \varphi) = \frac{\ell}{2\rho} \psi(\rho, \theta, \varphi), \quad (79)$$

where  $\Psi^B = \psi(\rho, \theta, \varphi)$  and the connection between the dimensionless and dimensionfull eigenvalues, respectively,  $\ell$  and  $E_a$  with  $e = 1 = m = \hbar$  is given by [43]

$$\ell = \frac{Z}{\sqrt{-2E_a}}, \quad \rho = \alpha r, \quad \alpha = \sqrt{-8E_a}, \quad (80)$$

where  $E_a$  is the energy of the electron Hydrogen-like atom,  $(r, \theta, \varphi)$  stand for the spherical polar coordinates of the position vector  $\vec{r} = (x_1, x_2, x_3)$  of the electron in relative to the nucleons of charge  $Z$  together with  $s^2 = \rho$ . We see then from equations (75), (76), (79) and (80) that the complete energy spectrum and eigenfunctions for the Hydrogen-like atom given by

$$\frac{\lambda}{2} = \frac{E_{osc}^B}{4} \Rightarrow [E_a]_\ell^{(m)} = [E_a]^{(N)} = -\frac{Z^2}{2N^2}, \quad (N = 1, 2, \dots) \quad (81)$$



and

$$[\psi(\rho, \theta, \varphi)]_{\ell; \alpha, m}^{(m)} \propto \rho^\ell \exp\left(-\frac{\rho}{2}\right) L_m^{(2\ell+1)}(\rho) \begin{cases} y_+(\theta, \varphi) \\ y_-(\theta, \varphi) \end{cases} \quad (82)$$

where  $E_{osc}^B$  is given by Eq. (75).

Here,  $N = \ell + m + 1$  ( $\ell = 0, 1, 2, \dots, N - 1; m = 0, 1, 2, \dots$ ) is the principal quantum number. Kosteletsky and Nieto shown that the supersymmetry in non-relativistic quantum mechanics may be realized in atomic systems [44].

### 5. The superconformal quantum mechanics from WH algebra

The superconformal quantum mechanics has been examined in [35]. Another application for these models is in the study of the radial motion of test particle near the horizon of extremal Reissner-Nordström black holes [35, 37]. Also, another interesting application of the superconformal symmetry is the treatment of the Dirac oscillator, in the context of the superconformal quantum mechanics [39–42, 46].

In this section we introduce the explicit supersymmetry for the conformal Hamiltonian in the WH-algebra picture. Let us consider the supersymmetric generalization of  $H$ , given by

$$\mathcal{H} = \frac{1}{2} \{Q_c, Q_c^\dagger\}, \quad (83)$$

where the new supercharge operators are given in terms of the momentum Yang representation

$$\begin{aligned} Q_c &= \left(-ip_x + \frac{\sqrt{g}}{x}\right) \Psi^\dagger, \\ Q_c^\dagger &= \Psi \left(ip_x + \frac{\sqrt{g}}{x}\right), \end{aligned} \quad (84)$$

with  $\Psi$  and  $\Psi^\dagger$  being Grassmannian operators so that its anticommutator is  $\{\Psi, \Psi^\dagger\} = \Psi\Psi^\dagger + \Psi^\dagger\Psi = 1$ .

Explicitly the superconformal Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2} \left( \mathbf{1} p_x^2 + \frac{\mathbf{1} g + \sqrt{g} B (1 - c \mathbf{P})}{x^2} \right) \quad (85)$$

where  $B = [\Psi^\dagger, \Psi]_-$ , so that the parity operator is conserved, i.e.,  $[\mathcal{H}, \mathbf{P}]_- = 0$ .

When one introduces the following operators

$$\begin{aligned} S &= x\Psi^\dagger, \\ S^\dagger &= \Psi x, \end{aligned} \tag{86}$$

it can be shown that these operators together with the conformal quantum mechanics operators  $D$  and  $K$

$$\begin{aligned} D &= \frac{1}{2}(xp_x + p_x x), \\ K &= \frac{1}{2}x^2, \end{aligned} \tag{87}$$

satisfy the deformed superalgebra  $osp(2|2)$  (Actually, this superalgebra is  $osp(2|2)$  when we fix  $\mathbf{P} = 1$  or  $\mathbf{P} = -1$ ), viz.,

$$\begin{aligned} [\mathcal{H}, D]_- &= -2i\mathcal{H}, \\ [\mathcal{H}, K]_- &= -iD, \\ [K, D]_- &= 2iK, \\ [Q_c, Q_c^\dagger]_+ &= 2\mathcal{H}, \\ [Q_c, S^\dagger]_+ &= -iD - \frac{1}{2}B(1 + c\mathbf{P}) + \sqrt{g}, \\ [Q_c^\dagger, S]_+ &= iD - \frac{1}{2}B(1 + c\mathbf{P}) + \sqrt{g}, \\ [Q_c^\dagger, D]_- &= -iQ_c^\dagger, \\ [Q_c^\dagger, K]_- &= S^\dagger, \\ [Q_c^\dagger, B]_- &= 2Q_c^\dagger, \\ [Q_c, K]_- &= -S, \\ [Q_c, B]_- &= -2Q_c, \\ [Q_c, D]_- &= -iQ_c, \\ [\mathcal{H}, S]_- &= Q_c, & [\mathcal{H}, S^\dagger]_- &= -Q_c^\dagger, \\ [B, S^\dagger]_- &= -2S^\dagger, & [B, S]_- &= 2S, \\ [D, S]_- &= -iS, & [D, S^\dagger]_- &= -iS^\dagger, \\ [S^\dagger, S]_+ &= 2K, \end{aligned} \tag{88}$$

where,  $\mathcal{H}, D, K, B$  are bosonic operators and  $Q_c, Q_c^\dagger, S, S^\dagger$  are fermionic ones. The supersymmetric extension of the Hamiltonian  $L_0$  (presented in the previous section) is

$$\mathcal{H}_0 = \frac{1}{2}(\mathcal{H} + K), \quad [\mathcal{H}_0, \mathbf{P}]_- = 0. \quad (89)$$

In general, superconformal quantum mechanics has interesting applications in supersymmetric black holes, for example in the problem of a quantum test particle moving in the black hole geometry.

## 6. Summary and conclusion

In this chapter, firstly we start by summarizing the R-deformed Heisenberg algebra or Wigner-Heisenberg algebraic technique for the Wigner quantum oscillator, based on the super-realization of the ladder operators effective spectral resolutions of general oscillator-related potentials.

We illustrate the applications of our operator method to the cases of the Hamiltonians of an isotonic oscillator (harmonic plus a centripetal barrier) system and a 3D isotropic harmonic oscillator for spin  $\frac{1}{2}$  embedded in the bosonic sector of a corresponding Wigner system.

Also, the energy eigenvalues and eigenfunctions of the hydrogen atom via Wigner-Heisenberg (WH) algebra in non-relativistic quantum mechanics, from the ladder operators for the 4-dimensional (4D) super Wigner system, ladder operators for the mapped super 3D system, and hence for hydrogen-like atom in bosonic sector, are deduced. The complete spectrum for the hydrogen atom is found with considerable simplicity by using the Kustaanheimo-Stiefel transformation. From the ladder operators for the four-dimensional (4D) super-Wigner system, ladder operators for the mapped super 3D system, and hence for the hydrogen-like atom in bosonic sector, can be deduced. Results of present investigations on these ladder operators will be reported separately.

For future directions, such a direct algebraic method considered in this chapter proves highly profitable for simpler algebraic treatment, as we shall show in subsequent publications, of other quantum mechanical systems with underlying oscillator connections like for example those of a relativistic electron in a Coulomb potential or of certain 3D SUSY oscillator models of the type of Celka and Hussin. This SUSY model has been reported in nonrelativistic context by Jayaraman and Rodrigues [10]. We will also demonstrate elsewhere the application of our method for a spectral resolution complete of the Pöschl-Teller I and II potentials by virtue of their hidden oscillator connections using the WH algebra operator technique developed in this chapter.

In the work of the Ref. [46], we analyze the Wigner-Heisenberg algebra to bosonic systems in connection with oscillators and, thus, we find a new representation for the Virasoro algebra. Acting the annihilation operator (creation operator) in the Fock basis  $|2m+1, \frac{\epsilon}{2}\rangle$  ( $|2m, \frac{\epsilon}{2}\rangle$ ) the eigenvalue of the ground state of the Wigner oscillator appears only in the excited states associated with the even(odd) quanta. We show that only in the case associated with one even index and one odd index in the operator  $L_n$  the Virasoro algebra is changed.

Recently, we have analyzed the connection between the conformal quantum mechanics and the Wigner-Heisenberg (WH) algebra [46]. With an appropriate relationship between the coupling constant  $g$  and Wigner parameter  $c$  one can identify the Wigner Hamiltonian with the simple Calogero Hamiltonian.

The important result is that the introduction of the WH algebra in the conformal quantum mechanics is still consistent with the conformal symmetry, and a realization of superconformal quantum mechanics in terms of deformed WH algebra is discussed. The spectra for the Casimir operator and the Hamiltonian  $L_0$  depend on the parity operator. The ladder operators depend on the parity operator, too. It is shown, for example, that the eigenvalues of Calogero-type Hamiltonian is dependent of the Wigner parameter  $c$  and the eigenvalues of the parity operator  $P$ . When  $c=0$  we obtain the usual conformal Hamiltonian structure.

We also investigated the supersymmetrization of this model, in that case we obtain a new spectrum for the supersymmetric Hamiltonian of the Calogero interaction's type.

In this case the spectra for the super-Casimir operator and the superhamiltonian depend also on the parity operator. Therefore, we have found a new realization of supersymmetric Calogero-type model on the quantum mechanics context in terms of deformed WH algebra.

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