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Highlighted Aspects from Black Box Fuzzy Modeling for Advanced Control Systems Design

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1. Introduction

This chapter presents an overview of a specific application of computational intelligence techniques, specifically, fuzzy systems: ***fuzzy model based advanced control systems design***. In the last two decades, fuzzy systems have been useful for identification and control of complex nonlinear dynamical systems. This rapid growth, and the interest in this discussion is motivated by the fact that the practical control design, due to the presence of nonlinearity and uncertainty in the dynamical system, fuzzy models are capable of representing the dynamic behavior well enough so that the real controllers designed based on such models can garantee, mathematically, stability and robustness of the control system (Åström et al., 2001; Castillo-Toledo & Meda-Campaña, 2004; Kadmiry & Driankov, 2004; Ren & Chen, 2004; Tong & Li, 2002; Wang & Luoh, 2004; Yoneyama, 2004).

Automatic control systems have become an essential part of our daily life. They are applied in an electroelectronic equipment and up to even at most complex problem as aircraft and rockets. There are different control systems schemes, but in common, all of them have the function to handle a dynamic system to meet certain performance specifications. An intermediate and important control systems design step, is to obtain some knowledge of the plant to be controlled, this is, the dynamic behavior of the plant under different operating conditions. If such knowledge is not available, it becomes difficult to create an efficient control law so that the control system presents the desired performance. A practical approach for controllers design is from the mathematical model of the plant to be controlled.

Mathematical modeling is a set of heuristic and/or computational procedures properly established on a real plant in order to obtain a mathematical equation (models) to represent accurately its dynamic behavior in operation. There are three basic approaches for mathematical modeling:

- `White box modeling`. In this case, such models can be satisfactorily obtained from the physical laws governing the dynamic behavior of the plant. However, this may be a limiting factor in practice, considering plants with uncertainties, nonlinearities, time delay, parametric variations, among other dynamic complexity characteristics. The poor understanding of physical phenomena that govern the plant behavior and the resulting model complexity, makes the white box approach a difficult and time consuming task.

In addition, a complete understanding of the physical behavior of a real plant is almost impossible in many practical applications.

- `Black box modeling`. In this case, if such models, from the physical laws, are difficult or even impossible to obtain, is necessary the task of extracting a model from experimental data related to dynamic behavior of the plant. The modeling problem consists in choosing an appropriate structure for the model, so that enough information about the dynamic behavior of the plant can be extracted efficiently from the experimental data. Once the structure was determined, there is the parameters estimation problem so that a quadratic cost function of the approximation error between the outputs of the plant and the model is minimized. This problem is known as ***systems identification*** and several techniques have been proposed for linear and nonlinear plant modeling. A limitation of this approach is that the structure and parameters of the obtained models usually do not have physical meaning and they are not associated to physical variables of the plant.
- `Gray box modeling`. In this case some information on the dynamic behavior of the plant is available, but the model structure and parameters must be determined from experimental data. This approach, also known as hybrid modeling, combines the features of the white box and black box approaches.

The area of mathematical modeling covers topics from linear regression up to sofisticated concepts related to qualitative information from expert, and great attention have been given to this issue in the academy and industry (Abonyi et al., 2000; Brown & Harris, 1994; Pedrycz & Gomide, 1998; Wang, 1996). A mathematical model can be used for:

- Analysis and better understanding of phenomena (models in engineering, economics, biology, sociology, physics and chemistry);
- Estimate quantities from indirect measurements, where no sensor is available;
- Hypothesis testing (fault diagnostics, medical diagnostics and quality control);
- Teaching through simulators for aircraft, plants in the area of nuclear energy and patients in critical conditions of health;
- Prediction of behavior (adaptive control of time-varying plants);
- Control and regulation around some operating point, optimal control and robust control;
- Signal processing (cancellation of noise, filtering and interpolation);

Modeling techniques are widely used in the control systems design, and successful applications have appeared over the past two decades. There are cases in which the identification procedure is implemented in real time as part of the controller design. This technique, known as adaptive control, is suitable for nonlinear and/or time varying plants. In adaptive control schemes, the plant model, valid in several operating conditions is identified on-line. The controller is designed in accordance to current identified model, in order to garantee the performance specifications. There is a vast literature on modeling and control design (Åström & Wittenmark, 1995; Keesman, 2011; Sastry & Bodson, 1989; Isermann & Münchhof, 2011; Zhu, 2011; Chalam, 1987; Ioannou, 1996; Lewis & Syrmos, 1995; Ljung, 1999; Söderström & Stoica, 1989; Van Overschee & De Moor, 1996; Walter & Pronzato, 1997). Most approaches have a focus on models and controllers described by linear differential or finite differences equations, based on transfer functions or state space representation. Moreover, motivated by the fact that all plant present some type of nonlinear behavior, there are several approaches to analysis, modeling and control of nonlinear plants (Tee et al., 2011; Isidori, 1995; Khalil, 2002; Sjöberg et al., 1995; Ogunfunmi, 2007; Vidyasagar, 2002), and one of the key elements for these applications are the fuzzy systems (Lee et al., 2011; Hellendoorn & Driankov, 1997; Grigorie, 2010; Vukadinovic, 2011; Michels, 2006; Serra & Ferreira, 2011; Nelles, 2011).

2. Fuzzy inference systems

The theory of fuzzy systems has been proposed by Lotfi A. Zadeh (Zadeh, 1965; 1973), as a way of processing vague, imprecise or linguistic information, and since 1970 presents wide industrial application. This theory provides the basis for knowledge representation and developing the essential mechanisms to infer decisions about appropriate actions to be taken on a real problem. Fuzzy inference systems are typical examples of techniques that make use of human knowledge and deductive process. Its structure allows the mathematical modeling of a large class of dynamical behavior, in many applications, and provides greater flexibility in designing high-performance control with a certain degree of transparency for interpretation and analysis, that is, they can be used to explain solutions or be built from expert knowledge in a particular field of interest. For example, although it does not know the exact mathematical model of an oven, one can describe their behavior as follows: " **IF** is applied more power on the heater **THEN** the temperature increases", where ***more*** and ***increases*** are linguistic terms that, while imprecise, they are important information about the behavior of the oven. In fact, for many control problems, an expert can determine a set of efficient control rules based on linguistic descriptions of the plant to be controlled. Mathematical models can not incorporate the traditional linguistic descriptions directly into their formulations. Fuzzy inference systems are powerful tools to achieve this goal, since the logical structure of its **IF** <*antecedent proposition*> **THEN** <*consequent proposition*> rules facilitates the understanding and analysis of the problem in question. According to consequent proposition, there are two types of fuzzy inference systems:

- *Mamdani Fuzzy Inference Systems*: In this type of fuzzy inference system, the antecedent and consequent propositions are linguistic informations.
- *Takagi-Sugeno Fuzzy Inference Systems*: In this type of fuzzy inference system, the antecedent proposition is a linguistic information and the consequent proposition is a functional expression of the linguistic variables defined in the antecedent proposition.

2.1 Mamdani fuzzy inference systems

The Mamdani fuzzy inference system was proposed by E. H. Mamdani (Mamdani, 1977) to capture the qualitative knowledge available in a given application. Without loss of generality, this inference system presents a set of rules of the form:

$$\Re^i : \textbf{IF } \tilde{x}_1 \text{ is } F^i_{j|\tilde{x}_1} \textbf{ AND } \dots \textbf{ AND } \tilde{x}_n \text{ is } F^i_{j|\tilde{x}_n} \textbf{ THEN } \tilde{y} \text{ is } G^i_{j|\tilde{y}} \tag{1}$$

In each rule $i \;|^{[i=1,2,\dots,l]}$, where l is the number of rules, $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ are the linguistic variables of the antecedent (input) and $\tilde{y}$ is the linguistic variable of the consequent (output), defined, respectively, in the own universe of discourse $\mathcal{U}_{\tilde{x}_1}, \ldots, \mathcal{U}_{\tilde{x}_n}$ e $\mathcal{Y}$. The fuzzy sets $F^i_{j|\tilde{x}_1}, F^i_{j|\tilde{x}_2}, \ldots, F^i_{j|\tilde{x}_n}$ e $G^i_{j|\tilde{y}}$, are the linguistic values (terms) used to partition the unierse of discourse of the linguistic variables of antecedent and consequent in the inference system, that is, $F^i_{j|\tilde{x}_t} \in \{F^i_{1|\tilde{x}_t}, F^i_{2|\tilde{x}_t}, \ldots, F^i_{p_{\tilde{x}_t}|\tilde{x}_t}\}^{t=1,2,\ldots,n}$ and $G^i_{j|\tilde{y}} \in \{G^i_{1|\tilde{y}}, G^i_{2|\tilde{y}}, \ldots, G^i_{p_{\tilde{y}}|\tilde{y}}\}$, where $p_{\tilde{x}_t}$ and $p_{\tilde{y}}$ are the partitions number of the universes of discourses associated to the linguistic variables $\tilde{x}_t$ and $\tilde{y}$, respectively. The variable $\tilde{x}_t$ belongs to the fuzzy set $F^i_{j|\tilde{x}_t}$ with a value $\mu^i_{F_{j|\tilde{x}_t}}$ defined by the membership function $\mu^i_{\tilde{x}_t} : R \to [0,1]$, where $\mu^i_{F_{j|\tilde{x}_t}} \in \{\mu^i_{F_{1|\tilde{x}_t}}, \mu^i_{F_{2|\tilde{x}_t}}, \ldots, \mu^i_{F_{p_{\tilde{x}_t}|\tilde{x}_t}}\}$. The variable $\tilde{y}$ belongs to the fuzzy set $G^i_{j|\tilde{y}}$ with a value $\mu^i_{G_{j|\tilde{y}}}$ defined by the membership function $\mu^i_{\tilde{y}} : R \to [0,1]$ where $\mu^i_{G_{j|\tilde{y}}} \in \{\mu^i_{G_{1|\tilde{y}}}, \mu^i_{G_{2|\tilde{y}}}, \ldots, \mu^i_{G_{p_{\tilde{y}}|\tilde{y}}}\}$. Each rule is interpreted by a fuzzy implication

$$\Re^i : \mu^i_{F_{j|\tilde{x}_1}} \star \mu^i_{F_{j|\tilde{x}_2}} \star \ldots \star \mu^i_{F_{j|\tilde{x}_n}} \to \mu^i_{G_{j|\tilde{y}}} \tag{2}$$

where $\star$ is a T-norm, $\mu^i_{F_{j|\tilde{x}_1}} \star \mu^i_{F_{j|\tilde{x}_2}} \star \ldots \star \mu^i_{F_{j|\tilde{x}_n}}$ is the fuzzy relation between the linguistic inputs, on the universes of discourses $\mathcal{U}_{\tilde{x}_1} \times \mathcal{U}_{\tilde{x}_2} \times \ldots \times \mathcal{U}_{\tilde{x}_n}$, and $\mu^i_{G_{j|\tilde{y}}}$ is the linguistic output defined on the universe of discourse $\mathcal{Y}$. The Mamdani inference systems can represent MISO (Multiple Input and Single Output) systems directly, and the set of implications correspond to a unique fuzzy relation in $\mathcal{U}_{\tilde{x}_1} \times \mathcal{U}_{\tilde{x}_2} \times \ldots \times \mathcal{U}_{\tilde{x}_n} \times \mathcal{Y}$ of the form

$$\Re_{MISO} : \bigvee_{i=1}^{l} [\mu^i_{F_{j|\tilde{x}_1}} \star \mu^i_{F_{j|\tilde{x}_2}} \star \ldots \star \mu^i_{F_{j|\tilde{x}_n}} \star \mu^i_{G_{j|\tilde{y}}}] \tag{3}$$

where $\bigvee$ is a S-norm.

The fuzzy output $m \mid^{[m=1,2,\ldots,r]}$ is given by

$$G(\tilde{y}_m) = \Re_{MISO} \circ (\mu^i_{F_{j|\tilde{x}^*_1}} \star \mu^i_{F_{j|\tilde{x}^*_2}} \star \ldots \star \mu^i_{F_{j|\tilde{x}^*_n}}) \tag{4}$$

where $\circ$ is a inference based composition operator, which can be of the type *max-min* or *max-product*, and $\tilde{x}^*_t$ is any point in $\mathcal{U}_{x_t}$. The Mamdani inference systems can represent MIMO (Multiple Input and Multple Output) systems of r outputs by a set of r MISO sub-rules coupled base $\Re^j_{MISO} \mid^{[j=1,2,\ldots,l]}$, that is,

$$\boldsymbol{G}(\tilde{\boldsymbol{y}}) = \Re_{MIMO} \circ (\mu^i_{F_{j|\tilde{x}^*_1}} \star \mu^i_{F_{j|\tilde{x}^*_2}} \star \ldots \star \mu^i_{F_{j|\tilde{x}^*_n}}) \tag{5}$$

with $\boldsymbol{G}(\tilde{\boldsymbol{y}}) = [G(\tilde{y}_1), \ldots, G(\tilde{y}_r)]^T$ and

$$\Re_{MIMO} : \bigcup_{m=1}^{r} \{ \bigvee_{i=1}^{l} [\mu^i_{F_{j|\tilde{x}_1}} \star \mu^i_{F_{j|\tilde{x}_2}} \star \ldots \star \mu^i_{F_{j|\tilde{x}_n}} \star \mu^i_{G_{j|\tilde{y}_m}}]\} \tag{6}$$

where the operator $\bigcup$ represents the set of all fuzzy relations $\Re^j_{MISO}$ associated to each output $\tilde{y}_m$.

2.2 Takagi-Sugeno fuzzy inference systems

The Takagi-Sugeno fuzzy inference system uses in the consequent proposition, a functional expression of the linguistic variables defined in the antecedent proposition (Takagi & Sugeno, 1985). Without loss of generality, the $i\ |^{[i=1,2,...,l]}$-th rule of this inference system, where l is the maximum number of rules, is given by:

$$R^i : \textbf{IF } \tilde{x}_1 \text{ is } F^i_{j|\tilde{x}_1} \textbf{ AND } \ldots \textbf{ AND } \tilde{x}_n \text{ is } F^i_{j|\tilde{x}_n} \textbf{ THEN } \tilde{y}_i = f_i(\tilde{\boldsymbol{x}}) \quad (7)$$

The vector $\tilde{\boldsymbol{x}} \in \Re^n$ contains the linguistic variables of the antecedent proposition. Each linguistic variable has its own universe of discourse $\mathcal{U}_{\tilde{x}_1}, \ldots, \mathcal{U}_{\tilde{x}_n}$ partitioned by fuzzy sets which represent the linguistic terms. The variable $\tilde{x}_t\ |^{t=1,2,...,n}$ belongs to the fuzzy set $F^i_{j|\tilde{x}_t}$ with value $\mu^i_{F_{j|\tilde{x}_t}}$ defined by a membership function $\mu^i_{\tilde{x}_t} : R \rightarrow [0,1]$, with $\mu^i_{F_{j|\tilde{x}_t}} \in \{\mu^i_{F_{1|\tilde{x}_t}}, \mu^i_{F_{2|\tilde{x}_t}}, \ldots, \mu^i_{F_{p_{\tilde{x}_t}|\tilde{x}_t}}\}$, where $p_{\tilde{x}_t}$ is the partitions number of the universe of discourse associated to the linguistic variable $\tilde{x}_t$. The activation degree h_i of the rule i is given by:

$$h_i(\tilde{\boldsymbol{x}}) = \mu^i_{F_{j|\tilde{x}^*_1}} \star \mu^i_{F_{j|\tilde{x}^*_2}} \star \ldots \star \mu^i_{F_{j|\tilde{x}^*_n}} \quad (8)$$

where $\tilde{x}^*_t$ is any point in $\mathcal{U}_{\tilde{x}_t}$. The normalized activation degree of the rule i is defined as:

$$\gamma_i(\tilde{\boldsymbol{x}}) = \frac{h_i(\tilde{\boldsymbol{x}})}{\sum_{r=1}^{l} h_r(\tilde{\boldsymbol{x}})} \quad (9)$$

This normalization implies that

$$\sum_{i=1}^{l} \gamma_i(\tilde{\boldsymbol{x}}) = 1 \quad (10)$$

The response of the Takagi-Sugeno fuzzy inference system is a weighted sum of the functional expressions defined on the consequent proposition of each rule, that is, a convex combination of local functions f_i:

$$y = \sum_{i=1}^{l} \gamma_i(\tilde{\boldsymbol{x}}) f_i(\tilde{\boldsymbol{x}}) \quad (11)$$

Such inference system can be seen as linear parameter varying system. In this sense, the Takagi-Sugeno fuzzy inference system can be considered as a mapping from antecedent space (input) to the convex region (polytope) defined on the local functional expressions in the consequent space. This property allows the analysis of the Takagi-Sugeno fuzzy inference system as a robust system which can be applied in modeling and controllers design for complex plants.

3. Fuzzy computational modeling based control

Many human skills are learned from examples. Therefore, it is natural establish this "didactic principle" in a computer program, so that it can learn how to provide the desired output as function of a given input. The Computational intelligence techniques, basically derived from the theory of Fuzzy Systems, associated to computer programs, are able to process numerical data and/or linguistic information, whose parameters can be adjusted from examples. The examples represent what these systems should respond when subjected to a particular input. These techniques use a numeric representation of knowledge, demonstrate adaptability and fault tolerance in contrast to the classical theory of artificial intelligence that uses symbolic representation of knowledge. The human knowledge, in turn, can be classified into two categories:

1. *Objective knowledge*: This kind of knowledge is used in the engineering problems formulation and is defined by mathematical equations (mathematical model of a submarine, aircraft or robot; statistics analysis of the communication channel behaviour; Newton's laws for motion analysis and Kirchhoff's Laws for circuit analysis).
2. *Subjective knowledge*: This kind of knowledge represents the linguistic informations defined through set of rules, knowledge from expert and design specifications, which are usually impossible to be described quantitatively.

Fuzzy systems are able to coordinate both types of knowledge to solve real problems. The necessity of expert and engineers to deal with increasingly complex control systems problems, has enabled via computational intelligence techniques, the identification and control of real plants with difficult mathematical modeling. The computational intelligence techniques, once related to classical and modern control techniques, allow the use of constraints in its formulation and satisfaction of robustness and stability requirements in an efficient and practical form. The implementation of intelligent systems, especially from 70's, has been characterized by the growing need to improve the efficiency of industrial control systems in the following aspects: increasing product quality, reduced losses, and other factors related to the improvement of the disabilities of the identification and control methods. The intelligent identification and control methodologies are based on techniques motivated by biological systems, human intelligence, and have been introduced exploring alternative representations schemes from the natural language, rules, semantic networks or qualitative models.

The research on fuzzy inference systems has been developed in two main directions. The first direction is the linguistic or qualitative information, in which the fuzzy inference system is developed from a collection of rules (propositions). The second direction is the quantitative information and is related to the theory of classical and modern systems. The combination of the qualitative and quantitative informations, which is the main motivation for the use of intelligent systems, has resulted in several contributions on stability and robustness of advanced control systems. In (Ding, 2011) is addressed the output feedback predictive control for a fuzzy system with bounded noise. The controller optimizes an infinite-horizon objective function respecting the input and state constraints. The control law is parameterized as a dynamic output feedback that is dependent on the membership functions, and the closed-loop stability is specified by the notion of quadratic boundedness. In (Wang et al., 2011) is considered the problem of fuzzy control design for a class of nonlinear distributed parameter systems that is described by first-order hyperbolic partial differential equations (PDEs), where the control actuators are continuously distributed in space. The goal of this methodology is to develop a fuzzy state-feedback control design methodology for these systems by employing a combination of PDE theory and concepts from Takagi-Sugeno fuzzy control. First, the Takagi-Sugeno fuzzy hyperbolic PDE model is proposed to accurately represent the nonlinear first-order hyperbolic PDE system. Subsequently, based on the Takagi-Sugeno fuzzy-PDE model, a Lyapunov technique is used to analyze the closed-loop exponential stability with a given decay rate. Then, a fuzzy state-feedback control design procedure is developed in terms of a set of spatial differential linear matrix inequalities (SDLMIs) from the resulting stability conditions. The developed design methodology is successfully applied to the control of a nonisothermal plug-flow reactor. In (Sadeghian & Fatehi, 2011) is used a nonlinear system identification method to predict and detect process fault of a cement rotary kiln from the White Saveh Cement Company. After selecting proper inputs and output, an inputŰoutput locally linear neuro-fuzzy (LLNF) model is identified for the plant in various operation points in the kiln. In (Li & Lee, 2011) an observer-based adaptive controller is developed from a hierarchical fuzzy-neural network (HFNN) is employed to solve the controller time-delay problem for a class of multi-input multi-output(MIMO) non-affine nonlinear systems under the constraint that only system outputs are available for measurement. By using the implicit function theorem and Taylor series expansion, the observer-based control law and the weight update law of the HFNN adaptive controller are derived. According to the design of the HFNN hierarchical fuzzy-neural network, the observer-based adaptive controller can alleviate the online computation burden and can guarantee that all signals involved are bounded and that the outputs of the closed-loop system track asymptotically the desired output trajectories.

Fuzzy inference systems are widely found in the following areas: Control Applications - aircraft (Rockwell Corp.), cement industry and motor/valve control (Asea Brown Boveri Ltd.), water treatment and robots control (Fuji Electric), subway system (Hitachi), board control (Nissan), washing machines (Matsushita, Hitachi), air conditioning system (Mitsubishi); Medical Technology - cancer diagnosis (Kawasaki medical School); Modeling and Optimization - prediction system for earthquakes recognition (Institute of Seismology Bureau of Metrology, Japan); Signal Processing For Adjustment and Interpretation - vibration compensation in video camera (Matsushita), video image stabilization (Matsushita / Panasonic), object and voice recognition (CSK, Hitachi Hosa Univ., Ricoh), adjustment of images on TV (Sony). Due to the development, the many practical possibilities and the commercial success of their applications, the theory of fuzzy systems have a wide acceptance in academic community as well as industrial applications for modeling and advanced control systems design.

4. Takagi-Sugeno fuzzy black box modeling

This section aims to illustrate the problem of black box modeling, well known as systems identification, addressing the use of Takagi-Sugeno fuzzy inference systems. The nonlinear input-output representation is often used for building TS fuzzy models from data, where the regression vector is represented by a finite number of past inputs and outputs of the system. In this work, the nonlinear autoregressive with exogenous input (NARX) structure model is used. This model is applied in most nonlinear identification methods such as neural networks, radial basis functions, cerebellar model articulation controller (CMAC), and also fuzzy logic. The NARX model establishes a relation between the collection of past scalar input-output data and the predicted output

$$y_{k+1} = F[y_k, \ldots, y_{k-n_y+1}, u_k, \ldots, \ldots, u_{k-n_u+1}] \tag{12}$$

where k denotes discrete time samples, n_y and n_u are integers related to the system's order. In terms of rules, the model is given by

$$R^i : \text{IF } y_k \text{ is } F_1^i \text{ AND } \cdots \text{ AND } y_{k-n_y+1} \text{ is } F_{n_y}^i \text{ AND } u_k \text{ is } G_1^i \text{ AND } \cdots \text{ AND } u_{k-n_u+1} \text{ is } G_{n_u}^i$$
$$\text{THEN } \hat{y}_{k+1}^i = \sum_{j=1}^{n_y} a_{i,j} y_{k-j+1} + \sum_{j=1}^{n_u} b_{i,j} u_{k-j+1} + c_i \tag{13}$$

where $a_{i,j}$, $b_{i,j}$ and c_i are the consequent parameters to be determined. The inference formula of the TS fuzzy model is a straightforward extension of (11) and is given by

$$y_{k+1} = \frac{\sum_{i=1}^{l} h_i(\boldsymbol{x}) \hat{y}_{k+1}^i}{\sum_{i=1}^{l} h_i(\boldsymbol{x})} \tag{14}$$

or

$$y_{k+1} = \sum_{i=1}^{l} \gamma_i(\boldsymbol{x}) \hat{y}_{k+1}^i \tag{15}$$

with

$$\boldsymbol{x} = [y_k, \ldots, y_{k-n_y+1}, u_k, \ldots, u_{k-n_u+1}] \tag{16}$$

and $h_i(\boldsymbol{x})$ is given as (8). This NARX model represents multiple input and single output (MISO) systems directly and multiple input and multiple output (MIMO) systems in a decomposed form as a set of coupled MISO models.

4.1 Antecedent parameters estimation problem

The experimental data based antecedent parameters estimation can be done by fuzzy clustring algorithms. A cluster is a group of similar objects. The term "similarity" should be understood as mathematical similarity measured in some well-define sense. In metric spaces, similarity is often defined by means of a distance norm. Distance can be measured from data vector to some cluster prototypical (center). Data can reveal clusters of different geometric shapes, sizes and densities. The clusters also can be characterized as linear and nonlinear subspaces of the data space.

The objective of clustering is partitioning the data set $\boldsymbol{Z}$ into c clusters. Assume that c is known, based on priori knowledge. The fuzzy partition of $\boldsymbol{Z}$ can be defined as a family of subsets $\{A_i | 1 \leq i \leq c\} \subset P(Z)$, with the following properties:

$$\bigcup_{i=1}^{c} A_i = Z \tag{17}$$

$$A_i \cap A_j = 0 \tag{18}$$

$$0 \subset A_i \subset Z_i \tag{19}$$

Equation (17) means that the subsets A_i collectively contain all the data in $\boldsymbol{Z}$. The subsets must be disjoint, as stated by (18), and none off them is empty nor contains all the data in $\boldsymbol{Z}$, as stated by (19). In terms of membership functions, μ_{A_i} is the membership function of A_i. To simplifly the notation, in this paper is used μ_{ik} instead of $\mu_i(z_k)$. The $c \times N$ matrix $\boldsymbol{U} = [\mu_{ik}]$ represents a fuzzy partitioning space if and only if:

$$M_{fc} = \left\{ \boldsymbol{U} \in \Re^{c \times N} | \mu_{ik} \in [0,1], \forall i, k; \sum_{i=1}^{c} \mu_{ik} = 1, \forall k; 0 < \sum_{k=1}^{N} \mu_{ik} < N, \forall i \right\} \tag{20}$$

The i-th row of the fuzzy partition matrix $\boldsymbol{U}$ contains values of the i-th membership function of the fuzzy subset A_i of $\boldsymbol{Z}$. The clustering algorithm optimizes an initial set of centroids by minimizing a cost function J in an iterative process. This function is usually formulated as:

$$\mathbf{J}(\mathbf{Z}; \mathbf{U}, \mathbf{V}, \mathbf{A}) = \sum_{i=1}^{c} \sum_{k=1}^{N} \mu_{ik}^{m} D_{ikA_i}^{2} \tag{21}$$

where, $\mathbf{Z} = \{z_1, z_2, \cdots, z_N\}$ is a finite data set. $\mathbf{U} = [\mu_{ik}] \in M_{fc}$ is a fuzzy partition of $\mathbf{Z}$. $\mathbf{V} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_c\}, \mathbf{v}_i \in \Re^n$, is a vector of cluster prototypes (centers). $\mathbf{A}$ denote a c-tuple of the norm-induting matrices: $\boldsymbol{A} = (\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_c)$. $D^2_{ik\boldsymbol{A}_i}$ is a square inner-product distance norm. The $m \in [1, \infty)$ is a weighting exponent which determines the fuzziness of the clusters. The clustering algorithms differ in the choice of the norm distance. The norm metric influences the clustering criterion by changing the measure of dissimilarity. The Euclidean norm induces hiperspherical clusters. It's characterizes the FCM algorithm, where the norm-inducing matrix $\boldsymbol{A}_{i_{FCM}}$ is equal to identity matrix ($\boldsymbol{A}_{i_{FCM}} = \boldsymbol{I}$), which strictly imposes a circular shape to all clusters. The Euclidean Norm is given by:

$$D^2_{ik_{FCM}} = (z_k - v_i)^T \mathbf{A}_{i_{FCM}} (z_k - v_i) \tag{22}$$

An adaptative distance norm in order to detect clusters of different geometrical shapes in a data set characterizes the ***GK** algorithm*:

$$D^2_{ik_{GK}} = (z_k - v_i)^T \boldsymbol{A}_{i_{GK}} (z_k - v_i) \tag{23}$$

In this algorithm, each cluster has its own norm-inducing matrix $\boldsymbol{A}_{i_{GK}}$, where each cluster adapts the distance norm to the local topological structure of the data set. $\boldsymbol{A}_{i_{GK}}$ is given by:

$$\boldsymbol{A}_{i_{GK}} = [\rho_i det(\boldsymbol{F}_i)]^{1/n} \boldsymbol{F}_i^{-1}, \tag{24}$$

where ρ_i is cluster volume, usually fixed in 1. The n is data dimension. The $\boldsymbol{F}_i$ is fuzzy covariance matrix of the i-th cluster defined by:

$$\boldsymbol{F}_i = \frac{\sum_{k=1}^{N} (\mu_{ik})^m (z_k - v_i)(z_k - v_i)^T}{\sum_{k=1}^{N} (\mu_{ik})^m} \tag{25}$$

The eigenstructure of the cluster covariance matrix provides information about the shape and orientation cluster. The ratio of the hyperellipsoid axes is given by the ratio of the square roots of the eigenvalues of $\mathbf{F}_i$. The directions of the axes are given by the eigenvectores of $\mathbf{F}_i$. The eigenvector corresponding to the smallest eigenvalue determines the normal to the hyperplane, and it can be used to compute optimal local linear models from the covariance matrix. The fuzzy maximum likelihood estimates (FLME) algorithm employs a distance norm based on maximum lekelihood estimates:

$$D_{ik_{FLME}} = \frac{\sqrt{\boldsymbol{G}_{i_{FLME}}}}{P_i} \exp\left[\frac{1}{2}\left(z_k - v_i\right)^T \boldsymbol{F}_{i_{FLME}}^{-1} \left(z_k - v_i\right)\right] \tag{26}$$

Note that, contrary to the GK algorithm, this distance norm involves an exponential term and decreases faster than the inner-product norm. The $\boldsymbol{F}_{i_{FLME}}$ denotes the fuzzy covariance matrix of the i-th cluster, given by (25). When m is equal to 1, it has a strict algorithm FLME. If m is greater than 1, it has a extended algorithm FLME, or Gath-Geva (GG) algorithm. Gath and Geva reported that the FLME algorithm is able to detect clusters of varying shapes, sizes and densities. This is because the cluster covariance matrix is used in conjuncion with an "exponential" distance, and the clusters are not constrained in volume. P_i is the prior probability of selecting cluster i, given by:

$$\boldsymbol{P}_i = \frac{1}{N} \sum_{k=1}^{N} \left(\mu_{ik}\right)^m \tag{27}$$

4.2 Consequent parameters estimation problem

The inference formula of the TS fuzzy model in (15) can be expressed as

$$\begin{aligned} y_{k+1} = {} & \gamma_1(\boldsymbol{x}_k)[a_{1,1}y_k + \ldots + a_{1,ny}y_{k-n_y+1} + b_{1,1}u_k + \ldots + b_{1,nu}u_{k-n_u+1} + c_1] + \\ & \gamma_2(\boldsymbol{x}_k)[a_{2,1}y_k + \ldots + a_{2,ny}y_{k-n_y+1} + b_{2,1}u_k + \ldots + b_{2,nu}u_{k-n_u+1} + c_2] + \\ & \vdots \\ & \gamma_l(\boldsymbol{x}_k)[a_{l,1}y_k + \ldots + a_{l,ny}y_{k-n_y+1} + b_{l,1}u_k + \ldots + b_{l,nu}u_{k-n_u+1} + c_l] \end{aligned} \tag{28}$$

which is linear in the consequent parameters: $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. For a set of N input-output data pairs $\{(\boldsymbol{x}_k, y_k) | i = 1, 2, \ldots, N\}$ available, the following vetorial form is obtained

$$\boldsymbol{Y} = [\boldsymbol{\psi}_1\boldsymbol{X}, \boldsymbol{\psi}_2\boldsymbol{X}, \ldots, \boldsymbol{\psi}_l\boldsymbol{X}]\boldsymbol{\theta} + \Xi \tag{29}$$

where $\boldsymbol{\psi}_i = diag(\gamma_i(\boldsymbol{x}_k)) \in \Re^{N\times N}$, $\boldsymbol{X} = [\boldsymbol{y}_k, \ldots, \boldsymbol{y}_{k-ny+1}, \boldsymbol{u}_k, \ldots, \boldsymbol{u}_{k-nu+1}, \mathbf{1}] \in \Re^{N\times(n_y+n_u+1)}$, $\boldsymbol{Y} \in \Re^{N\times 1}$, $\Xi \in \Re^{N\times 1}$ and $\boldsymbol{\theta} \in \Re^{l(n_y+n_u+1)\times 1}$ are the normalized membership degree matrix of (9), the data matrix, the output vector, the approximation error vector and the estimated parameters vector, respectively. If the unknown parameters associated variables are *exactly known* quantities, then the least squares method can be used efficiently. However, in practice, and in the present context, the elements of $\boldsymbol{X}$ are no exactly known quantities so that its value can be expressed as

$$y_k = \chi_k^T\boldsymbol{\theta} + \eta_k \tag{30}$$

where, at the k-th sampling instant, $\boldsymbol{\chi}_k^T = [\gamma_k^1(\boldsymbol{x}_k + \boldsymbol{\xi}_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \boldsymbol{\xi}_k)]$ is the vector of the data with error in variables, $\boldsymbol{x}_k = [y_{k-1}, \ldots, y_{k-n_y}, u_{k-1}, \ldots, u_{k-n_u}, 1]^T$ is the vector of the data with exactly known quantities, e.g., free noise input-output data, $\boldsymbol{\xi}_k$ is a vector of noise associated with the observation of $\boldsymbol{x}_k$, and $\boldsymbol{\eta}_k$ is a disturbance noise.

The normal equations are formulated as

$$[\sum_{j=1}^{k} \boldsymbol{\chi}_j \boldsymbol{\chi}_j^T]\hat{\boldsymbol{\theta}}_k = \sum_{j=1}^{k} \boldsymbol{\chi}_j y_j \tag{31}$$

and multiplying by $\frac{1}{k}$ gives

$$\{\frac{1}{k}\sum_{j=1}^{k}[\gamma_j^1(\boldsymbol{x}_j + \boldsymbol{\xi}_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \boldsymbol{\xi}_j)][\gamma_j^1(\boldsymbol{x}_j + \boldsymbol{\xi}_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \boldsymbol{\xi}_j)]^T\}\hat{\boldsymbol{\theta}}_k = \frac{1}{k}\sum_{j=1}^{k}[\gamma_j^1(\boldsymbol{x}_j + \boldsymbol{\xi}_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \boldsymbol{\xi}_j)]y_j \tag{32}$$

Noting that $y_j = \boldsymbol{\chi}_j^T \boldsymbol{\theta} + \eta_j$,

$$\{\frac{1}{k}\sum_{j=1}^{k}[\gamma_j^1(\boldsymbol{x}_j + \boldsymbol{\xi}_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \boldsymbol{\xi}_j)][\gamma_j^1(\boldsymbol{x}_j + \boldsymbol{\xi}_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \boldsymbol{\xi}_j)]^T\}\hat{\boldsymbol{\theta}}_k = \frac{1}{k}\sum_{j=1}^{k}[\gamma_j^1(\boldsymbol{x}_j + \boldsymbol{\xi}_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \boldsymbol{\xi}_j)][\gamma_j^1(\boldsymbol{x}_j + \boldsymbol{\xi}_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \boldsymbol{\xi}_j)]^T\boldsymbol{\theta} + \frac{1}{k}\sum_{j=1}^{k}[\gamma_j^1(\boldsymbol{x}_j + \boldsymbol{\xi}_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \boldsymbol{\xi}_j)]\eta_j \tag{33}$$

and

$$\tilde{\boldsymbol{\theta}}_k = \{\frac{1}{k}\sum_{j=1}^{k}[\gamma_j^1(\boldsymbol{x}_j + \boldsymbol{\xi}_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \boldsymbol{\xi}_j)][\gamma_j^1(\boldsymbol{x}_j + \boldsymbol{\xi}_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \boldsymbol{\xi}_j)]^T\}^{-1}\frac{1}{k}\sum_{j=1}^{k}[\gamma_j^1(\boldsymbol{x}_j + \boldsymbol{\xi}_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \boldsymbol{\xi}_j)]\eta_j \tag{34}$$

where $\tilde{\boldsymbol{\theta}}_k = \hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}$ is the parameter error. Taking the probability in the limit as $k \to \infty$,

$$\text{p.lim}\,\tilde{\boldsymbol{\theta}}_k = \text{p.lim}\,\{\frac{1}{k}\boldsymbol{C}_k^{-1}\frac{1}{k}\boldsymbol{b}_k\} \tag{35}$$

with

$$\boldsymbol{C}_k = \sum_{j=1}^{k}[\gamma_j^1(\boldsymbol{x}_j + \boldsymbol{\xi}_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \boldsymbol{\xi}_j)][\gamma_j^1(\boldsymbol{x}_j + \boldsymbol{\xi}_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \boldsymbol{\xi}_j)]^T$$

$$\boldsymbol{b}_k = \sum_{j=1}^{k}[\gamma_j^1(\boldsymbol{x}_j + \boldsymbol{\xi}_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \boldsymbol{\xi}_j)]\eta_j$$

Applying Slutsky's theorem and assuming that the elements of $\frac{1}{k}\boldsymbol{C}_k$ and $\frac{1}{k}\boldsymbol{b}_k$ converge in probability, we have

$$\text{p.lim}\,\tilde{\boldsymbol{\theta}}_k = \text{p.lim}\,\frac{1}{k}\boldsymbol{C}_k^{-1}\,\text{p.lim}\,\frac{1}{k}\boldsymbol{b}_k \tag{36}$$

Thus,

$$\text{p.lim}\,\frac{1}{k}\boldsymbol{C}_k = \text{p.lim}\,\frac{1}{k}\sum_{j=1}^{k}[\gamma_j^1(\boldsymbol{x}_j+\boldsymbol{\xi}_j),\ldots,\gamma_j^l(\boldsymbol{x}_j+\boldsymbol{\xi}_j)][\gamma_j^1(\boldsymbol{x}_j+\boldsymbol{\xi}_j),\ldots,\gamma_j^l(\boldsymbol{x}_j+\boldsymbol{\xi}_j)]^T$$

$$\text{p.lim}\,\frac{1}{k}\boldsymbol{C}_k = \text{p.lim}\,\frac{1}{k}\sum_{j=1}^{k}(\gamma_j^1)^2(\boldsymbol{x}_j+\boldsymbol{\xi}_j)(\boldsymbol{x}_j+\boldsymbol{\xi}_j)^T+\ldots+\text{p.lim}\,\frac{1}{k}\sum_{j=1}^{k}(\gamma_j^l)^2(\boldsymbol{x}_j+\boldsymbol{\xi}_j)(\boldsymbol{x}_j+\boldsymbol{\xi}_j)^T$$

Assuming $\boldsymbol{x}_j$ and $\boldsymbol{\xi}_j$ statistically independent,

$$\text{p.lim}\,\frac{1}{k}\boldsymbol{C}_k = \text{p.lim}\,\frac{1}{k}\sum_{j=1}^{k}(\gamma_j^1)^2[\boldsymbol{x}_j\boldsymbol{x}_j^T+\boldsymbol{\xi}_j\boldsymbol{\xi}_j^T]+\ldots+\text{p.lim}\,\frac{1}{k}\sum_{j=1}^{k}(\gamma_j^l)^2[\boldsymbol{x}_j\boldsymbol{x}_j^T+\boldsymbol{\xi}_j\boldsymbol{\xi}_j^T]$$

$$\text{p.lim}\,\frac{1}{k}\boldsymbol{C}_k = \text{p.lim}\,\frac{1}{k}\sum_{j=1}^{k}\boldsymbol{x}_j\boldsymbol{x}_j^T[(\gamma_j^1)^2+\ldots+(\gamma_j^l)^2]+\text{p.lim}\,\frac{1}{k}\sum_{j=1}^{k}\boldsymbol{\xi}_j\boldsymbol{\xi}_j^T[(\gamma_j^1)^2+\ldots+(\gamma_j^l)^2] \tag{37}$$

with $\sum_{i=1}^{l}\gamma_j^i = 1$. Hence, the asymptotic analysis of the TS fuzzy model consequent parameters estimation is based in a weighted sum of the fuzzy covariance matrices of $\boldsymbol{x}$ and $\boldsymbol{\xi}$. Similarly,

$$\text{p.lim}\,\frac{1}{k}\boldsymbol{b}_k = \text{p.lim}\,\frac{1}{k}\sum_{j=1}^{k}[\gamma_j^1(\boldsymbol{x}_j+\boldsymbol{\xi}_j),\ldots,\gamma_j^l(\boldsymbol{x}_j+\boldsymbol{\xi}_j)]\eta_j$$

$$\text{p.lim}\,\frac{1}{k}\boldsymbol{b}_k = \text{p.lim}\,\frac{1}{k}\sum_{j=1}^{k}[\gamma_j^1\boldsymbol{\xi}_j\eta_j,\ldots,\gamma_j^l\boldsymbol{\xi}_j\eta_j] \tag{38}$$

Substituting from (37) and (38) in (36), results

$$\text{p.lim}\,\tilde{\boldsymbol{\theta}}_k = \{\text{p.lim}\,\frac{1}{k}\sum_{j=1}^{k}\boldsymbol{x}_j\boldsymbol{x}_j^T[(\gamma_j^1)^2+\ldots+(\gamma_j^l)^2]+\text{p.lim}\,\frac{1}{k}\sum_{j=1}^{k}\boldsymbol{\xi}_j\boldsymbol{\xi}_j^T[(\gamma_j^1)^2+\ldots$$

$$+(\gamma_j^l)^2]\}^{-1}\text{p.lim}\,\frac{1}{k}\sum_{j=1}^{k}[\gamma_j^1\boldsymbol{\xi}_j\eta_j,\ldots,\gamma_j^l\boldsymbol{\xi}_j\eta_j] \tag{39}$$

with $\sum_{i=1}^{l}\gamma_j^i = 1$. For the case of only one rule ($l=1$), the analysis is simplified to the linear one, with $\gamma_j^i\,|_{j=1,\ldots,k}^{i=1} = 1$. Thus, this analysis, which is a contribution of this article, is an extension of the standard linear one, from which can result several studies for fuzzy filtering and modeling in a noisy environment, fuzzy signal enhancement in communication channel, and so forth.

Provided that the input u_k continues to excite the process and, at the same time, the coefficients in the submodels from the consequent are not all zero, then the output y_k will exist for all k observation intervals. As a result, the fuzzy covariance matrix $\sum_{j=1}^{k} \boldsymbol{x}_j \boldsymbol{x}_j^T[(\gamma_j^1)^2 + \ldots + (\gamma_j^l)^2]$ will also be non-singular and its inverse will exist. Thus, the only way in which the asymptotic error can be zero is for $\boldsymbol{\xi}_j \eta_j$ identically zero. But, in general, $\boldsymbol{\xi}_j$ and η_j are correlated, the asymptotic error will not be zero and the least squares estimates will be asymptotically biased to an extent determined by the relative ratio of noise to signal variances. In other words, least squares method is not appropriate to estimate the TS fuzzy model consequent parameters in a noisy environment because the estimates will be inconsistent and the bias error will remain no matter how much data can be used in the estimation.

As a consequence of this analysis, the definition of the vector $[\beta_j^1 \mathbf{z}_j, \ldots, \beta_j^l \mathbf{z}_j]$ as *fuzzy instrumental variable vector* or simply the *fuzzy instrumental variable* (FIV) is proposed. Clearly, with the use of the FIV vector in the form suggested, becomes possible to eliminate the asymptotic bias while preserving the existence of a solution. However, the statistical efficiency of the solution is dependent on the degree of correlation between $[\beta_j^1 \boldsymbol{z}_j, \ldots, \beta_j^l \boldsymbol{z}_j]$ and $[\gamma_j^1 \boldsymbol{x}_j, \ldots, \gamma_j^l \boldsymbol{x}_j]$. In particular, the lowest variance estimates obtained from this approach occur only when $\boldsymbol{z}_j = \boldsymbol{x}_j$ and $\beta_j^i \,|_{j=1,\ldots,k}^{i=1,\ldots,l} = \gamma_j^i \,|_{j=1,\ldots,k}^{i=1,\ldots,l}$, i.e., when the $\boldsymbol{z}_j$ are equal to the dynamic system "free noise" variables, which are unavailable in practice. According to situation, several fuzzy instrumental variables can be chosen. An effective choice of FIV would be the one based on the delayed input sequence

$$\boldsymbol{z}_j = [u_{k-\tau}, \ldots, u_{k-\tau-n}, u_k, \ldots, u_{k-n}]^T$$

where τ is chosen so that the elements of the fuzzy covariance matrix $\boldsymbol{C_{zx}}$ are maximized. In this case, the input signal is considered persistently exciting, e.g., it continuously perturbs or excites the system. Another FIV would be the one based on the delayed input-output sequence

$$\boldsymbol{z}_j = [y_{k-1-dl}, \cdots, y_{k-n_y-dl}, u_{k-1-dl}, \cdots, u_{k-n_u-dl}]^T$$

where dl is the applied delay. Other FIV could be the one based in the input-output from a "fuzzy auxiliar model" with the same structure of the one used to identify the nonlinear dynamic system. Thus,

$$\boldsymbol{z}_j = [\hat{y}_{k-1}, \cdots, \hat{y}_{k-n_y}, u_{k-1}, \cdots, u_{k-n_u}]^T$$

where $\hat{y}_k$ is the output of the fuzzy auxiliar model, and u_k is the input of the dynamic system. The inference formula of this fuzzy auxiliar model is given by

$$\begin{aligned}
\hat{y}_{k+1} = {} & \beta_1(\boldsymbol{z}_k)[\alpha_{1,1}\hat{y}_k + \ldots + \alpha_{1,ny}\hat{y}_{k-n_y+1} + \rho_{1,1}u_k + \ldots + \rho_{1,nu}u_{k-n_u+1} + \delta_1] + \\
& \beta_2(\boldsymbol{z}_k)[\alpha_{2,1}\hat{y}_k + \ldots + \alpha_{2,ny}\hat{y}_{k-n_y+1} + \rho_{2,1}u_k + \ldots + \rho_{2,nu}u_{k-n_u+1} + \delta_2] + \\
& \vdots \\
& \beta_l(\boldsymbol{z}_k)[\alpha_{l,1}\hat{y}_k + \ldots + \alpha_{l,ny}\hat{y}_{k-n_y+1} + \rho_{l,1}u_k + \ldots + \rho_{l,nu}u_{k-n_u+1} + \delta_l]
\end{aligned} \tag{40}$$

which is also linear in the consequent parameters: α, ρ and δ. The closer these parameters are to the actual, but unknown, system parameters ($\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$), more correlated $\boldsymbol{z}_k$ and $\boldsymbol{x}_k$ will be, and the obtained FIV estimates closer to the optimum.

4.2.1 Batch processing scheme

The normal equations are formulated as

$$\sum_{j=1}^{k}[\beta_j^1\boldsymbol{z}_j,\ldots,\beta_j^l\boldsymbol{z}_j][\gamma_j^1(\boldsymbol{x}_j+\xi_j),\ldots,\gamma_j^l(\boldsymbol{x}_j+\xi_j)]^T\hat{\theta}_k-\sum_{j=1}^{k}[\beta_j^1\boldsymbol{z}_j,\ldots,\beta_j^l\boldsymbol{z}_j]y_j=0 \quad (41)$$

or, with $\zeta_j=[\beta_j^1\boldsymbol{z}_j,\ldots,\beta_j^l\boldsymbol{z}_j]$,

$$[\sum_{j=1}^{k}\zeta_j\chi_j^T]\hat{\theta}_k-\sum_{j=1}^{k}\zeta_j y_j=0 \quad (42)$$

so that the FIV estimate is obtained as

$$\hat{\theta}_k=\{\sum_{j=1}^{k}[\beta_j^1\boldsymbol{z}_j,\ldots,\beta_j^l\boldsymbol{z}_j][\gamma_j^1(\boldsymbol{x}_j+\xi_j),\ldots,\gamma_j^l(\boldsymbol{x}_j+\xi_j)]^T\}^{-1}\sum_{j=1}^{k}[\beta_j^1\boldsymbol{z}_j,\ldots,\beta_j^l\boldsymbol{z}_j]y_j \quad (43)$$

and, in vectorial form, the interest problem may be placed as

$$\hat{\theta}=(\Gamma^T\Sigma)^{-1}\Gamma^T\boldsymbol{Y} \quad (44)$$

where $\Gamma^T\in\Re^{l(n_y+n_u+1)\times N}$ is the fuzzy extended instrumental variable matrix with rows given by ζ_j, $\Sigma\in\Re^{N\times l(n_y+n_u+1)}$ is the fuzzy extended data matrix with rows given by χ_j and $\boldsymbol{Y}\in\Re^{N\times 1}$ is the output vector and $\hat{\theta}\in\Re^{l(n_y+n_u+1)\times 1}$ is the parameters vector. The models can be obtained by the following two approaches:

- *Global approach* : In this approach all linear consequent parameters are estimated simultaneously, minimizing the criterion:

$$\hat{\theta}=\arg\min \parallel \Gamma^T\Sigma\theta-\Gamma^T\boldsymbol{Y}\parallel_2^2 \quad (45)$$

- *Local approach* : In this approach the consequent parameters are estimated for each rule i, and hence independently of each other, minimizing a set of weighted local criteria ($i=1,2,\ldots,l$):

$$\hat{\theta}_i=\arg\min \parallel \boldsymbol{Z}^T\Psi_i\boldsymbol{X}\theta_i-\boldsymbol{Z}^T\Psi_i\boldsymbol{Y}\parallel_2^2 \quad (46)$$

  where $\boldsymbol{Z}^T$ has rows given by $\boldsymbol{z}_j$ and Ψ_i is the normalized membership degree diagonal matrix according to $\boldsymbol{z}_j$.

Example 1. So that the readers can understand the definitions of global and local fuzzy modeling estimations, consider the following second-order polynomial given by

$$y=2u_k^2-4u_k+3 \quad (47)$$

where u_k is the input and y_k is the output, respectively. The TS fuzzy model used to approximate this polynomial has the following structure with 2 rules:

$$R^i: \text{ IF } u_k \text{ is } F_i \text{ THEN } \hat{y}_k = a_0 + a_1 u_k$$

where $i = 1, 2$. It was choosen the points $u_k = -0.5$ and $u_k = 0.5$ to analysis the consequent models obtained by global and local estimation, and it was defined triangular membership functions for $-0.5 \leq u_k \leq 0.5$ in the antecedent. The following rules were obtained:

Local estimation:

$$R^1: \text{ IF } u_k \text{ is } -0.5 \text{ THEN } \hat{y} = 3.1000 - 4.4012u_k$$
$$R^2: \text{ IF } u_k \text{ is } +0.5 \text{ THEN } \hat{y} = 3.1000 - 3.5988u_k$$

Global estimation:

$$R^1: \text{ IF } u_k \text{ is } -0.5 \text{ THEN } \hat{y} = 4.6051 - 1.7503u_k$$
$$R^2: \text{ IF } u_k \text{ is } +0.5 \text{ THEN } \hat{y} = 1.3464 + 0.3807u_k$$

The application of local and global estimation to the TS fuzzy model results in the consequent models given in Fig. 1. The consequent models obtained by local estimation describe properly the local behavior of the function and the fuzzy model can easily be interpreted in terms of the local behavior (the rule consequents). The consequent models obtained by global estimation are not relevant for the local behavior of the nonlinear function. The fit of the function is

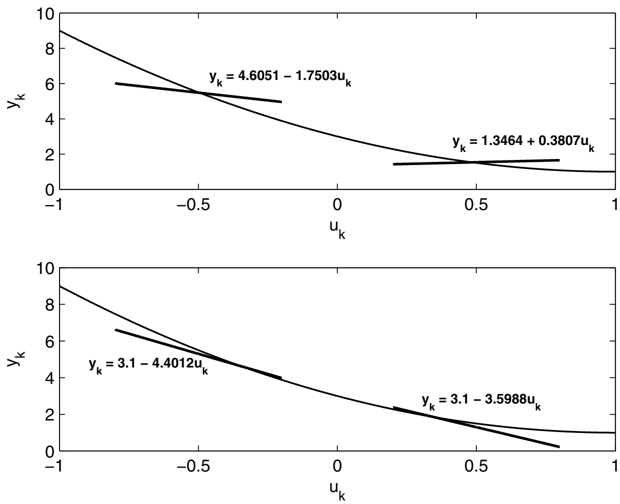

Fig. 1. The nonlinear function and the result of global (top) and local (bottom) estimation of the consequent parameters of the TS fuzzy models.

shown in Fig. 2. The global estimation gives a good fit and a minimal prediction error, but it bias the estimates of the consequent as parameters of local models. In the local estimation a larger prediction error is obtained than with global estimation, but it gives locally relevant parameters of the consequent. This is the tradeoff between local and global estimation. All the results of the Example 1 can be extended for any nonlinear estimation problem and they would be considered for computational and experimental results analysis in this paper.

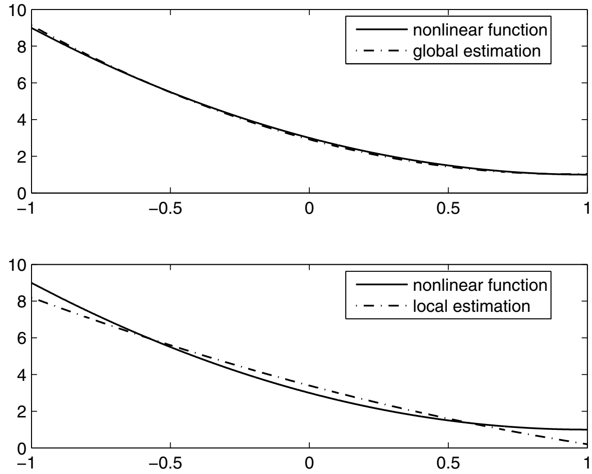

Fig. 2. The nonlinear function approximation result by global (top) and local (bottom) estimation of the consequent parameters of the TS fuzzy models.

4.2.2 Recursive processing scheme

An on line FIV scheme can be obtained by utilizing the recursive solution to the FIV equations and then updating the fuzzy auxiliar model continuously on the basis of these recursive consequent parameters estimates. The FIV estimate in (43) can take the form

$$\hat{\theta}_k = \boldsymbol{P}_k \boldsymbol{b}_k \tag{48}$$

where

$$\boldsymbol{P}_k = \sum_{j=1}^{k} [\beta_j^1 \boldsymbol{z}_j, \ldots, \beta_j^l \boldsymbol{z}_j][\gamma_j^1(\boldsymbol{x}_j + \xi_j), \ldots, \gamma_j^l(\boldsymbol{x}_j + \xi_j)]^T\}^{-1}$$

and

$$\boldsymbol{b}_k = \sum_{j=1}^{k} [\beta_j^1 \boldsymbol{z}_j, \ldots, \beta_j^l \boldsymbol{z}_j] y_j$$

which can be expressed as

$$\boldsymbol{P}_k^{-1} = \boldsymbol{P}_{k-1}^{-1} + [\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k][\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \tag{49}$$

and

$$\boldsymbol{b}_k = \boldsymbol{b}_{k-1} + [\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_k^l \boldsymbol{z}_k] y_k \tag{50}$$

respectively. Pre-multiplying (49) by $\boldsymbol{P}_k$ and post-multiplying by $\boldsymbol{P}_{k-1}$ gives

$$\boldsymbol{P}_{k-1} = \boldsymbol{P}_k + \boldsymbol{P}_k[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k][\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \boldsymbol{P}_{k-1} \tag{51}$$

then post-multiplying (51) by the FIV vector $[\beta_j^1 \boldsymbol{z}_j, \ldots, \beta_j^l \boldsymbol{z}_j]$, results

$$\boldsymbol{P}_{k-1}[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k] = \boldsymbol{P}_k[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k] + \boldsymbol{P}_k[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k][\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \boldsymbol{P}_{k-1}[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k] \tag{52}$$

$$\boldsymbol{P}_{k-1}[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k] = \boldsymbol{P}_k[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k]\{1 + [\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \boldsymbol{P}_{k-1} [\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k]\} \tag{53}$$

Then, post-multiplying by

$$\{1 + [\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \boldsymbol{P}_{k-1}[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k]\}^{-1}[\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \boldsymbol{P}_{k-1} \tag{54}$$

we obtain

$$\boldsymbol{P}_{k-1}[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k]\{1 + [\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \boldsymbol{P}_{k-1}[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k]\}^{-1} [\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \boldsymbol{P}_{k-1} = \boldsymbol{P}_k[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k][\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \boldsymbol{P}_{k-1} \tag{55}$$

Substituting (51) in (55), we have

$$\boldsymbol{P}_k = \boldsymbol{P}_{k-1} - \boldsymbol{P}_{k-1}[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k]\{1 + [\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \boldsymbol{P}_{k-1} [\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k]\}^{-1}[\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \boldsymbol{P}_{k-1} \tag{56}$$

Substituting (56) and (50) in (48), the recursive consequent parameters estimates will be:

$$\hat{\theta}_k = \{\boldsymbol{P}_{k-1} - \boldsymbol{P}_{k-1}[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k]\{1 + [\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \boldsymbol{P}_{k-1} [\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k]\}^{-1}[\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \boldsymbol{P}_{k-1}\}\{\boldsymbol{b}_{k-1} + [\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_k^l \boldsymbol{z}_k] y_k\} \tag{57}$$

so that finally,

$$\hat{\theta}_k = \hat{\theta}_{k-1} - \boldsymbol{K}_k\{[\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \hat{\theta}_{k-1} - y_k\} \tag{58}$$

where

$$\boldsymbol{K}_k = \boldsymbol{P}_{k-1}[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_k^l \boldsymbol{z}_k]\{1 + [\gamma_j^1(\boldsymbol{x}_k + \xi_k), \ldots, \gamma_k^l(\boldsymbol{x}_k + \xi_k)]^T \boldsymbol{P}_{k-1}[\beta_k^1 \boldsymbol{z}_k, \ldots, \beta_j^l \boldsymbol{z}_k]\}^{-1} \tag{59}$$

Equations (56)-(59) compose the recursive algorithm to be implemented so the consequent parameters of a Takagi-Sugeno fuzzy model can be estimated from experimental data.

5. Results

In the sequel, some results will be presented to demonstrate the effectiveness of black box fuzzy modeling for advanced control systems design.

5.1 Computational results

5.1.1 Stochastic nonlinear SISO system identification

The plant to be identified consists on a second order highly nonlinear discrete-time system

$$y_{k+1} = \frac{y_k y_{k-1}(y_k + 2.5)}{1 + y_k^2 + y_{k-1}^2} + u_k + e_k \tag{60}$$

which is a benchmark problem in neural and fuzzy modeling, where y_k is the output and $u_k = \sin(\frac{2\pi k}{25})$ is the applied input. In this case e_k is a white noise with zero mean and variance σ^2. The TS model has two inputs y_k and y_{k-1} and a single output y_{k+1}, and the antecedent part of the fuzzy model (the fuzzy sets) is designed based on the evolving clustering method (ECM). The model is composed of rules of the form:

$$R^i: \text{ IF } y_k \text{ is } F_1^i \text{ AND } y_{k-1} \text{ is } F_2^i \text{ THEN}$$
$$\hat{y}_{k+1}^i = a_{i,1}y_k + a_{i,2}y_{k-1} + b_{i,1}u_k + c_i \tag{61}$$

where $F_{1,2}^i$ are gaussian fuzzy sets.

Experimental data sets of N points each are created from (60), with $\sigma^2 \in [0, 0.20]$. This means that the noise applied take values between 0 and $\pm 30\%$ of the output nominal value, which is an acceptable practical percentage of noise. These data sets are presented to the proposed algorithm, for obtaining an IV fuzzy model, and to the LS based algorithm, for obtaining a LS fuzzy model. The models are obtained by the global and local approaches as in (45) and (46), repectively. The noise influence is analized according to the difference between the outputs of the fuzzy models, obtained from the noisy experimental data, and the output of the plant without noise. The antecedent parameters and the structure of the fuzzy models are the same in the experiments, while the consequent parameters are obtained by the proposed method and by the LS method. Thus, the obtained results are due to these algorithms and accuracy conclusions will be derived about the proposed algorithm performance in the presence of noise. Two criteria, widely used in experimental data analysis, are applied to avaliate the obtained fuzzy models fit: Variance Accounted For (VAF)

$$\mathbf{VAF}(\%) = 100 \times \left[1 - \frac{var(\mathbf{Y} - \hat{\mathbf{Y}})}{var(\mathbf{Y})}\right] \tag{62}$$

where $\mathbf{Y}$ is the nominal plant output, $\hat{\mathbf{Y}}$ is the fuzzy model output and var means signal variance, and Mean Square Error (MSE)

$$\mathbf{MSE} = \frac{1}{N}\sum_{k=1}^{N}(y_k - \hat{y}_k)^2 \tag{63}$$

where y_k is the nominal plant output, $\hat{y}_k$ is the fuzzy model output and N is the number of points. Once obtained these values, a comparative analysis will be established between the proposed algorithm, based on IV, and the algorithm based on LS according to the approaches presented above. In the performance of the TS models obtained off-line according to (45) and (46), the number of points is 500, the proposed algorithm used λ equal to 0.99; the number of rules is 4, the structure is the presented in (61) and the antecedent parameters are obtained by the ECM method for both algorithms. The proposed algorithm performs better than the LS algorithm for the two approaches as it is more robust to noise. This is due to the chosen instrumental variable matrix, with $dl = 1$, to satisfy the convergence conditions as well as possible. In the global approach, for low noise variance, both algorithms presented similar performance with VAF and MSE of 99.50% and 0.0071 for the proposed algorithm and of 99.56% and 0.0027 for the LS based algorithm, respectively. However, when the noise variance increases, the chosen instrumental variable matrix satisfies the convergence conditions, and as a consequence the proposed algorithm becomes more robust to the noise with VAF and MSE of 98.81% and 0.0375. On the other hand the LS based algorithm presented VAF and MSE of 82.61% and 0.4847, respectively, that represents a poor performance. Similar analysis can be applied to the local approach: increasing the noise variance, both algorithms present good performances where the VAF and MSE values increase too. This is due to the polytope property, where the obtained models can represent local approximations giving more flexibility curves fitting. The proposed algorithm presented VAF and MSE values of 93.70% and 0.1701 for the worst case and of 96.3% and 0.0962 for the better case. The LS based algorithm presented VAF and MSE values of 92.4% and 0.2042 for the worst case and of 95.5% and 0.1157 for the better case. The worst case of noisy data set was still used by the algorithm proposed in (Wang & Langari, 1995), where the VAF and MSE values were of 92.6452% and 0.1913, and by the algorithm proposed in (Pedrycz, 2006) where the VAF and MSE values were of 92.5216% and 0.1910, respectively. These results, considering the local approach, show that they have an intermediate performance between the proposed method in this paper and the LS based algorithm. For the global approach, the VAF and MSE values are 96.5% and 0.09 for the proposed method and of 81.4% and 0.52 for the LS based algorithm, respectively. For the local approach, the VAF and MSE values are 96.0% and 0.109 for the proposed method and of 95.5% and 0.1187 for the LS based algorithm, respectively. In sense to be clear to the reader, the results of local and global estimation to the TS fuzzy model from the stochastic SISO nonlinear system identification, it has the following conclusions: When interpreting TS fuzzy models obtained from data, one has to be aware of the tradeoffs between local and global estimation. The TS fuzzy models estimated by local approach describe properly the local behavior of the nonlinear system, but not give a good fit; for the global approach, the opposite holds - a perfect fit is obtained, but the TS fuzzy models are not relevant for the local behavior of the nonlinear system. This is the tradeoffs between local and global estimation. To illustrate the robustness of the FIV algorithm, it was performed a numerical experiment based on 300 different realizations of noise. The numerical experiment followed a particular computational pattern:

- Define a domain with 300 different sequences of noise;
- Generate a realization of noise randomly from the domain, and perform the identification procedure for the IV and LS based algorithms;

- Aggregate the results of IV and LS algorithms according to VAF and MSE criteria into the final result from histograms, indicating the number of its occurences (frequency) during the numerical experiment.

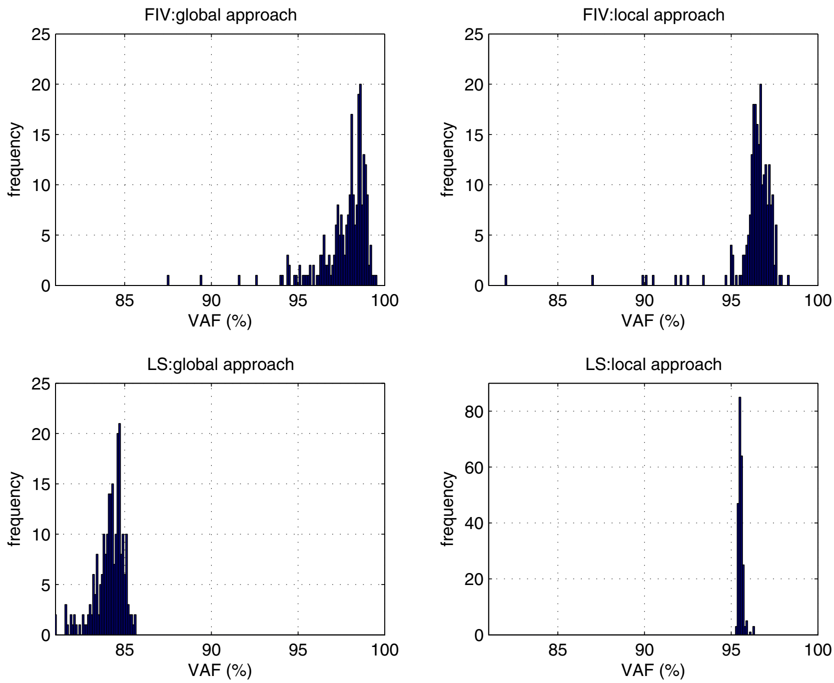

Fig. 3. Robustness analysis: Histogram of VAF for the IV and LS based algorithms.

The IV and LS based algorithms were submitted to these different conditions of noise at same time and the efficiency was observed through VAF and MSE criteria according to the histograms shown on Fig. 3 and Fig. 4, respectively. Clearly, the proposed method presented the best results compared with LS based algorithm. For the global approach, the results of VAF and MSE values are of $98.60 \pm 1.25\%$ and 0.037 ± 0.02 for the proposed method and of $84.70 \pm 0.65\%$ and 0.38 ± 0.15 for the LS based algorithm, respectively. For the local approach, the results of VAF and MSE values are of $96.70 \pm 0.55\%$ and 0.07 ± 0.015 for the proposed method and of $95.30 \pm 0.15\%$ and 0.1150 ± 0.005 for the LS based algorithm, respectively. In general, from the results shown in Tab. 1, it can conclude that the proposed method has favorable results compared with existing techniques and good robustness properties for identification of stochastic nonlinear systems.

5.2 Experimental results

In this section, the experimental results on adaptive model based control of a multivariable (two inputs and one output) nonlinear pH process, commonly found in industrial environment, are presented.

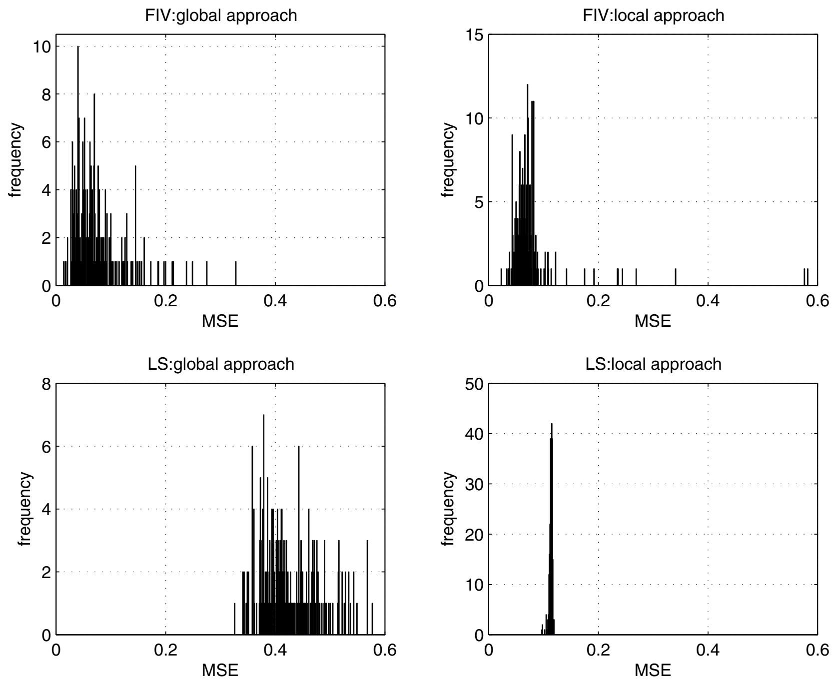

Fig. 4. Robustness analysis: Histogram of MSE for the IV and LS based algorithms.

5.2.1 Fuzzy adaptive black box fuzzy model based control of pH neutralization process

The input-output experimental data set of the nonlinear plant were obtained from DAISY[1] (Data Acquisition For Identification of Systems) plataform.

This plant presents the following input-output variables:

- $u_1(t)$: acid flow (l);
- $u_2(t)$: base flow (l);
- $y(t)$: level of pH in the tank.

Figure 5 shows the open loop temporal response of the plant, considering a sampling time of 10 seconds. These data will be used for modeling of the process. The obtained fuzzy model will be used for indirect multivariable adaptive fuzzy control design. The TS fuzzy inference system uses a functional expression of the pH level in the tank. The $i\ |^{i=1,2,\ldots,l}$-th rule of the multivariable TS fuzzy model, where l is the number of rules is given by:

$$R^i : \textbf{IF } \tilde{Y}(z)z^{-1} \text{ is } F^i_{j|\tilde{Y}(z)z^{-1}} \textbf{ THEN}$$
$$Y^i(z) = \frac{b^i_1}{1-a^i_1 z^{-1}-a^i_2 z^{-2}} U_1(z) + \frac{b^i_2}{1-a^i_1 z^{-1}-a^i_2 z^{-2}} U_2(z) \quad (64)$$

[1] accessed in http://homes.esat.kuleuven.be/ smc/daisy.

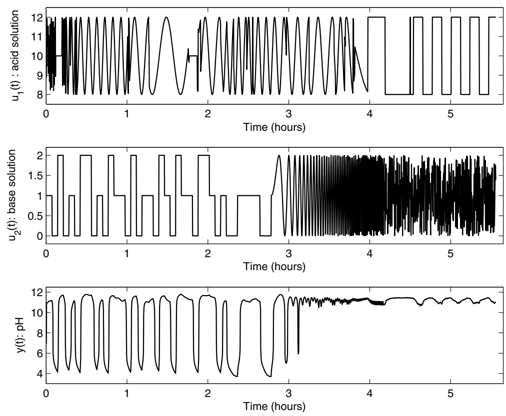

Fig. 5. Open loop temporal response of the nonlinear pH process

The C-means fuzzy clustering algorithm was used to estimate the antecedent parameters of the TS fuzzy model. The fuzzy recursive instrumental variable algorithm based on QR factorization, was used to estimate the consequent submodels parameters of the TS fuzzy model. For initial estimation was used 100 points, the number of rules was $l = 2$, and the fuzzy frequency response validation method was used for fuzzy controller design based on the inverse model (Serra & Ferreira, 2011).

The parameters of the submodels in the consequent proposition of the multivariable TS fuzzy model are shown in Figure 6. It is observed that in addition to nonlinearity, the pH neutralization process presents uncertainty behavior in order to commit any application of fix control design.

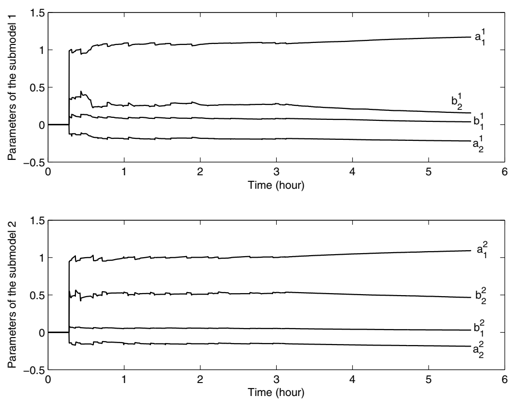

Fig. 6. TS fuzzy model parameters estimated by fuzzy instrumental variable algortihm based on QR factoration

The TS multivariable fuzzy model, at last sample, is given by:

$$
\begin{aligned}
&R^1 : \textbf{IF } \tilde{y}(k-1) \text{ is } F^1 \textbf{THEN} \\
&y^1(k) = 1.1707y(k-1) - 0.2187y(k-2) + 0.0372u_1(k) + 0.1562u_2(k) \\
&R^2 : \textbf{IF } \tilde{y}(k-1) \text{ is } F^2 \textbf{THEN} \\
&y^2(k) = 1.0919y(k-1) - 0.1861y(k-2) + 0.0304u_1(k) + 0.4663u_2(k)
\end{aligned}
\quad (65)
$$

The validation of the TS fuzzy model, according to equation (65) via fuzzy frequency response

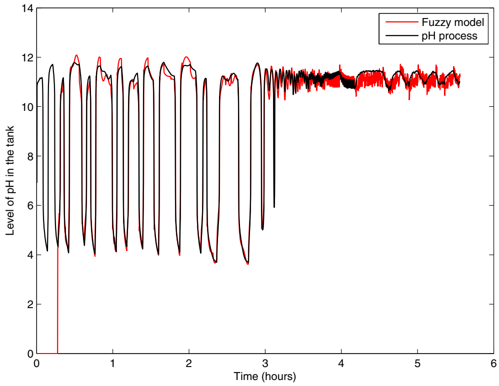

Fig. 7. Recursive estimation processing for submodels parameters in the TS multivariable fuzzy model consequent proposition.

is shown in Figure 8. It can be observed the efficiency of the proposed identification algorithm to track the output variable of pH neutralization process. This result has fundamental importance for multivariable adaptive fuzzy controller design step. The region of uncertainty defined by fuzzy frequency response for the identified model contains the frequency response of the pH process. It means that the fuzzy model represents the dynamic behavior perfectly, considering the uncertainties and nonlinearities of the pH neutralization process. Consequently, the model based control design presents robust stability characteristic. The adaptive control design methodology adopted in this paper consists of a control action based on the inverse model. Once the plant model becomes known precisely by the rules of multivariable TS fuzzy model, considering the fact that the submodels are stable, one can develop a strategy to control the flow of acid and base, in order to maintain the pH level of 7. Thus, the multivariable fuzzy controller is designed so that the control system closed-loop presents unity gain and the output is equal to the reference. So, it yields:

$$
G_{MF}(z) = \frac{R(z)}{Y(z)} = \frac{G^i_{c_1} G^i_{p_1} + G^i_{c_2} G^i_{p_2}}{1 + G^i_{c_1} G^i_{p_1} + G^i_{c_2} G^i_{p_2}} \quad (66)
$$

where $G^i_{c_1}$ e $G^i_{c_2}$ are the transfer functions of the controllers in the i-th rule, as $G^i_{p_1}$ and $G^i_{p_2}$ are submodels in the consequent proposition from the output $Y(z)$ to inputs $U_1(z)$ and $U_2(z)$,

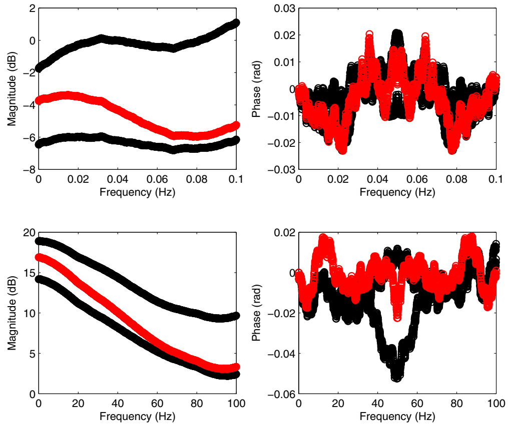

Fig. 8. Validation step of the multivariable TS fuzzy model: (a) - (b) Fuzzy frequency response of the TS fuzzy model (black curve) representing the dynamic behavior of the pH level and the flow of acid solution (red curve), (c) - (d) Fuzzy frequency response of the TS fuzzy model (black curve) representing the dynamic behavior of the pH level and flow of the base (red curve).

respectively. Considering

$$G^i_{c_1} = \frac{1}{G^i_{p_1}}$$

and

$$G^i_{c_2} = \frac{1}{G^i_{p_2}}$$

results:

$$G_{MF}(z) = \frac{R(z)}{Y(z)} = \frac{2}{3} \tag{67}$$

this is,

$$Y(z) = \frac{2}{3}R(z) \tag{68}$$

For compensation this closed loop gain of the control system, it is necessary generate a reference signal so that $Y(z) = R(z)$. Therefore, adopting the new reference signal $R'(z) = \frac{3}{2}R(z)$, it yields:

$$Y(z) = \frac{2}{3}R'(z) \tag{69}$$

$$Y(z) = \frac{2}{3}\frac{3}{2}R(z) \tag{70}$$

$$Y(z) = R(z) \tag{71}$$

For the inverse model based indirect multivariable fuzzy control design, one adopte a new reference signal given by $R'(z) = \frac{3}{2}R(z)$. The TS fuzzy multivariable controller presents the folowing structure:

$$R^i : \textbf{IF } \tilde{Y}(z)z^{-1} \text{ is } F^i_{j|\tilde{Y}(z)z^{-1}} \textbf{ THEN}$$
$$G^i_{c_1} = \frac{1 - \hat{a}^i_1 z^{-1} - \hat{a}^i_2 z^{-2}}{\hat{b}^i_1} E(z)$$
$$G^i_{c_2} = \frac{1 - \hat{a}^i_1 z^{-1} - \hat{a}^i_2 z^{-2}}{\hat{b}^i_2} E(z) \quad (72)$$

The temporal response of the TS fuzzy multivariable adaptive control is shown in Fig. 9. It can be observed the control system track the reference signal, $pH = 7$, because the controller can tune itself based on the identified TS fuzzy multivariable model.

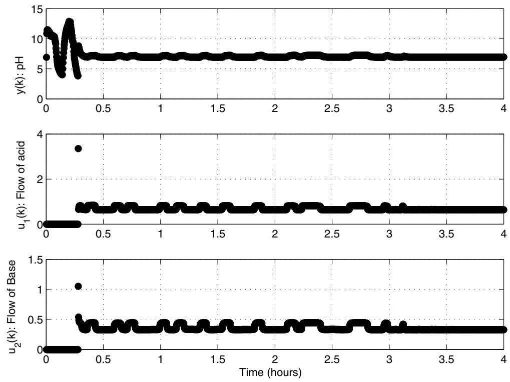

Fig. 9. Performance of the TS fuzzy multivariable adaptive control system.

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