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The Bicomplex Heisenberg Uncertainty Principle

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1. Introduction

Quantum mechanics is one of the two fundamental pillar of modern physics. The success of the theory can be found everywhere in our everyday life and essentially in every new product that we build. We just have to remember that every semiconductor chip usually uses a quantum behavior in an essential way, for example quantum tunneling, to work. Until now, none of the thousand of experiments realized have succeeded to contradicted or to find a problem with the predictions given by quantum mechanics.

However, in spite of this incredible success, many profound questions are still open. For example, we have some problems understanding the measurement, the coherence and the decoherence process, as well as the interpretation of what the theory tell us about the world we live in (Schlosshauer, 2005).

Among the possible ways of investigation that we have, we think that stressing the foundations of the theory at the level of the mathematical structure, on which the theory stands, could be a good way to understand why and how the theory works. The mathematical structure of quantum mechanics consists in Hilbert spaces defined over the field of complex numbers (Birkhoff & Von Neumann, 1936). The success of the theory has led a number of investigators, over many decades, to look for general principles or arguments that would lead quite inescapably to the complex Hilbert space structure. It has been argued (Stueckelberg, 1960; Stueckelberg & Guenin, 1961), for instance, that the formulation of an uncertainty principle, heavily motivated by experiment, implies that a real Hilbert space can in fact be endowed with a complex structure. The proof, however, involves a number of additional hypotheses that may not be so directly connected with experiment. In fact Reichenbach (Reichenbach, 1944) has shown that a theory is not straightforwardly deduced from experiments, but rather arrived at by a process involving a good deal of instinctive inferences. This was also pointed out more recently by Penrose (Penrose, 2005, p. 59);

> *In the development of mathematical ideas, one important initial driving force has always been to find mathematical structures that accurately mirror the behaviour of the physical world. But it is normally not possible to examine the physical world itself in such precise detail that appropriately clear-cut mathematical notions can be abstracted directly from it.*

Moreover, in the last decade, some of the efforts to derive the complex Hilbert space structure have focused on information-theoretic principles (Clifton et al., 2003; Fuchs, 2002). The general principles assumed at the outset are no doubt attractive, but yet open to questioning (Marchildon, 2004).

The upshot is that there is no compelling argument restricting the number system on which quantum mechanics is built to the field of complex numbers. The justification of the theory lie rather in its ability to correctly describe and explain experiments.
We think that all this justifies the investigation of a quantum mechanics standing on a different algebra than the usual one, not necessarily in the aim of replacing the actual theory, but in the aim of a better understanding of the actual theory by meticulously compare the two descriptions. Moreover, it does not exclude that a quantum mechanics standing on a different algebra can end with some new predictions.
This is with those things in mind that we would like to introduced this chapter on *bicomplex quantum mechanics* and on the *bicomplex Heisenberg uncertainty principle*.
In section 2, we present the bicomplex numbers, that are a generalization of complex numbers by means of entities specified by four real numbers. Bicomplex numbers are commutative but do not form a division algebra. Division algebras do not have zero divisors, that is, nonzero elements whose product is zero. We also present some algebraic properties of bicomplex numbers, modules, scalar product and linear operator. In the recent years, bicomplex numbers have founded application in quantum mechanics (Gervais Lavoie et al., 2010b; Rochon & Tremblay, 2004; 2006), in pure mathematics (Charak et al., 2009; Gervais Lavoie et al., 2010a; 2011; Rochon, 2003; 2004; Rochon & Shapiro, 2004) as well as in the construction of three dimensional fractals (Garant-Pelletier & Rochon, 2009; Martineau & Rochon, 2005; Rochon, 2000).
The section 3 presents some important results on infinite-dimentional bicomplex Hilbert spaces.
In section 4, we give a sketch of some fundamentals aspect of bicomplex quantum mechanics. We also present our solution for the problem of the bicomplex harmonic oscillator. These results are already given in (Gervais Lavoie et al., 2010b), but we present them here with a new approach, the differential one. We also plot some of the eigenfunctions that we found and give some new representation of them by means of hyperbolic sinus and cosinus functions.
Section 5 is the main part of this chapter. We work out, in details, the bicomplex Heisenberg uncertainty principle. This will give an explicit and fully detailed example of the kind of computation that arise in bicomplex quantum mechanics.

2. Preliminaries

This section summarizes basic properties of bicomplex numbers and modules defined over them. The notions of scalar product and linear operators are also introduced. Proofs and additional material can be found in (Gervais Lavoie et al., 2010a;b; 2011; Price, 1991; Rochon & Shapiro, 2004; Rochon & Tremblay, 2004; 2006).

2.1 Bicomplex numbers

The set $\mathbb{T}$ of *bicomplex numbers* can be define essentially in two equivalent way as

$$\mathbb{T} := \{w = w_e + w_{\mathbf{i_1}}\mathbf{i_1} + w_{\mathbf{i_2}}\mathbf{i_2} + w_{\mathbf{j}}\mathbf{j} \mid w_e, w_{\mathbf{i_1}}, w_{\mathbf{i_2}}, w_{\mathbf{j}} \in \mathbb{R}\} \tag{1}$$

$$\equiv \{w = z + z'\mathbf{i_2} \mid z, z' \in \mathbb{C}(\mathbf{i_1})\}, \tag{2}$$

where $\mathbf{i_1}$, $\mathbf{i_2}$ and $\mathbf{j}$ are (complex) *imaginary* and *hyperbolic* units such that

$$\mathbf{i}_1^2 = -1 = \mathbf{i}_2^2 \qquad \text{and} \qquad \mathbf{j}^2 = 1. \tag{3}$$

The product of units is commutative and defined as

$$\mathbf{i_1 i_2} = \mathbf{j}, \qquad \mathbf{i_1 j} = -\mathbf{i_2} \qquad \text{and} \qquad \mathbf{i_2 j} = -\mathbf{i_1}. \tag{4}$$

It is obvious that definition (1) and (2) imply that $z = w_e + w_{\mathbf{i_1}}\mathbf{i_1}$ and $z' = w_{\mathbf{i_2}} + w_{\mathbf{j}}\mathbf{i_1}$ are both in $\mathbb{C}(\mathbf{i_1})$.
Three important subsets of $\mathbb{T}$ can be specified as

$$\mathbb{C}(\mathbf{i_k}) := \{x + y\mathbf{i_k} \mid x, y \in \mathbb{R}\}, \qquad k = 1, 2; \tag{5}$$

$$\mathbb{D} := \{x + y\mathbf{j} \mid x, y \in \mathbb{R}\}. \tag{6}$$

Each of the sets $\mathbb{C}(\mathbf{i_k})$ is isomorphic to the field of complex numbers, while $\mathbb{D}$ is the set of so-called *hyperbolic numbers*.
With the addition and multiplication of two bicomplex numbers defined in the obvious way, the set $\mathbb{T}$ makes up a commutative ring.

2.1.1 Complexification

In addition to the formal definition, it is instructive to see how the set of bicomplex numbers can be construct. Let us define the action $\xrightarrow{k}$ that add up an imaginary part (with respect to k) to all the real variables. For $x, y \in \mathbb{R}$, we thus have

$$x \xrightarrow{\mathbf{i}} x + y\mathbf{i} \ \in \mathbb{C}, \tag{7}$$

$$x \xrightarrow{\mathbf{i_1}} x + y\mathbf{i_1} \ \in \mathbb{C}(\mathbf{i_1}) \simeq \mathbb{C}, \tag{8}$$

$$x \xrightarrow{\mathbf{i_2}} x + y\mathbf{i_2} \ \in \mathbb{C}(\mathbf{i_2}) \simeq \mathbb{C}. \tag{9}$$

The action $\xrightarrow{k}$ will be call a *complexification*. Let us now applied a complexification on $x + y\mathbf{i_1}$. There are essentially two possibilities, the first one is $(s, t \in \mathbb{R})$

$$x + y\mathbf{i_1} \xrightarrow{\mathbf{i_1}} (x + s\mathbf{i_1}) + (y + t\mathbf{i_1})\mathbf{i_1} = (x - t) + (s + y)\mathbf{i_1} \in \mathbb{C}(\mathbf{i_1}). \tag{10}$$

This complexification is trivial in the sense that it maps $\mathbb{C}(\mathbf{i_1})$ to $\mathbb{C}(\mathbf{i_1})$. The second one is more interesting

$$x + y\mathbf{i_1} \xrightarrow{\mathbf{i_2}} (x + s\mathbf{i_2}) + (y + t\mathbf{i_2})\mathbf{i_1} = x + y\mathbf{i_1} + s\mathbf{i_2} + t\mathbf{i_2 i_1}. \tag{11}$$

Here, because $\mathbf{i_1}$ and $\mathbf{i_2}$ are two independent imaginary units, we cannot write $\mathbf{i_2 i_1} = -1$. However, one can remark that

$$(\mathbf{i_2 i_1})^2 = \mathbf{i_2 i_1 i_2 i_1} = \mathbf{i_2^2 i_1^2} = (-1)(-1) = 1. \tag{12}$$

This means that $\mathbf{i_2 i_1}$ have the same behavior as an hyperbolic unit and then, we can write $\mathbf{j} := \mathbf{i_2 i_1} = \mathbf{i_1 i_2}$. We finally ends with

$$x + y\mathbf{i_1} \xrightarrow{\mathbf{i_2}} x + y\mathbf{i_1} + s\mathbf{i_2} + t\mathbf{j}, \tag{13}$$

which is the set of bicomplex numbers.

The complexification process can be applied again to generate the *tricomplex numbers*, and so on. For n successive complexification, we talk of a *multicomplex number* of order n, and we noted it by $\mathbb{MC}_n$ (Garant-Pelletier & Rochon, 2009; Price, 1991; Vaijac & Vaijac, to appear). Then, it is not hard to see that

$$\mathbb{MC}_0 \equiv \mathbb{R}, \qquad \mathbb{MC}_1 \equiv \mathbb{C} \qquad \text{and} \qquad \mathbb{MC}_2 \equiv \mathbb{T}. \tag{14}$$

For an arbitrary multicomplex number $s \in \mathbb{MC}_{n>0}$, s is 2^n-dimensionnal (in the sense that we need 2^n real numbers to specify it), posses 2^{n-1} independent imaginary units, and $2^{n-1} - 1$ independent hyperbolic units.
The set $\mathbb{T}$ of bicomplex numbers can also be construct by applying the complexification process on the set of hyperbolic numbers, or by applying an *hyperbolisation process* (the process that add up an hyperbolic term instead of a imaginary one) on the set of complex numbers.
In Fig. 1, we give a sketch of some generalization of the real numbers. The set $\mathbb{P}$ stand for the set of *parabolic* or *dual* numbers defined by

$$\mathbb{P} := \left\{ p = x + y\varepsilon \mid x, y \in \mathbb{R}, \quad \varepsilon^2 = 0 \right\}. \tag{15}$$

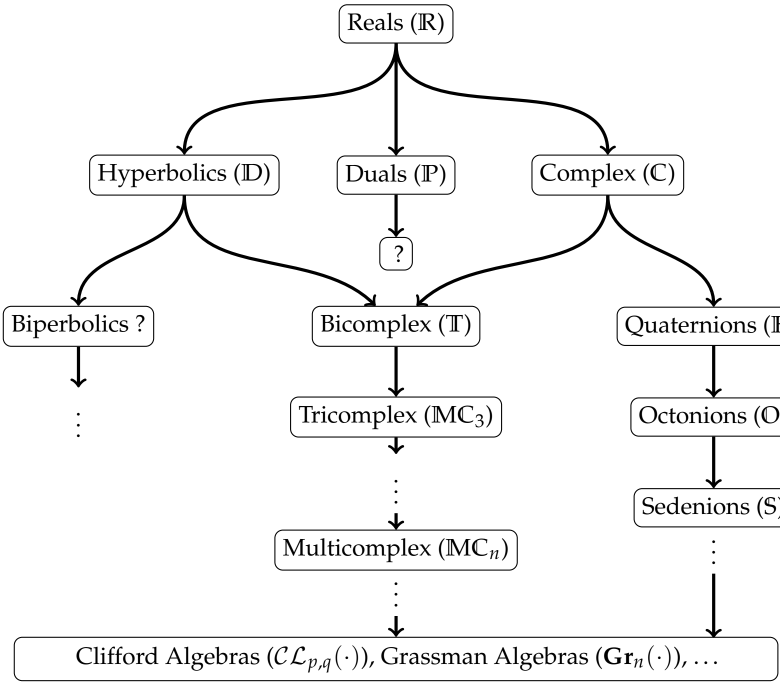


Fig. 1. Generalization of real numbers

2.1.2 Algebraic properties of bicomplex numbers

Bicomplex algebra is considerably simplified by the introduction of two bicomplex numbers $\mathbf{e_1}$ and $\mathbf{e_2}$ defined as

$$\mathbf{e_1} := \frac{1+\mathbf{j}}{2} \qquad \text{and} \qquad \mathbf{e_2} := \frac{1-\mathbf{j}}{2}. \tag{16}$$

One easily checks that

$$\mathbf{e_1}^2 = \mathbf{e_1}, \qquad \mathbf{e_2}^2 = \mathbf{e_2}, \qquad \mathbf{e_1} + \mathbf{e_2} = 1 \qquad \text{and} \qquad \mathbf{e_1}\mathbf{e_2} = 0. \tag{17}$$

Any bicomplex number w can be written uniquely as

$$w = z_{\widehat{1}}\mathbf{e_1} + z_{\widehat{2}}\mathbf{e_2}, \tag{18}$$

where $z_{\widehat{1}}$ and $z_{\widehat{2}}$ both belong to $\mathbb{C}(\mathbf{i_1})$. Specifically,

$$z_{\widehat{1}} = (w_e + w_{\mathbf{j}}) + (w_{\mathbf{i_1}} - w_{\mathbf{i_2}})\mathbf{i_1} \qquad \text{and} \qquad z_{\widehat{2}} = (w_e - w_{\mathbf{j}}) + (w_{\mathbf{i_1}} + w_{\mathbf{i_2}})\mathbf{i_1}. \tag{19}$$

The numbers $\mathbf{e_1}$ and $\mathbf{e_2}$ make up the so-called *idempotent basis* of the bicomplex numbers (Price, 1991). Note that the last of (17) illustrates the fact that $\mathbb{T}$ has zero divisors which are nonzero elements whose product is zero. The caret notation ($\widehat{1}$ and $\widehat{2}$) will be used systematically in connection with idempotent decompositions, with the purpose of easily distinguishing different types of indices.

As a consequence of (17) and (18), one can check that if $\sqrt[n]{z_{\widehat{1}}}$ is an nth root of $z_{\widehat{1}}$ and $\sqrt[n]{z_{\widehat{2}}}$ is an nth root of $z_{\widehat{2}}$, then $\sqrt[n]{z_{\widehat{1}}}\,\mathbf{e_1} + \sqrt[n]{z_{\widehat{2}}}\,\mathbf{e_2}$ is an nth root of w.

The uniqueness of the idempotent decomposition allows the introduction of two projection operators as

$$P_1 : w \in \mathbb{T} \mapsto z_{\widehat{1}} \in \mathbb{C}(\mathbf{i_1}), \tag{20}$$

$$P_2 : w \in \mathbb{T} \mapsto z_{\widehat{2}} \in \mathbb{C}(\mathbf{i_1}). \tag{21}$$

The P_k ($k = 1, 2$) satisfy

$$[P_k]^2 = P_k, \qquad P_1\mathbf{e_1} + P_2\mathbf{e_2} = \mathbf{Id}, \tag{22}$$

and, for $s, t \in \mathbb{T}$,

$$P_k(s+t) = P_k(s) + P_k(t) \qquad \text{and} \qquad P_k(s \cdot t) = P_k(s) \cdot P_k(t). \tag{23}$$

The product of two bicomplex numbers w and w' can be written in the idempotent basis as

$$w \cdot w' = (z_{\widehat{1}}\mathbf{e_1} + z_{\widehat{2}}\mathbf{e_2}) \cdot (z'_{\widehat{1}}\mathbf{e_1} + z'_{\widehat{2}}\mathbf{e_2}) = z_{\widehat{1}}z'_{\widehat{1}}\mathbf{e_1} + z_{\widehat{2}}z'_{\widehat{2}}\mathbf{e_2}. \tag{24}$$

Since 1 is uniquely decomposed as $\mathbf{e_1} + \mathbf{e_2}$, we can see that $w \cdot w' = 1$ if and only if $z_{\widehat{1}}z'_{\widehat{1}} = 1 = z_{\widehat{2}}z'_{\widehat{2}}$. Thus w has an inverse if and only if $z_{\widehat{1}} \neq 0 \neq z_{\widehat{2}}$, and the inverse w^{-1} is then equal to $z_{\widehat{1}}^{-1}\mathbf{e_1} + z_{\widehat{2}}^{-1}\mathbf{e_2}$. A nonzero w that does not have an inverse has the property that either $z_{\widehat{1}} = 0$ or $z_{\widehat{2}} = 0$, and such a w is a divisor of zero. Zero divisors make up the so-called *null cone* ($\mathcal{NC}$). That terminology comes from the fact that when w is written as $z + z'\mathbf{i_2}$, zero divisors are such that $z^2 + (z')^2 = 0$.

2.1.3 Bicomplex numbers are not quaternions

We would like to point out that even if bicomplex numbers and quaternions are both given by four real elements, they form two completely different algebras. First, bicomplex numbers are commutative while quaternions are not. Secondly, quaternion numbers form a division algebra, but not the bicomplex numbers. A division algebra is characterized by the fact that every nonzero element have a multiplicative inverse. Let us give the multiplication table of the two algebra to clearly see the difference. Let $x_1 \dots x_4 \in \mathbb{R}$,

Bicomplex $\mathbb{T}$

$$x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{j},$$
$$\exists\, a, b \in \mathbb{T} \mid a \cdot b = 0,\ a \neq 0 \neq b,$$

$\cdot$	1	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{j}$
1	1	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{j}$
$\mathbf{i_1}$	$\mathbf{i_1}$	-1	$\mathbf{j}$	$-\mathbf{i_2}$
$\mathbf{i_2}$	$\mathbf{i_2}$	$\mathbf{j}$	-1	$-\mathbf{i_1}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{i_2}$	$-\mathbf{i_1}$	1

Quaternions $\mathbb{H}$

$$x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k},$$
$$\forall a, b \in \mathbb{H} \mid a \cdot b = 0 \Leftrightarrow a = 0 \text{ or } b = 0,$$

$\cdot$	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	-1	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	-1

(25)

For a complete treatment of quantum mechanics define over the field of quaternions, the reader can consult (Adler, 1995).

2.1.4 Conjugation of bicomplex numbers

Three different conjugation can be defines on bicomplex numbers, consistent with the fact that we have two independent imaginary unit (we can conjugate one unit, the other or the two at the same time). However, in the present work, we will consider only one of them.
We define the conjugate $w^\dagger$ of the bicomplex number $w = z_{\widehat{1}}\mathbf{e_1} + z_{\widehat{2}}\mathbf{e_2}$ as

$$w^\dagger := \bar{z}_{\widehat{1}}\mathbf{e_1} + \bar{z}_{\widehat{2}}\mathbf{e_2}, \tag{26}$$

where the bar denotes the usual complex conjugation on $\mathbb{C}(\mathbf{i_1})$. Operation $w^\dagger$ was denoted by $w^{\dagger_3}$ in (Gervais Lavoie et al., 2010a; 2011; Rochon & Tremblay, 2004; 2006), consistent with the fact that at least two other types of conjugation can be defined with bicomplex numbers. Making use of (24), we immediately see that

$$w \cdot w^\dagger = z_{\widehat{1}}\bar{z}_{\widehat{1}}\mathbf{e_1} + z_{\widehat{2}}\bar{z}_{\widehat{2}}\mathbf{e_2}. \tag{27}$$

Furthermore, for any $s, t \in \mathbb{T}$,

$$(s+t)^\dagger = s^\dagger + t^\dagger, \qquad (s^\dagger)^\dagger = s \qquad \text{and} \qquad (s \cdot t)^\dagger = s^\dagger \cdot t^\dagger. \tag{28}$$

It can be noted that with our choice of conjugation, we have $\mathbf{j}^\dagger = (\overline{\mathbf{i_2}})(\overline{\mathbf{i_1}}) = (-\mathbf{i_2})(-\mathbf{i_1}) = \mathbf{j}$ (another choice of conjugation would have lead us to a different expression here). This also imply that $\mathbf{e_k}^\dagger = \mathbf{e_k},\ k = 1, 2$.
The real modulus $|w|$ of a bicomplex number w can be defined as

$$|w| := \sqrt{w_e^2 + w_{\mathbf{i_1}}^2 + w_{\mathbf{i_2}}^2 + w_{\mathbf{j}}^2} = \sqrt{(z_{\widehat{1}}\bar{z}_{\widehat{1}} + z_{\widehat{2}}\bar{z}_{\widehat{2}})/2} = \sqrt{Re(w \cdot w^\dagger)}\,. \tag{29}$$

This coincides with the Euclidean norm on $\mathbb{R}^4$. Clearly, $|\cdot| : \mathbb{T} \to \mathbb{R}$, $|w| \geq 0$, with $|w| = 0$ if and only if $w = 0$ and for any $s, t \in \mathbb{T}$,

$$|s+t| \leq |s| + |t| \qquad \text{and} \qquad |\lambda \cdot t| = |\lambda| \cdot |t|, \tag{30}$$

for $\lambda \in \mathbb{C}(\mathbf{i_1})$ or $\mathbb{C}(\mathbf{i_2})$. Moreover,

$$|s \cdot t| \leq \sqrt{2}|s| \cdot |t|. \tag{31}$$

As the reader can see in the last of (30), we will used the same symbol $|\cdot|$ to designated the Euclidean norm on different set. For example here, $|t|$ is the Euclidean $\mathbb{R}^4$-norm on $\mathbb{T}$ while $|\lambda|$ is the Euclidean $\mathbb{R}^2$-norm on $\mathbb{C}(\mathbf{i_k})$.
In the idempotent basis, any hyperbolic number can be written as $x_{\widehat{1}}\mathbf{e_1} + x_{\widehat{2}}\mathbf{e_2}$, with $x_{\widehat{1}}$ and $x_{\widehat{2}}$ in $\mathbb{R}$. We define the set $\mathbb{D}^+$ of positive hyperbolic numbers as

$$\mathbb{D}^+ := \{x_{\widehat{1}}\mathbf{e_1} + x_{\widehat{2}}\mathbf{e_2} \mid x_{\widehat{1}}, x_{\widehat{2}} \geq 0\}. \tag{32}$$

Clearly, $w \cdot w^{\dagger} \in \mathbb{D}^+$ for any w in $\mathbb{T}$.

2.2 $\mathbb{T}$-Module, scalar product and linear operators

The set of bicomplex numbers is a commutative ring. Just like vector spaces are defined over fields, modules are defined over rings. A module M defined over the ring of bicomplex numbers is called a $\mathbb{T}$-*module* (Gervais Lavoie et al., 2010a; 2011; Rochon & Tremblay, 2006).
Let $\{|u_l\rangle \mid l = 1 \ldots n\}$ be a $\mathbb{T}$-basis (a set of elements of M that form a basis), then the $\mathbb{T}$-module M is given by the set

$$M = \left\{ \sum_{l=1}^{n} w_l |u_l\rangle \;\middle|\; w_l \in \mathbb{T} \right\}. \tag{33}$$

For $k = 1, 2$, we define V_k as the set of all elements of the form $\mathbf{e_k}|\psi\rangle$, with $|\psi\rangle \in M$. Succinctly, $V_1 := \mathbf{e_1}M$ and $V_2 := \mathbf{e_2}M$. In fact, V_k, $k = 1, 2$ are vector spaces over $\mathbb{C}(\mathbf{i_1})$ and any element $|v_k\rangle \in V_k$ satisfies $|v_k\rangle = \mathbf{e_k}|v_k\rangle$.
For arbitrary $\mathbb{T}$-modules, vector spaces V_1 and V_2 bear no structural similarities. For more specific modules, however, they may share structure. It was shown in (Gervais Lavoie et al., 2011) that if M is a finite-dimensional free $\mathbb{T}$-module, then V_1 and V_2 have the same dimension. For any $|\psi\rangle \in M$, there exist a unique decomposition

$$|\psi\rangle = \mathbf{e_1}P_1(|\psi\rangle) + \mathbf{e_2}P_2(|\psi\rangle), \tag{34}$$

where $\mathbf{e_k}P_k(|\psi\rangle) \in V_k$, $k = 1, 2$. One can show that ket projectors and idempotent-basis projectors (denoted with the same symbol) satisfy the following, for $k = 1, 2$:

$$P_k(s|\psi\rangle + t|\phi\rangle) = P_k(s)\,P_k(|\psi\rangle) + P_k(t)\,P_k(|\phi\rangle), \qquad s, t \in \mathbb{T}. \tag{35}$$

It will be useful to rewrite (34) as

$$|\psi\rangle = \mathbf{e_1}|\psi_{\widehat{1}}\rangle + \mathbf{e_2}|\psi_{\widehat{2}}\rangle, \tag{36}$$

where

$$|\psi_{\widehat{1}}\rangle := P_1(|\psi\rangle) \qquad \text{and} \qquad |\psi_{\widehat{2}}\rangle := P_2(|\psi\rangle). \tag{37}$$

The $\mathbb{T}$-module M can be viewed as a vector space M' over $\mathbb{C}(\mathbf{i_1})$, and $M' = V_1 \oplus V_2$. From a set-theoretical point of view, M and M' are identical. In this sense we can say, perhaps improperly, that the **module** M can be decomposed into the direct sum of two vector spaces over $\mathbb{C}(\mathbf{i_1})$, i.e. $M = V_1 \oplus V_2$.

2.2.1 Bicomplex scalar product

A *bicomplex scalar product* maps two arbitrary kets $|\psi\rangle$ and $|\phi\rangle$ into a bicomplex number $(|\psi\rangle, |\phi\rangle)$, so that the following always holds ($s \in \mathbb{T}$):

1. $(|\psi\rangle, |\phi\rangle + |\chi\rangle) = (|\psi\rangle, |\phi\rangle) + (|\psi\rangle, |\chi\rangle)$;
2. $(|\psi\rangle, s|\phi\rangle) = s(|\psi\rangle, |\phi\rangle)$;
3. $(|\psi\rangle, |\phi\rangle) = (|\phi\rangle, |\psi\rangle)^{\dagger}$;
4. $(|\psi\rangle, |\psi\rangle) = 0 \iff |\psi\rangle = 0$.

Property 3 implies that $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}$, while properties 2 and 3 together imply that $(s|\psi\rangle, |\phi\rangle) = s^{\dagger}(|\psi\rangle, |\phi\rangle)$. However, in this work we will also require the bicomplex scalar product $(\cdot, \cdot)$ to be *hyperbolic positive*, i.e.

$$(|\psi\rangle, |\psi\rangle) \in \mathbb{D}^{+}, \ \forall |\psi\rangle \in M. \tag{38}$$

This is a necessary condition if we want to recover the standard quantum mechanics from the bicomplex one.
Noted that the following projection of a bicomplex scalar product:

$$(\cdot, \cdot)_{\widehat{k}} := P_k((\cdot, \cdot)) : M \times M \longrightarrow \mathbb{C}(\mathbf{i_1}) \tag{39}$$

is a **standard scalar product** on V_k, for $k = 1, 2$. One easily shows (Gervais Lavoie et al., 2010a, (3.12)) that

$$\begin{aligned}(|\psi\rangle, |\phi\rangle) &= \mathbf{e_1} P_1\left((|\psi_{\widehat{1}}\rangle, |\phi_{\widehat{1}}\rangle)\right) + \mathbf{e_2} P_2\left((|\psi_{\widehat{2}}\rangle, |\phi_{\widehat{2}}\rangle)\right) \\ &= \mathbf{e_1}\left(|\psi_{\widehat{1}}\rangle, |\phi_{\widehat{1}}\rangle\right)_{\widehat{1}} + \mathbf{e_2}\left(|\psi_{\widehat{2}}\rangle, |\phi_{\widehat{2}}\rangle\right)_{\widehat{2}}. \end{aligned} \tag{40}$$

As the reader can see, the caret notation $(\widehat{k})$ will be used systematically to distinguish idempotent projection of ket, scalar product as well as scalar. In fact, this notation is simply a convenient way to deal with the idempotent representation $P_k(\cdot)$ in a more compact form.
We point out that a bicomplex scalar product is **completely characterized** by the two standard scalar products $(\cdot, \cdot)_{\widehat{k}}$ on V_k. In fact, if $(\cdot, \cdot)_{\widehat{k}}$ is an arbitrary scalar product on V_k, for $k = 1, 2$, then $(\cdot, \cdot)$ defined as in (40) is a bicomplex scalar product on M.
In this work, we will used the Dirac notation

$$(|\psi\rangle, |\phi\rangle) = \langle\psi|\phi\rangle = \mathbf{e_1}\langle\psi_{\widehat{1}}|\phi_{\widehat{1}}\rangle_{\widehat{1}} + \mathbf{e_2}\langle\psi_{\widehat{2}}|\phi_{\widehat{2}}\rangle_{\widehat{2}} \tag{41}$$

for the scalar product. The one-to-one correspondence between *bra* $\langle \cdot |$ and *ket* $| \cdot \rangle$ can be establish from the bicomplex Riesz theorem (Gervais Lavoie et al., 2010a, Th. 3.7) that we will present in section 3.

2.2.2 Bicomplex linear operators

A *bicomplex linear operator* A is a mapping from M to M such that, for any $s, t \in \mathbb{T}$ and any $|\psi\rangle, |\phi\rangle \in M$

$$A(s|\psi\rangle + t|\phi\rangle) = sA|\psi\rangle + tA|\phi\rangle. \tag{42}$$

A bicomplex linear operator A can always be written as $A = \mathbf{e_1} A_{\hat{1}} + \mathbf{e_2} A_{\hat{2}}$ and then,

$$A|\psi\rangle = \mathbf{e_1} A_{\hat{1}} |\psi_{\hat{1}}\rangle + \mathbf{e_2} A_{\hat{2}} |\psi_{\hat{2}}\rangle \tag{43}$$

where

$$A_{\hat{k}} |\psi_{\hat{k}}\rangle := P_k(A) |\psi_{\hat{k}}\rangle = P_k(A|\psi\rangle), \qquad \forall |\psi\rangle \in M, \qquad k = 1, 2. \tag{44}$$

The bicomplex *adjoint* operator A^* of A is the operator defined so that for any $|\psi\rangle, |\phi\rangle \in M$

$$(|\psi\rangle, A|\phi\rangle) = (A^*|\psi\rangle, |\phi\rangle). \tag{45}$$

One can show that in finite-dimensional free $\mathbb{T}$-modules, the adjoint always exists, is linear and satisfies (Rochon & Tremblay, 2006, Sec. 8.1)

$$(A^*)^* = A, \qquad (sA + tB)^* = s^\dagger A^* + t^\dagger B^* \qquad \text{and} \qquad (AB)^* = B^* A^*. \tag{46}$$

The reader can noted that we will used the same symbol for the adjoint operator in M or in V_k ;

$$A^* = \mathbf{e_1} A^*_{\hat{1}} + \mathbf{e_2} A^*_{\hat{2}}. \tag{47}$$

We shall say that a ket $|\psi\rangle$ belongs to the null cone ($\mathcal{NC}$) if either $|\psi_{\hat{1}}\rangle = 0$ or $|\psi_{\hat{2}}\rangle = 0$, and that a linear operator A belongs to the null cone ($\mathcal{NC}$) if either $A_{\hat{1}} = 0$ or $A_{\hat{2}} = 0$.
A bicomplex *self-adjoint* operator is a linear operator H such that

$$(|\psi\rangle, H|\phi\rangle) = (H|\psi\rangle, |\phi\rangle) \tag{48}$$

for all $|\psi\rangle$ and $|\phi\rangle$ in M.
Let $A : M \to M$ be a bicomplex linear operator. If there exists $\lambda \in \mathbb{T}$ and a ket $|\psi\rangle \in M$ such that $|\psi\rangle \notin \mathcal{NC}$ and that

$$A|\psi\rangle = \lambda|\psi\rangle \tag{49}$$

holds, then λ is called a bicomplex *eigenvalue* of A and $|\psi\rangle$ is called an *eigenket* of A corresponding to the eigenvalue λ. It was shown in (Rochon & Tremblay, 2006, Th. 14) that the eigenvalues of a self-adjoint operator acting in a finite-dimensional free $\mathbb{T}$-module, associated with eigenkets not in the null cone, are hyperbolic numbers.

Moreover, the eigenket equation (49) is equivalent to the system of two eigenket equations given by

$$A_{\widehat{k}}|\psi_{\widehat{k}}\rangle = \lambda_{\widehat{k}}|\psi_{\widehat{k}}\rangle\,, \qquad k = 1,2, \tag{50}$$

where $\lambda = \mathbf{e_1}\lambda_{\widehat{1}} + \mathbf{e_2}\lambda_{\widehat{2}}$, $\lambda_{\widehat{1}}, \lambda_{\widehat{2}} \in \mathbb{C}(\mathbf{i_1})$ and $|\psi\rangle = \mathbf{e_1}|\psi_{\widehat{1}}\rangle + \mathbf{e_2}|\psi_{\widehat{2}}\rangle$. We say that $|\psi\rangle$ is an *eigenket* of A rather then an eigenvector because element of M are modules instead of vectors. For a complete treatment of the Module Theory, see (Bourbaki, 2006).
The reader can remark that the element $|\psi_{\widehat{k}}\rangle$ was noted by $|\psi\rangle_{\widehat{k}}$ in (Gervais Lavoie et al., 2010a; 2011). However, the notation $|\psi_{\widehat{k}}\rangle$ is more appropriated here with scalar product in the Dirac notation.

3. Infinite-dimensional bicomplex Hilbert spaces

The mathematical structure of standard quantum mechanics (SQM) consists in Hilbert spaces, frequently infinite-dimensional ones, defined over the field of complex numbers (Birkhoff & Von Neumann, 1936). In the case of bicomplex quantum mechanics (BQM), the natural extension is to deal with infinite-dimensional bicomplex Hilbert spaces. We will sketched some important results here but proof and additional material can be found in (Gervais Lavoie et al., 2010a).

Result 1. *Let M be a $\mathbb{T}$-module and let $(\cdot,\cdot)$ be a bicomplex scalar product define on M. The space $\{M,(\cdot,\cdot)\}$ is called a* $\mathbb{T}$-inner product space*, or* bicomplex pre-Hilbert space*. When no confusion arise, we will noted $\{M,(\cdot,\cdot)\}$ as M.*

We defined a *bicomplex Hilbert space* as a $\mathbb{T}$-inner product space (bicomplex pre-Hilbert space) which is complete (with respect to the $\mathbb{T}$-norm induced by the bicomplex scalar product $(\cdot,\cdot)$).

Result 2. *Because $M = V_1 \oplus V_2$, and $(\cdot,\cdot) = (\cdot,\cdot)_{\widehat{1}}\mathbf{e_1} + (\cdot,\cdot)_{\widehat{2}}\mathbf{e_2}$, we have that $\{M,(\cdot,\cdot)\}$ is a bicomplex Hilbert space if and only if $\left\{V_k,(\cdot,\cdot)_{\widehat{k}}\right\}$ is complete, $k = 1,2$.*

As a corollary of this result, if $\{M,(\cdot,\cdot)\}$ is a bicomplex Hilbert space, then $\left\{V_k,(\cdot,\cdot)_{\widehat{k}}\right\}$ is a complex (in $\mathbb{C}(\mathbf{i_1})$) Hilbert space for $k = 1,2$.
A direct application of this corollary leads to the bicomplex Riesz representation theorem as follow.

Result 3 (Riesz). *Let $\{M,(\cdot,\cdot)\}$ be a bicomplex Hilbert space and let $f : M \to \mathbb{T}$ be a continuous linear functional on M. Then, there exist a unique $|\psi\rangle \in M$ such that $\forall|\phi\rangle \in M$, $f(|\phi\rangle) = (|\psi\rangle,|\phi\rangle) = \langle\psi|\phi\rangle$.*

The bicomplex Riesz theorem means that for an arbitrary bicomplex Hilbert space M, the dual space M^* of continuous linear functionals on M can be identified with M through the bicomplex scalar product $(\cdot,\cdot)$.
Let us take a look at the orthonormalization of elements of M. Let $\{|s_l\rangle\}$ be a countable basis of M. Then, $\{|s_l\rangle\}$ can always be orthonormalized.
It is interesting to note that the normalizability of kets requires that the scalar product belongs to $\mathbb{D}^+$. To see this, let us write $(|m_1\rangle,|m_1\rangle) = a_{\widehat{1}}\mathbf{e_1} + a_{\widehat{2}}\mathbf{e_2}$ with $a_{\widehat{1}}, a_{\widehat{2}} \in \mathbb{R}$, and let

$$|m_1'\rangle = (z_{\widehat{1}}\mathbf{e_1} + z_{\widehat{2}}\mathbf{e_2})|m_1\rangle,$$

with $z_{\widehat{1}}, z_{\widehat{2}} \in \mathbb{C}(\mathbf{i_1})$ and $z_{\widehat{1}} \neq 0 \neq z_{\widehat{2}}$. We get

$$\begin{aligned}(|m_1'\rangle, |m_1'\rangle) &= (|z_{\widehat{1}}|^2\mathbf{e_1} + |z_{\widehat{2}}|^2\mathbf{e_2})\,(|m_1\rangle, |m_1\rangle)\\ &= (|z_{\widehat{1}}|^2\mathbf{e_1} + |z_{\widehat{2}}|^2\mathbf{e_2})(a_{\widehat{1}}\mathbf{e_1} + a_{\widehat{2}}\mathbf{e_2})\\ &= c_{\widehat{1}}a_{\widehat{1}}\mathbf{e_1} + c_{\widehat{2}}a_{\widehat{2}}\mathbf{e_2}, \end{aligned} \tag{51}$$

with $c_{\widehat{k}} = |z_{\widehat{k}}|^2 \in \mathbb{R}^+$. The normalization condition of $|m_1'\rangle$ becomes

$$c_{\widehat{1}}a_{\widehat{1}}\mathbf{e_1} + c_{\widehat{2}}a_{\widehat{2}}\mathbf{e_2} = 1, \tag{52}$$

or $c_{\widehat{1}}a_{\widehat{1}} = 1 = c_{\widehat{2}}a_{\widehat{2}}$. This is possible only if $a_{\widehat{1}} > 0$ and $a_{\widehat{2}} > 0$. Hence, in particular $(|m_1\rangle, |m_1\rangle) \in \mathbb{D}^+$.
In fact, we will show here that the bicomplex normalization is a more restricting condition than the complex one. Let us try to normalized a ket $|m_2\rangle \in \mathcal{NC}$. Suppose that $|m_2\rangle = \mathbf{e_1}|m_2\rangle$ (which means that the part in $\mathbf{e_2}$ is $|0\rangle$) and let us write $(|m_2\rangle, |m_2\rangle) = a_{\widehat{1}}\mathbf{e_1} + a_{\widehat{2}}\mathbf{e_2}$ as previously. From the properties of the bicomplex scalar product 2.2.1, we can write

$$(|m_2\rangle, |m_2\rangle) = (|m_2\rangle, \mathbf{e_1}|m_2\rangle) = \mathbf{e_1}(|m_2\rangle, |m_2\rangle), \tag{53}$$

which directly imply that

$$a_{\widehat{1}}\mathbf{e_1} + a_{\widehat{2}}\mathbf{e_2} = \mathbf{e_1}\left(a_{\widehat{1}}\mathbf{e_1} + a_{\widehat{2}}\mathbf{e_2}\right) = a_{\widehat{1}}\mathbf{e_1}. \tag{54}$$

In other words, $a_{\widehat{2}} = 0$, but in this case, we cannot satisfy the condition (52) ($\mathbf{e_1}$ is not invertible) and then, $|m_2\rangle$ is not normalizable.
To state this another way, the requirement to be not in the $\mathcal{NC}$ is embedded in the normalization requirement. In this sense, we can say that the bicomplex normalization is more restrictive than the complex one, because it exclude an infinite number of elements of M, those in the $\mathcal{NC}$ instead of only one in the complex case, the vector $|0\rangle$. However, in practice, this is not a big glitch because we naturally avoid the $\mathcal{NC}$ to avoid the "trivial" situation where $M \simeq \mathbf{e_k}V_k$.

4. Bicomplex quantum mechanics

Bicomplex quantum mechanics was first investigated in (Rochon & Tremblay, 2004; 2006). In (Rochon & Tremblay, 2004), the bicomplex Schrödinger equation was introduced and the continuity equations and symmetries was derived. The bicomplex Born probability formulas was studied by extracting some real moduli. In (Rochon & Tremblay, 2006), the concept of free modules over the ring of bicomplex numbers was developed, bicomplex scalar product, Dirac notation and linear operator was also investigated.
Motivated by these results, the problem of the bicomplex quantum harmonic oscillator was worked out in details in (Gervais Lavoie et al., 2010b), and the eigenvalues and eigenfunctions was obtained. The section 4.1 is a summary of important results on the bicomplex harmonic oscillator.
First of all, we will state a fundamental postulate on which the BQM stands.

Postulate 1. *There exist two operators X and P (called the bicomplex* position *and* momentum *operators respectively) in M such that X and P are self-adjoint and their commutation relation is a multiple of the identity.*

Mathematically, this postulate means that

$$[X,P] = wI, \qquad w \in \mathbb{T}, \qquad X,P,I \in M, \qquad X^* = X \qquad \text{and} \qquad P^* = P. \tag{55}$$

Without lost of generality, we can rewrite w as $\mathbf{i_1}\hbar\xi$, $\xi \in \mathbb{T}$. Let $|E\rangle \notin \mathcal{NC}$ be a normalizable element of M. The properties of the bicomplex scalar product 2.2.1 allow us to write

$$\begin{aligned}
\mathbf{i_1}\hbar\xi(|E\rangle,|E\rangle) &= (|E\rangle,\mathbf{i_1}\hbar\xi I|E\rangle) \\
&= (|E\rangle,XP|E\rangle) - (|E\rangle,PX|E\rangle) \\
&= (X|E\rangle,P|E\rangle) - (P|E\rangle,X|E\rangle) \\
&= (PX|E\rangle,|E\rangle) - (XP|E\rangle,|E\rangle) \\
&= (-\mathbf{i_1}\hbar\xi I|E\rangle,|E\rangle) \\
&= \mathbf{i_1}\hbar\xi^{\dagger}(|E\rangle,|E\rangle).
\end{aligned} \tag{56}$$

Because $|E\rangle$ is normalizable, $(|E\rangle,|E\rangle) \notin \mathcal{NC}$ and we have that $\xi = \xi^{\dagger}$ which signify that $\xi \in \mathbb{D}$, or $\xi = \xi_{\hat{1}}\mathbf{e_1} + \xi_{\hat{2}}\mathbf{e_2}$ with $\xi_{\hat{1}}, \xi_{\hat{2}} \in \mathbb{R}$.
As the reader can see, the assumptions made here on X, P, ξ and $|E\rangle$ are very general ones, and are closely related to the assumptions made in SQM. The main idea beyond all this is to build the BQM standing on as least assumptions as possible. For example, we could postulate that in BQM, $[X,P] = \mathbf{i_1}\hbar I$ as in the standard case, without questioning itself. However, as we see later, if we had done that, we would have neglected an apparently nontrivial part of the solution.

4.1 The bicomplex quantum harmonic oscillator

We start this section with a little calculation that allow us to restrict further the constant ξ. This derivation is given in (Gervais Lavoie et al., 2010b), but we think that it is instructive to give it again here.
First of all, to work out the quantum harmonic oscillator problem, we need an Hamiltonian. We will consider the following

$$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2X^2, \tag{57}$$

as the Hamiltonian of the bicomplex harmonic oscillator, where m and ω are positive real numbers and X and P are the bicomplex self-adjoint operators defined previously. Clearly, this imply $H : M \to M$ and that H is self-adjoint.
Secondly, we will ask the following: Is it possible to further restrict meaningful values of ξ, for instance by a simple rescaling of X and P? To answer this question, let us write

$$X = (\alpha_{\hat{1}}\mathbf{e_1} + \alpha_{\hat{2}}\mathbf{e_2})X', \qquad P = (\beta_{\hat{1}}\mathbf{e_1} + \beta_{\hat{2}}\mathbf{e_2})P', \tag{58}$$

with nonzero $\alpha_{\widehat{k}}$ and $\beta_{\widehat{k}}$ ($k = 1,2$). For X' and P' to be self-adjoint, $\alpha_{\widehat{k}}$ and $\beta_{\widehat{k}}$ must be real. Making use of (57) we find that

$$\begin{aligned} H =& \frac{1}{2m}(\beta_{\widehat{1}}^2\mathbf{e_1} + \beta_{\widehat{2}}^2\mathbf{e_2})(P')^2 + \frac{1}{2}m\omega^2(\alpha_{\widehat{1}}^2\mathbf{e_1} + \alpha_{\widehat{2}}^2\mathbf{e_2})(X')^2 \\ =& \frac{1}{2m'}(P')^2 + \frac{1}{2}m'(\omega')^2(X')^2. \end{aligned} \tag{59}$$

For m' and ω' to be positive real numbers, $\alpha_{\widehat{1}}^2\mathbf{e_1} + \alpha_{\widehat{2}}^2\mathbf{e_2}$ and $\beta_{\widehat{1}}^2\mathbf{e_1} + \beta_{\widehat{2}}^2\mathbf{e_2}$ must also belong to $\mathbb{R}^+$. This entails that $\alpha_{\widehat{1}}^2 = \alpha_{\widehat{2}}^2$ and $\beta_{\widehat{1}}^2 = \beta_{\widehat{2}}^2$, or equivalently $\alpha_{\widehat{1}} = \pm\alpha_{\widehat{2}}$ and $\beta_{\widehat{1}} = \pm\beta_{\widehat{2}}$. Hence we can write

$$\begin{aligned} \mathbf{i_1}\hbar(\xi_{\widehat{1}}\mathbf{e_1} + \xi_{\widehat{2}}\mathbf{e_2})I &= [X,P] \\ &= [(\alpha_{\widehat{1}}\mathbf{e_1} + \alpha_{\widehat{2}}\mathbf{e_2})X', (\beta_{\widehat{1}}\mathbf{e_1} + \beta_{\widehat{2}}\mathbf{e_2})P'] \\ &= (\alpha_{\widehat{1}}\beta_{\widehat{1}}\mathbf{e_1} + \alpha_{\widehat{2}}\beta_{\widehat{2}}\mathbf{e_2})[X', P']. \end{aligned} \tag{60}$$

But this in turn implies that

$$[X', P'] = \mathbf{i_1}\hbar\left(\frac{\xi_{\widehat{1}}}{\alpha_{\widehat{1}}\beta_{\widehat{1}}}\mathbf{e_1} + \frac{\xi_{\widehat{2}}}{\alpha_{\widehat{2}}\beta_{\widehat{2}}}\mathbf{e_2}\right)I = \mathbf{i_1}\hbar(\xi'_{\widehat{1}}\mathbf{e_1} + \xi'_{\widehat{2}}\mathbf{e_2})I. \tag{61}$$

This equation shows that $\alpha_{\widehat{1}}$, $\alpha_{\widehat{2}}$, $\beta_{\widehat{1}}$ and $\beta_{\widehat{2}}$ can always be picked so that $\xi'_{\widehat{1}}$ and $\xi'_{\widehat{2}}$ are positive. Furthermore, we can choose $\alpha_{\widehat{1}}$ and $\beta_{\widehat{1}}$ so as to make $\xi'_{\widehat{1}}$ equal to 1. But since $|\alpha_{\widehat{1}}\beta_{\widehat{1}}| = |\alpha_{\widehat{2}}\beta_{\widehat{2}}|$, we have no control over the norm of $\xi'_{\widehat{2}}$. The upshot is that we can always write H as in (57), with the commutation relation of X and P given by

$$[X,P] = \mathbf{i_1}\hbar\xi I = \mathbf{i_1}\hbar(\xi_{\widehat{1}}\mathbf{e_1} + \xi_{\widehat{2}}\mathbf{e_2})I \qquad \text{with} \qquad \xi_{\widehat{1}}, \xi_{\widehat{2}} \in \mathbb{R}^+. \tag{62}$$

We also have the freedom of setting either $\xi_{\widehat{1}} = 1$ or $\xi_{\widehat{2}} = 1$, but not both. In all this work, we assumed that $\xi \notin \mathcal{NC}$ (which means $\xi_{\widehat{k}} \neq 0$, $k = 1,2$). Otherwise, BQM is reduced to SQM time a constant.

In (Gervais Lavoie et al., 2010b), we work out the bicomplex harmonic oscillator problem in the algebraic way in full details. Here, to present our results, we will give a sketch of the differential solution and show that it's lead to the same eigenfunctions.

First of all, we need to compute the action of the operators X and P in their functional form. To do this, let us assume that

$$X|x\rangle = x|x\rangle, \qquad X: M \to M, \qquad |x\rangle \in M \qquad \text{and} \qquad x \in \mathbb{R}. \tag{63}$$

This signify that $|x\rangle$ is an eigenket of X and that x is the real eigenvalue of X associate with the ket $|x\rangle$. Because $|x\rangle$ is an eigenket of the position operator, it is reasonable to write $\langle x|x'\rangle = \delta(x - x')$, with $\delta(x - x')$ the real Dirac delta function. Let us now consider the following

$$\langle x|[X,P]|x'\rangle = \langle x|\mathbf{i_1}\hbar\xi I|x'\rangle = \mathbf{i_1}\hbar\xi\delta(x - x'). \tag{64}$$

On the other hand,

$$\begin{aligned}\langle x|[X,P]|x'\rangle &= \langle x|XP|x'\rangle - \langle x|PX|x'\rangle \\ &= \langle x'|PX|x\rangle^{\dagger} - x'\langle x|P|x'\rangle \\ &= x^{\dagger}\langle x|P|x'\rangle - x'\langle x|P|x'\rangle \\ &= (x^{\dagger}-x')\langle x|P|x'\rangle \\ &= (x-x')\langle x|P|x'\rangle. \end{aligned} \tag{65}$$

Putting the two results together, we get

$$(x-x')\langle x|P|x'\rangle = \mathbf{i_1}\hbar\xi\delta(x-x'). \tag{66}$$

In SQM, we know that $(x-x')\frac{d}{dx}\delta(x-x') = -\delta(x-x')$ (Marchildon, 2002, chap. 5). But we can also use this result here because $x \in \mathbb{R}$. This lead to

$$\langle x|P|x'\rangle = -\mathbf{i_1}\hbar\xi\frac{d}{dx}\delta(x-x'). \tag{67}$$

At this point, it is easy to see that the functional form of the position and momentum bicomplex oparators are given by

$$X \to x, \qquad\qquad P \to -\mathbf{i_1}\hbar\xi\frac{d}{dx}. \tag{68}$$

With these representations, we can rewrite the Hamiltonian (57) as a differential equation. Let $\phi_n(x)$ be a normalisable eigenfunction of H (in the coordinate representation). Then, we have

$$\begin{aligned} &\frac{1}{2m}P^2\phi_n(x) + \frac{1}{2}m\omega^2X^2\phi_n(x) = H\phi_n(x) \\ \Rightarrow \qquad &-\frac{\hbar^2\xi^2}{2m}\frac{d^2}{dx^2}\phi_n(x) + \frac{1}{2}m\omega^2x^2\phi_n(x) = E_n\phi_n(x). \end{aligned} \tag{69}$$

A priori, this equation is a bicomplex equation of the real variable x. Taking $\xi = \mathbf{e_1}\xi_{\widehat{1}} + \mathbf{e_2}\xi_{\widehat{2}}$, $E_n = \mathbf{e_1}E_{n\widehat{1}} + \mathbf{e_2}E_{n\widehat{2}}$ and $\phi_n(x) = \mathbf{e_1}\phi_{n\widehat{1}}(x) + \mathbf{e_2}\phi_{n\widehat{2}}(x)$, we get

$$-\frac{\hbar^2\xi_{\widehat{k}}^2}{2m}\frac{d^2}{dx^2}\phi_{n\widehat{k}}(x) + \frac{1}{2}m\omega^2x^2\phi_{n\widehat{k}}(x) = E_{n\widehat{k}}\phi_{n\widehat{k}}(x) \qquad \text{with} \qquad k=1,2. \tag{70}$$

In this equation, $\xi_{\widehat{k}} \in \mathbb{R}^+$ because of (62), $E_{n\widehat{k}} \in \mathbb{R}$ because E_n is the eigenvalue of a self-adjoint operator, and $\phi_{n\widehat{k}}(x)$ is a complex function of the real variable x. In fact, (70) is exactly the differential equation of the standard quantum harmonic oscillator with $\hbar$ replaced by $\hbar\xi_{\widehat{k}}$. This also mean that we already know the solutions for $\phi_{n\widehat{k}}(x)$ and for $E_{n\widehat{k}}$, they are given by (Marchildon, 2002, chap. 5)

$$\phi_{n\widehat{k}}(x) = \left[\sqrt{\frac{m\omega}{\pi\hbar\xi_{\widehat{k}}}}\frac{1}{2^n n!}\right]^{1/2} \exp\left\{-\frac{m\omega}{2\hbar\xi_{\widehat{k}}}x^2\right\} H_n\left(\sqrt{\frac{m\omega}{\hbar\xi_{\widehat{k}}}}x\right), \tag{71}$$

$$E_{n\widehat{k}} = \hbar\xi_{\widehat{k}}\omega\left(n+\frac{1}{2}\right), \tag{72}$$

with $H_n(x)$ the Hermite polynomial of order n in the real variable x. Let us define the variable $\theta_{\widehat{k}}$ for convenience as

$$\theta_{\widehat{k}} := \sqrt{\frac{m\omega}{\hbar \xi_{\widehat{k}}}} x \qquad \text{for} \qquad k = 1, 2. \tag{73}$$

It can be shown (Price, 1991) that for any bicomplex number $w = z_{\widehat{1}}\mathbf{e_1} + z_{\widehat{2}}\mathbf{e_2}$,

$$\mathrm{e}^{w} = \mathbf{e_1}\mathrm{e}^{z_{\widehat{1}}} + \mathbf{e_2}\mathrm{e}^{z_{\widehat{2}}}. \tag{74}$$

This holds also for any polynomial function $Q(w)$, that is,

$$Q(z_{\widehat{1}}\mathbf{e_1} + z_{\widehat{2}}\mathbf{e_2}) = \mathbf{e_1}Q(z_{\widehat{1}}) + \mathbf{e_2}Q(z_{\widehat{2}}). \tag{75}$$

Moreover, if $\xi = \xi_{\widehat{1}}\mathbf{e_1} + \xi_{\widehat{2}}\mathbf{e_2}$ with $\xi_{\widehat{1}}$ and $\xi_{\widehat{2}}$ positive, we have

$$\frac{1}{\xi^{1/4}} = \frac{\mathbf{e_1}}{\xi_{\widehat{1}}^{1/4}} + \frac{\mathbf{e_2}}{\xi_{\widehat{2}}^{1/4}}. \tag{76}$$

From (72), we have that the energy E_n of the bicomplex harmonic oscillator is given by

$$E_n = E_{n\widehat{1}}\mathbf{e_1} + E_{n\widehat{2}}\mathbf{e_2} = \mathbf{e_1}\hbar\xi_{\widehat{1}}\omega\left(n + \frac{1}{2}\right) + \mathbf{e_2}\hbar\xi_{\widehat{2}}\omega\left(n + \frac{1}{2}\right) = \hbar\omega\left(n + \frac{1}{2}\right)\xi. \tag{77}$$

For the eigenfunctions, (71) imply that $\phi_n(x)$ will be given by

$$\begin{aligned}
\phi_n(x) &= \phi_{n\widehat{1}}(x)\mathbf{e_1} + \phi_{n\widehat{2}}(x)\mathbf{e_2} \\
&= \mathbf{e_1}\left[\sqrt{\frac{m\omega}{\pi\hbar\xi_{\widehat{1}}}}\frac{1}{2^n n!}\right]^{1/2} \mathrm{e}^{-\theta_{\widehat{1}}^2/2} H_n\left(\theta_{\widehat{1}}\right) + \mathbf{e_2}\left[\sqrt{\frac{m\omega}{\pi\hbar\xi_{\widehat{2}}}}\frac{1}{2^n n!}\right]^{1/2} \mathrm{e}^{-\theta_{\widehat{2}}^2/2} H_n\left(\theta_{\widehat{2}}\right) \\
&= \left\{\mathbf{e_1}\left[\sqrt{\frac{m\omega}{\pi\hbar\xi_{\widehat{1}}}}\frac{1}{2^n n!}\right]^{1/2} + \mathbf{e_2}\left[\sqrt{\frac{m\omega}{\pi\hbar\xi_{\widehat{2}}}}\frac{1}{2^n n!}\right]^{1/2}\right\} \\
&\quad \cdot \left\{\mathbf{e_1}\mathrm{e}^{-\theta_{\widehat{1}}^2/2} + \mathbf{e_2}\mathrm{e}^{-\theta_{\widehat{2}}^2/2}\right\}\left\{\mathbf{e_1}H_n(\theta_{\widehat{1}}) + \mathbf{e_2}H_n(\theta_{\widehat{2}})\right\}.
\end{aligned} \tag{78}$$

Moreover, we the help of (74) and (76), we obtain

$$\phi_n(x) = \left[\sqrt{\frac{m\omega}{\pi\hbar\xi}}\frac{1}{2^n n!}\right]^{1/2} \mathrm{e}^{-\theta^2/2} H_n(\theta), \tag{79}$$

where

$$H_n(\theta) := \mathbf{e_1}H_n(\theta_{\widehat{1}}) + \mathbf{e_2}H_n(\theta_{\widehat{2}}) \tag{80}$$

is a hyperbolic Hermite polynomial of order n.
Equation (79) expresses normalized eigenfunctions of the bicomplex harmonic oscillator Hamiltonian purely in terms of hyperbolic constants and functions, with no reference to a particular representation like $\{\mathbf{e_k}\}$. Indeed ξ can be viewed as a $\mathbb{D}^+$ constant, θ is equal to $\sqrt{m\omega/\hbar\xi}\, x$ and $H_n(\theta)$ is just the Hermite polynomial in θ.

In (Gervais Lavoie et al., 2010a), we show that the set $\{\phi_n(x) \mid n = 0, 1, \ldots\}$ form a $\mathbb{T}$-basis of M, and that M is a bicomplex Hilbert space with the following decomposition for an arbitrary $\psi(x) \in M$;

$$\psi(x) = \sum_n w_n \phi_n(x) \qquad \text{with} \qquad w_n \in \mathbb{T}. \tag{81}$$

Moreover, in (Gervais Lavoie et al., 2010b), we show that the most general eigenfunction of H is given by a linear combination, in the idempotent basis, of two functions $\phi_{n\widehat{k}}(x)$ with some coefficient, and possibly different order n, such as

$$\phi(x) = \mathbf{e_1} w_{l\widehat{1}} \phi_{l\widehat{1}}(x) + \mathbf{e_2} w_{n\widehat{2}} \phi_{n\widehat{2}} \tag{82}$$

with $w_{l\widehat{1}}$ and $w_{n\widehat{2}}$ in $\mathbb{C}(\mathbf{i_1})$ and $l, n = 0, 1, \ldots$. The associated energy is then

$$E = \hbar\omega \left\{ \left(l + \frac{1}{2} \right) \mathbf{e_1} \xi_{\widehat{1}} + \left(n + \frac{1}{2} \right) \mathbf{e_2} \xi_{\widehat{2}} \right\}. \tag{83}$$

The eigenfunction (82) can be written explicitly as

$$\phi(x) = \left[\frac{m\omega}{\pi\hbar} \right]^{1/4} \left\{ \mathbf{e_1} \frac{w_{l\widehat{1}} \mathrm{e}^{-\theta_{\widehat{1}}^2/2}}{\sqrt{2^l l! \sqrt{\xi_{\widehat{1}}}}} H_l(\theta_{\widehat{1}}) + \mathbf{e_2} \frac{w_{n\widehat{2}} \mathrm{e}^{-\theta_{\widehat{2}}^2/2}}{\sqrt{2^n n! \sqrt{\xi_{\widehat{2}}}}} H_n(\theta_{\widehat{2}}) \right\}. \tag{84}$$

The function ϕ is normalized, *i.e.* $(\phi, \phi) = 1$, if

$$|w_{l\widehat{1}}|^2 \mathbf{e_1} + |w_{n\widehat{2}}|^2 \mathbf{e_2} = 1. \tag{85}$$

$\phi(x)$ can also be rewrite in term of 1 and $\mathbf{j}$. From (16), we only have to rewrite the idempotent basis in term of 1 and $\mathbf{j}$ to find (we take ϕ normalized for simplicity)

$$\begin{aligned} \phi(x) = \frac{1}{2} \left[\frac{m\omega}{\pi\hbar} \right]^{1/4} & \left\{ \left(\frac{\mathrm{e}^{-\theta_{\widehat{1}}^2/2}}{\sqrt{2^l l! \sqrt{\xi_{\widehat{1}}}}} H_l(\theta_{\widehat{1}}) + \frac{\mathrm{e}^{-\theta_{\widehat{2}}^2/2}}{\sqrt{2^n n! \sqrt{\xi_{\widehat{2}}}}} H_n(\theta_{\widehat{2}}) \right) \right. \\ & \left. + \mathbf{j} \left(\frac{\mathrm{e}^{-\theta_{\widehat{1}}^2/2}}{\sqrt{2^l l! \sqrt{\xi_{\widehat{1}}}}} H_l(\theta_{\widehat{1}}) - \frac{\mathrm{e}^{-\theta_{\widehat{2}}^2/2}}{\sqrt{2^n n! \sqrt{\xi_{\widehat{2}}}}} H_n(\theta_{\widehat{2}}) \right) \right\}. \end{aligned} \tag{86}$$

This last equation however is a kind of hybrid between the representation $\{1, \mathbf{j}\}$ and $\{\mathbf{e_1}, \mathbf{e_2}\}$. Indeed, $\theta_{\widehat{k}}$ and $\xi_{\widehat{k}}$ are define in the idempotent basis. But, from (19), it is not hard to see that we can rewrite $\xi_{\widehat{k}}$ in term of new parameters α and β (that have nothing to do with those of (58)) as

$$\xi_{\widehat{1}} = \alpha + \beta, \qquad \xi_{\widehat{2}} = \alpha - \beta \qquad \text{such that} \qquad \xi = \alpha + \beta\mathbf{j}, \qquad \alpha, \beta \in \mathbb{R}. \tag{87}$$

From this, we have that

$$\theta_{\widehat{1}} = \sqrt{\frac{m\omega}{\hbar(\alpha + \beta)}} x \qquad \text{and} \qquad \theta_{\widehat{2}} = \sqrt{\frac{m\omega}{\hbar(\alpha - \beta)}} x. \tag{88}$$

Using (87) and (88) in (86), we can rewrite $\phi(x)$ purely in term of 1 and $\mathbf{j}$, without any allusion to the idempotent basis. We find

$$\phi(x) = \frac{1}{2}\left[\frac{m\omega}{\pi\hbar}\right]^{1/4}$$
$$\cdot\left\{\left(\frac{\exp\left\{\frac{-m\omega}{2\hbar(\alpha+\beta)}x^2\right\}}{\sqrt{2^l l!\sqrt{\alpha+\beta}}}H_l\left(\sqrt{\frac{m\omega}{\hbar(\alpha+\beta)}}x\right) + \frac{\exp\left\{\frac{-m\omega}{2\hbar(\alpha-\beta)}x^2\right\}}{\sqrt{2^n n!\sqrt{\alpha-\beta}}}H_n\left(\sqrt{\frac{m\omega}{\hbar(\alpha-\beta)}}x\right)\right)\right.$$
$$\left.+\mathbf{j}\left(\frac{\exp\left\{\frac{-m\omega}{2\hbar(\alpha+\beta)}x^2\right\}}{\sqrt{2^l l!\sqrt{\alpha+\beta}}}H_l\left(\sqrt{\frac{m\omega}{\hbar(\alpha+\beta)}}x\right) - \frac{\exp\left\{\frac{-m\omega}{2\hbar(\alpha-\beta)}x^2\right\}}{\sqrt{2^n n!\sqrt{\alpha-\beta}}}H_n\left(\sqrt{\frac{m\omega}{\hbar(\alpha-\beta)}}x\right)\right)\right\}. \quad (89)$$

One can remark that the conditions $\xi \in \mathbb{D}^+$ and $\xi \notin \mathcal{NC}$ are express as $\alpha+\beta > 0$ and $\alpha-\beta > 0$ for the parameters α and β.
Another way to express our eigenfunctions in term of real and hyperbolic part is to rewrite the hyperbolic exponential $e^{-\theta^2/2}$ in term of real hyperbolic sinus and cosinus. Indeed, from (Rochon & Tremblay, 2004), we can write

$$\begin{aligned} e^{-\theta^2/2} &= e^{-\frac{(\theta_1^2+\theta_2^2)}{2}} e^{-\theta_1\theta_2\mathbf{j}} \\ &= e^{-\frac{(\theta_1^2+\theta_2^2)}{2}}\{\cosh\theta_1\theta_2 - \mathbf{j}\sinh\theta_1\theta_2\} \qquad \text{with} \qquad \theta = \theta_1+\theta_2\mathbf{j}. \end{aligned} \quad (90)$$

Taking

$$\xi = \alpha + \beta\mathbf{j}, \quad (91)$$

we have that

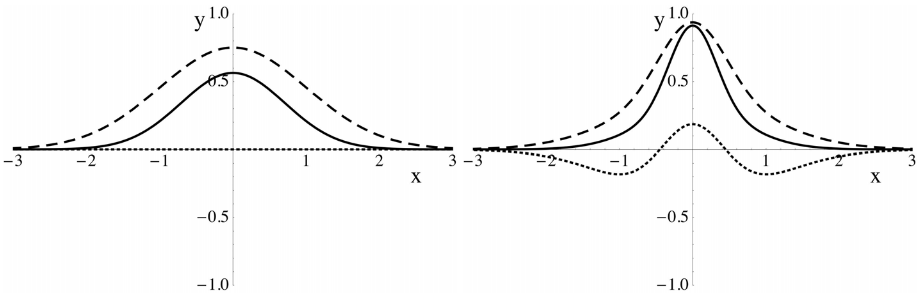
$$\xi^{-1/4} = \frac{(\alpha+\beta)^{-1/4} + (\alpha-\beta)^{1/4}}{2} + \mathbf{j}\frac{(\alpha+\beta)^{-1/4} - (\alpha-\beta)^{1/4}}{2} = \alpha' + \beta'\mathbf{j}. \quad (92)$$

For the normalized eigenfunction (79), we can then write

$$\phi_n(x) = \left[\sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{2^n n!}\right]^{1/2} e^{-\frac{(\theta_1^2+\theta_2^2)}{2}}$$
$$\cdot\left\{\left[\big(\alpha'\cosh\theta_1\theta_2 - \beta'\sinh\theta_1\theta_2\big)\mathrm{Re}\left(H_n(\theta)\right) + \big(\beta'\cosh\theta_1\theta_2 - \alpha'\sinh\theta_1\theta_2\big)\mathrm{Hy}\left(H_n(\theta)\right)\right]\right.$$
$$\left.\mathbf{j}\left[\big(\alpha'\cosh\theta_1\theta_2 - \beta'\sinh\theta_1\theta_2\big)\mathrm{Hy}\left(H_n(\theta)\right) + \big(\beta'\cosh\theta_1\theta_2 - \alpha'\sinh\theta_1\theta_2\big)\mathrm{Re}\left(H_n(\theta)\right)\right]\right\}, \quad (93)$$

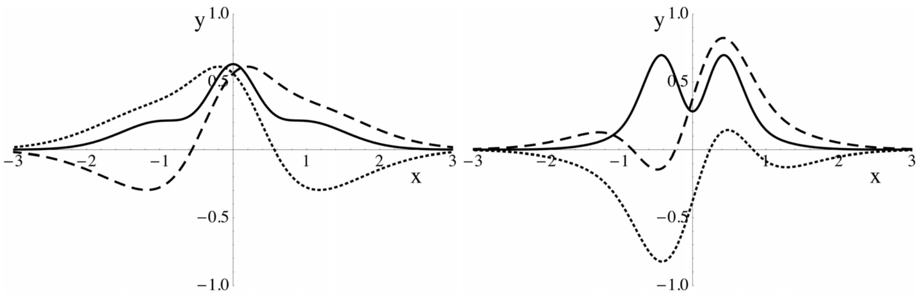
where $\mathrm{Re}\left(H_n(\theta)\right)$ and $\mathrm{Hy}\left(H_n(\theta)\right)$ stand for the real and the hyperbolic part of $H_n(\theta)$, respectively.
Finally, it is not so hard to see that if we take $\xi_{\hat{1}} = 1 = \xi_{\hat{2}}$ (resp. $\alpha = 1$ and $\beta = 0$) and $l = n$ (indirectly $X_{\hat{1}} = X_{\hat{2}}$, $P_{\hat{1}} = P_{\hat{2}}$ and so on), we recover the usual eigenfunctions and energy of the standard quantum harmonic oscillator.
We end this section with some plots of the eigenfunction $\phi(x)$ for different value of $\xi_{\hat{1}}$, $\xi_{\hat{2}}$, l and n. In Fig. 2 to 4, the dashed line stands for the real part, the dotted line for the hyperbolic

part and the full line is the probability density $|\phi(x)|^2 = |\phi_{\hat{1}}(x)|^2/2 + |\phi_{\hat{2}}(x)|^2/2$. We also take $m\omega/\hbar = 1$ on the y-axe for simplicity.



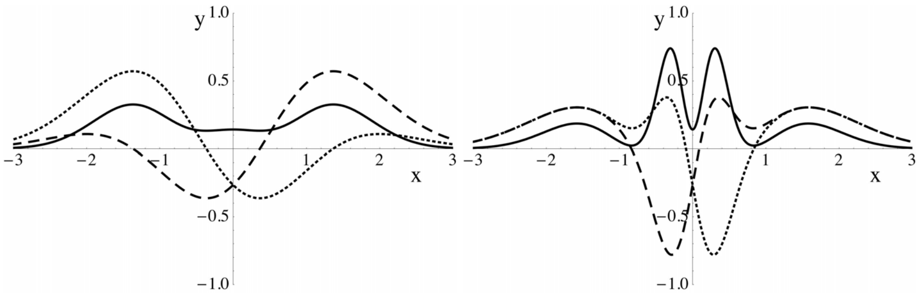
(a) $l = 0 = n$ and $\xi_{\hat{1}} = 1 = \xi_{\hat{2}}$, (b) $l = 0 = n$ and $\xi_{\hat{1}} = 0.2$, $\xi_{\hat{2}} = 1$.

Fig. 2. Eigenfunction (86) with $l = 0 = n$. Fig. (a) show that eigenfunctions of the harmonic oscillator of the SQM can be recover from the bicomplex eigenfunction (86).



(a) $l = 0$ and $n = 1$. (b) $l = 1$ and $n = 0$.

Fig. 3. Eigenfunction (86) with $\xi_{\hat{1}} = 0.2$, $\xi_{\hat{2}} = 1$.



(a) $\xi_{\hat{1}} = 1$ and $\xi_{\hat{2}} = 1$. (b) $\xi_{\hat{1}} = 1$ and $\xi_{\hat{2}} = 0.1$.

Fig. 4. Eigenfunction (86) with $l = 2$, $n = 1$.

5. The bicomplex Heisenberg uncertainty principle

The uncertainty principle, due to Heisenberg, is a fundamental principle in quantum mechanics, but also in post-classical physics in general. The uncertainty principle establish a lower limit on the theoretical precision that one can, even in principle, reach about two non-commuting observable of a physical system. This limit on the absolute precision that can be achieve is one of the biggest cut between the classical and deterministic physics, and the probabilistic post-classical quantum physics.

From the fundamental aspect of the uncertainty principle, it seems natural that all the extensions of standard quantum mechanics try to establish their own. For example, in quaternionic quantum mechanics, the uncertainty principle can be formulated as (Adler, 1995) $(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} |\langle C \rangle|^2$, with $[A, B] = IC$, where A, B and C are self-adjoint (left-acting) operators and I is a left-acting anti-self-adjoint operator. Even if A, B, C and I are quaternionic operators, the quaternionic uncertainty principle have essentially the same form as the Heisenberg uncertainty principle in SQM.

In this section, we find, in an algebraic way, the bicomplex uncertainty principle of two non-commuting bicomplex self-adjoint operators. Let A' and B' be these two bicomplex self-adjoint operators. With none of the eigenkets of A' nor B' in the null-cone, we assumed that the eigenvalues of A' and B' are hyperbolic numbers.

We start with the same definition of the mean value of an operator as in SQM, that is a sum over the eigenvalues times the probability. However, we used the bicomplex Born formula (Rochon & Tremblay, 2004, Th. 1) $\mathcal{P}(\ \cdot\ ) = |\psi|^2$, with $|\cdot|^2$ the Euclidean $\mathbb{R}^4$-norm, to define the probability. Let $A' : M \to M$ be such that $A'|a'_i\rangle = a'_i|a'_i\rangle$, with $\{a'_i\}$ the set of hyperbolic eigenvalues and $\{|a'_i\rangle\}$ an orthonormalized $\mathbb{T}$-basis of eigenkets of A. We define

$$\langle A' \rangle_{BQM} = \sum_i a'_i \mathcal{P}\left(A' \to a'_i\right) = \sum_i a'_i \left|\langle a'_i|\psi\rangle\right|^2 \in \mathbb{D}. \tag{94}$$

The reader can remark that $\mathcal{P}\left(A' \to a'_i\right) = \left|\langle a'_i|\psi\rangle\right|^2$ is a real probability because it is restricted to $[0,1]$ as long as $|\psi\rangle$ is normalized, and the sum of all probability is equal to 1. We know from (29) that $|\cdot|^2 = \frac{1}{2}|P_1(\cdot)|^2 + \frac{1}{2}|P_2(\cdot)|^2$. From the property of the bicomplex scalar product 2.2.1 (particularly (40)), we can write

$$\begin{aligned}
\langle A' \rangle_{BQM} &= \sum_i a'_i \frac{\left|\langle a'_{i\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\right|^2 + \left|\langle a'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2}{2} \\
&= \frac{1}{2}\sum_i a'_i \left\{\langle a'_{i\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\overline{\langle a'_{i\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}} + \langle a'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\overline{\langle a'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}}\right\} \\
&= \frac{1}{2}\sum_i \left(\mathbf{e_1} a'_{i\widehat{1}} + \mathbf{e_2} a'_{i\widehat{2}}\right)\left\{\langle \psi_{\widehat{1}}|a'_{i\widehat{1}}\rangle_{\widehat{1}}\langle a'_{i\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}} + \langle \psi_{\widehat{2}}|a'_{i\widehat{2}}\rangle_{\widehat{2}}\langle a'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right\} \\
&= \frac{1}{2}\left\{\mathbf{e_1}\sum_i a'_{i\widehat{1}} P_1\left(\langle \psi_{\widehat{1}}|a'_{i\widehat{1}}\rangle\langle a'_{i\widehat{1}}|\psi_{\widehat{1}}\rangle\right) + \mathbf{e_2}\sum_i a'_{i\widehat{2}} P_1\left(\langle \psi_{\widehat{1}}|a'_{i\widehat{1}}\rangle\langle a'_{i\widehat{1}}|\psi_{\widehat{1}}\rangle\right)\right. \\
&\quad \left. + \mathbf{e_1}\sum_i a'_{i\widehat{1}} P_2\left(\langle \psi_{\widehat{2}}|a'_{i\widehat{2}}\rangle\langle a'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle\right) + \mathbf{e_2}\sum_i a'_{i\widehat{2}} P_2\left(\langle \psi_{\widehat{2}}|a'_{i\widehat{2}}\rangle\langle a'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle\right)\right\}.
\end{aligned} \tag{95}$$

The $\overline{\cdot}$ stand for the standard complex conjugation because $\langle\cdot|\cdot\rangle_{\widehat{k}} \in \mathbb{C}(\mathbf{i_1})$. We want to warn the reader here that we can write $\left|\langle a'_{i\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}\right|^2 = \langle a'_{i\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}\overline{\langle a'_{i\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}}$ only because $\langle a'_{i\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}} \in$

$\mathbb{C}(\mathbf{i_1})$, in other word, $\langle \cdot | \cdot \rangle_{\widehat{k}}$ is a **standard complex scalar product**. Otherwise, we cannot write $\left|\langle a_i' | \psi \rangle\right|^2 = \langle a_i' | \psi \rangle \overline{\langle a_i' | \psi \rangle}$ for $|a_i'\rangle, |\psi\rangle \in M$. Indeed, (29) imply that $|w|^2 = \mathrm{Re}\left(w \cdot w^\dagger\right)$ instead of $|w|^2 = w \cdot w^\dagger$ for arbitrary $w \in \mathbb{T}$.
Using the properties of the projections operators, the fact that $a_{i\widehat{k}}' \in \mathbb{R}$ and the standard spectral theorem on V_k, we can write

$$\sum_i a_{i\widehat{k}}' P_k \left(\langle \psi_{\widehat{k}} | a_{i\widehat{k}}' \rangle \langle a_{i\widehat{k}}' | \psi_{\widehat{k}} \rangle\right) = P_k \left(\langle \psi_{\widehat{k}} | \left[\sum_i a_{i\widehat{k}}' | a_{i\widehat{k}}' \rangle \langle a_{i\widehat{k}}' |\right] | \psi_{\widehat{k}} \rangle\right)$$
$$= P_k \left(\langle \psi_{\widehat{k}} | A_{\widehat{k}}' | \psi_{\widehat{k}} \rangle\right) = \langle \psi_{\widehat{k}} | A_{\widehat{k}}' | \psi_{\widehat{k}} \rangle_{\widehat{k}}. \tag{96}$$

Then, we obtain $\big($keeping in mind that $\langle \psi | A' | \psi \rangle = \mathbf{e_1} \langle \psi_{\widehat{1}} | A_{\widehat{1}}' | \psi_{\widehat{1}} \rangle_{\widehat{1}} + \mathbf{e_2} \langle \psi_{\widehat{2}} | A_{\widehat{2}}' | \psi_{\widehat{2}} \rangle_{\widehat{2}}\big)$

$$\langle A' \rangle_{BQM} = \frac{1}{2}\left\{ \langle \psi | A' | \psi \rangle + \mathbf{e_1} \sum_i a_{i\widehat{1}}' \left|\langle a_{i\widehat{2}}' | \psi_{\widehat{2}} \rangle_{\widehat{2}}\right|^2 + \mathbf{e_2} \sum_i a_{i\widehat{2}}' \left|\langle a_{i\widehat{1}}' | \psi_{\widehat{1}} \rangle_{\widehat{1}}\right|^2 \right\}. \tag{97}$$

Noted that the last two terms of (97) represent a bicomplex (hyperbolic in fact) interaction or coupling between V_1 and V_2. Indeed, if we want to restrict $BQM \to SQM$, we only have to take $a_{i\widehat{1}}' = a_{i\widehat{2}}'$ and $|a_{i\widehat{1}}'\rangle = |a_{i\widehat{2}}'\rangle$, and if we do that in (97), it is not hard to see that we recover the standard equation $\langle A \rangle_{SQM} = \langle \psi | A | \psi \rangle$.
For the term $\langle A'^2 \rangle_{BQM}$, the same steps will give us

$$\langle A'^2 \rangle_{BQM} = \frac{1}{2}\left\{ \langle \psi | A'^2 | \psi \rangle + \mathbf{e_1} \sum_i a_{i\widehat{1}}'^2 \left|\langle a_{i\widehat{2}}' | \psi_{\widehat{2}} \rangle_{\widehat{2}}\right|^2 + \mathbf{e_2} \sum_i a_{i\widehat{2}}'^2 \left|\langle a_{i\widehat{1}}' | \psi_{\widehat{1}} \rangle_{\widehat{1}}\right|^2 \right\}. \tag{98}$$

Let us now evaluate the product $\langle A'^2 \rangle \langle B'^2 \rangle$, with B' the bicomplex self-adjoint operator defined previously $\big(\{b_i'\}$ and $\{|b_i'\rangle\}$ are defined the same way as for $A'\big)$. For convenience, we will remove the $_{BQM}$ index

$$\begin{aligned}\langle A'^2 \rangle \langle B'^2 \rangle = \frac{1}{4}\Big\{ & \langle \psi | A'^2 | \psi \rangle \langle \psi | B'^2 | \psi \rangle + \mathbf{e_1} \langle \psi_{\widehat{1}} | A_{\widehat{1}}'^2 | \psi_{\widehat{1}} \rangle_{\widehat{1}} \sum_i b_{i\widehat{1}}'^2 \left|\langle b_{i\widehat{2}}' | \psi_{\widehat{2}} \rangle_{\widehat{2}}\right|^2 \\ & + \mathbf{e_2} \langle \psi_{\widehat{2}} | A_{\widehat{2}}'^2 | \psi_{\widehat{2}} \rangle_{\widehat{2}} \sum_i b_{i\widehat{2}}'^2 \left|\langle b_{i\widehat{1}}' | \psi_{\widehat{1}} \rangle_{\widehat{1}}\right|^2 \\ & + \mathbf{e_1} \langle \psi_{\widehat{1}} | B_{\widehat{1}}'^2 | \psi_{\widehat{1}} \rangle_{\widehat{1}} \sum_i a_{i\widehat{1}}'^2 \left|\langle a_{i\widehat{2}}' | \psi_{\widehat{2}} \rangle_{\widehat{2}}\right|^2 \\ & + \mathbf{e_2} \langle \psi_{\widehat{2}} | B_{\widehat{2}}'^2 | \psi_{\widehat{2}} \rangle_{\widehat{2}} \sum_i a_{i\widehat{2}}'^2 \left|\langle a_{i\widehat{1}}' | \psi_{\widehat{1}} \rangle_{\widehat{1}}\right|^2 \\ & + \mathbf{e_2} \sum_{i,j} a_{i\widehat{2}}'^2 \left|\langle a_{i\widehat{1}}' | \psi_{\widehat{1}} \rangle_{\widehat{1}}\right|^2 b_{j\widehat{2}}'^2 \left|\langle b_{j\widehat{1}}' | \psi_{\widehat{1}} \rangle_{\widehat{1}}\right|^2 \\ & + \mathbf{e_1} \sum_{i,j} a_{i\widehat{1}}'^2 \left|\langle a_{i\widehat{2}}' | \psi_{\widehat{2}} \rangle_{\widehat{2}}\right|^2 b_{j\widehat{1}}'^2 \left|\langle b_{j\widehat{2}}' | \psi_{\widehat{2}} \rangle_{\widehat{2}}\right|^2 \Big\}. \end{aligned} \tag{99}$$

We would like to apply the bicomplex Schwartz inequality (Gervais Lavoie et al., 2010a, Th. 3.8) directly to the first term on the right hand side of (99). However, it is not so clear how we

can do that. The reason is that in bicomplex quantum mechanics, the (real) "length" of the ket $|\psi\rangle$ is not given by $\langle\psi|\psi\rangle$, but by $|\langle\psi|\psi\rangle|$. In consequence, the bicomplex Schwartz inequality apply to $|\langle\psi|\psi\rangle|\,|\langle\phi|\phi\rangle|$ rather than $\langle\psi|\psi\rangle\langle\phi|\phi\rangle$. From the properties of the Euclidean norm on bicomplex, it doesn't seems possible, at first look, to inject a norm in (99) to build the term $\left|\langle\psi|A'^2|\psi\rangle\right|\left|\langle\phi|B'^2|\phi\rangle\right|$.
One way to avoid this difficulty is to work with idempotent projection. We will noted $\langle\cdot\rangle_{\widehat{k}}$ the projection $P_k(\langle\cdot\rangle)$. From (99), we find

$$\begin{aligned}\langle A'^2\rangle_{\widehat{1}}\langle B'^2\rangle_{\widehat{1}} = \frac{1}{4}\Bigg\{ & \langle\psi_{\widehat{1}}|A'^2_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\langle\psi_{\widehat{1}}|B'^2_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}} \\ & + \langle\psi_{\widehat{1}}|A'^2_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\sum_i b'^2_{i\widehat{1}}\left|\langle b'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2 \\ & + \langle\psi_{\widehat{1}}|B'^2_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\sum_i a'^2_{i\widehat{1}}\left|\langle a'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2 \\ & + \sum_{i,j} a'^2_{i\widehat{1}}\left|\langle a'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2 b'^2_{j\widehat{1}}\left|\langle b'_{j\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2\Bigg\},\end{aligned} \tag{100}$$

and equivalently for $\langle A'^2\rangle_{\widehat{2}}\langle B'^2\rangle_{\widehat{2}}$.
From the definition of the bicomplex scalar product 2.2.1, we know that $\langle\psi_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}$ is a **standard complex (in $\mathbb{C}(\mathbf{i_1})$) scalar product**. This imply that $\langle\psi_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}$ **is** the (real) "length" of the ket $|\psi_{\widehat{k}}\rangle$. From this, it becomes clear that we can apply the standard complex Schwartz inequality to the first term of (100), where the two kets are $A'_{\widehat{1}}|\psi_{\widehat{1}}\rangle$, $B'_{\widehat{1}}|\psi_{\widehat{1}}\rangle$ respectively. This leads to

$$\begin{aligned}\langle A'^2\rangle_{\widehat{1}}\langle B'^2\rangle_{\widehat{1}} \geq \frac{1}{4}\Bigg\{ & \left|\langle\psi_{\widehat{1}}|A'_{\widehat{1}}B'_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\right|^2 + \langle\psi_{\widehat{1}}|A'^2_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\sum_i b'^2_{i\widehat{1}}\left|\langle b'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2 \\ & + \langle\psi_{\widehat{1}}|B'^2_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\sum_i a'^2_{i\widehat{1}}\left|\langle a'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2 \\ & + \sum_{i,j} a'^2_{i\widehat{1}}\left|\langle a'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2 b'^2_{j\widehat{1}}\left|\langle b'_{j\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2\Bigg\}.\end{aligned} \tag{101}$$

It is important to remark that the $\geq$ sign is well used here because (101) is an equation over reals numbers. Indeed, on the left hand side, as long as A' and B' are bicomplex self-adjoint operators, theirs eigenvalues are hyperbolic numbers, and then, according to (94), the mean valued of the operators A' and B' (equivalently for A'^2 and B'^2) are hyperbolic numbers. This also means that the projections $\langle\cdot\rangle_{\widehat{k}}$ are real numbers.
On the right hand side of (101), $|\cdot|^2$ is the Euclidean $\mathbb{R}^2$-norm and is undoubtedly real. As we said previously, $\langle\cdot|\cdot\rangle_{\widehat{k}}$ is a standard complex scalar product. Then $\langle\psi_{\widehat{1}}|A'^2_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}$ is real. Finally, the idempotent projection of hyperbolic numbers, the eigenvalues of A' and B', are also real numbers.
Let us introduce four new operators

$$M'_{\widehat{k}} := \frac{1}{2}\left[A'_{\widehat{k}}\,,\,B'_{\widehat{k}}\right], \qquad N'_{\widehat{k}} := \frac{1}{2}\left(A'_{\widehat{k}}B'_{\widehat{k}} + B'_{\widehat{k}}A'_{\widehat{k}}\right) \qquad \text{for} \qquad k = 1,2. \tag{102}$$

It is easy to see that $M'^{*}_{\widehat{k}} = -M'_{\widehat{k}}$ and $N'^{*}_{\widehat{k}} = N'_{\widehat{k}}$. Let us write ($k = 1, 2$)

$$\begin{aligned}
\left|\langle\psi_{\widehat{k}}|A'_{\widehat{k}}B'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}\right|^2 &= \left|P_k\left(\langle\psi_{\widehat{k}}|M'_{\widehat{k}} + N'_{\widehat{k}}|\psi_{\widehat{k}}\rangle\right)\right|^2 \\
&= \left|\langle\psi_{\widehat{k}}|M'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}} + \langle\psi_{\widehat{k}}|N'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}\right|^2 \\
&= \left|\langle\psi_{\widehat{k}}|M'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}\right|^2 + \langle\psi_{\widehat{k}}|M'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}\overline{\langle\psi_{\widehat{k}}|N'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}} \\
&\quad + \left|\langle\psi_{\widehat{k}}|N'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}\right|^2 + \overline{\langle\psi_{\widehat{k}}|M'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}}\langle\psi_{\widehat{k}}|N'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}} \\
&= \left|\langle\psi_{\widehat{k}}|M'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}\right|^2 + \langle\psi_{\widehat{k}}|M'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}\langle\psi_{\widehat{k}}|N'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}} \\
&\quad + \left|\langle\psi_{\widehat{k}}|N'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}\right|^2 - \langle\psi_{\widehat{k}}|M'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}\langle\psi_{\widehat{k}}|N'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}} \\
&= \left|\langle\psi_{\widehat{k}}|M'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}\right|^2 + \left|\langle\psi_{\widehat{k}}|N'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}\right|^2. \qquad (103)
\end{aligned}$$

Here again, in the third line, we can use the property $|x|^2 = x \cdot \overline{x}$ only because $\langle\psi_{\widehat{k}}|M'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}$ and $\langle\psi_{\widehat{k}}|N'_{\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}$ are element of $\mathbb{C}(\mathbf{i_1})$. The argument is the same as for (95).
Now, using (103) in (101), we have

$$\begin{aligned}
\langle A'^2\rangle_{\widehat{1}}\langle B'^2\rangle_{\widehat{1}} \geq \frac{1}{4}\Bigg\{ &\left|\langle\psi_{\widehat{1}}|M'_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\right|^2 + \left|\langle\psi_{\widehat{1}}|N'_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\right|^2 \\
&+ \langle\psi_{\widehat{1}}|A'^2_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\sum_i b'^2_{i\widehat{1}}\left|\langle b'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2 \\
&+ \langle\psi_{\widehat{1}}|B'^2_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\sum_i a'^2_{i\widehat{1}}\left|\langle a'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2 \\
&+ \sum_{i,j} a'^2_{i\widehat{1}}\left|\langle a'_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2 b'^2_{j\widehat{1}}\left|\langle b'_{j\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2\Bigg\}. \qquad (104)
\end{aligned}$$

Because (104) is an inequality, we can remove strictly positives terms form the right-hand side, exactly as we do in SQM (Marchildon, 2002, chap. 6). It is not hard to see that in fact, all the right-hand side term's are strictly positive. Then, by choice, we can write

$$\langle A'^2\rangle_{\widehat{1}}\langle B'^2\rangle_{\widehat{1}} \geq \frac{1}{4}\left|\langle\psi_{\widehat{1}}|M'_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\right|^2. \qquad (105)$$

Let us now redefined the self-adjoint operator A'. We take $A'_{\widehat{k}} := A_{\widehat{k}} - \langle A\rangle_{\widehat{k}}I$, with $A_{\widehat{k}}$ self-adjoint and I the identity on V_k or M depending on context. Explicitly, for $A'_{\widehat{1}}$, we have

$$A'_{\widehat{1}} = A_{\widehat{1}} - \frac{1}{2}\left\{\langle\psi_{\widehat{1}}|A_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}} + \sum_i a_{i\widehat{1}}\left|\langle a_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2\right\} I. \qquad (106)$$

As we said previously, and from the definition of the means value of an operator (94), we know that $\langle A\rangle_{\widehat{k}} \in \mathbb{R}$ and $\langle A^2\rangle_{\widehat{k}} \in \mathbb{R}$. Because we modify the operator A' by only a constant operator ($\langle A\rangle_{\widehat{k}}I$), it seems clear that the eigenkets of A' will be the same as the eigenkets of

A (this simply correspond to a rescaling of the operator), and we write $|a_i'\rangle = |a_i\rangle$. Moreover ($k = 1, 2$),

$$A'_{\widehat{k}}|a_{i\widehat{k}}\rangle = \left(A_{\widehat{k}} - \langle A\rangle_{\widehat{k}} I\right)|a_{i\widehat{k}}\rangle = \left(a_{i\widehat{k}} - \langle A\rangle_{\widehat{k}}\right)|a_{i\widehat{k}}\rangle = a'_{i\widehat{k}}|a_{i\widehat{k}}\rangle. \tag{107}$$

Then, the eigenvalues of $A'_{\widehat{k}}$ will be transform as $a'_{i\widehat{k}} = a_{i\widehat{k}} - \langle A\rangle_{\widehat{k}} \in \mathbb{R}$. For A', we have

$$A' = \mathbf{e_1}\left(A_{\widehat{1}} - \langle A\rangle_{\widehat{1}} I\right) + \mathbf{e_2}\left(A_{\widehat{2}} - \langle A\rangle_{\widehat{2}} I\right) = A - \langle A\rangle I. \tag{108}$$

Let us rewrite (98) in term of A and $\langle A\rangle$;

$$\langle A'^2\rangle = \frac{1}{2}\Big\{\langle\psi|\left(A - \langle A\rangle I\right)^2|\psi\rangle + \mathbf{e_1}\sum_i \left(a_{i\widehat{1}} - \langle A\rangle_{\widehat{1}}\right)^2 \left|\langle a_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2$$
$$+ \mathbf{e_2}\sum_i \left(a_{i\widehat{2}} - \langle A\rangle_{\widehat{2}}\right)^2 \left|\langle a_{i\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\right|^2\Big\}. \tag{109}$$

Using the normalization of the kets $\{|\psi\rangle\}$ and $\{|a_{i\widehat{k}}\rangle\}$ (in fact, the orthonormalization can be assumed from (Gervais Lavoie et al., 2011, Sec. 4.3) and (Gervais Lavoie et al., 2010a, Sec. 3.2)) and the fact that $\sum_i |\langle a_{i\widehat{k}}|\psi_{\widehat{k}}\rangle_{\widehat{k}}|^2 = 1$, we can write

$$\langle A'^2\rangle = \frac{1}{2}\Big\{\langle\psi|A^2|\psi\rangle + \langle A\rangle^2 - 2\langle A\rangle\langle\psi|A|\psi\rangle$$
$$+ \mathbf{e_1}\sum_i a_{i\widehat{1}}^2 \left|\langle a_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2 + \mathbf{e_1}\langle A\rangle_{\widehat{1}}^2 - 2\mathbf{e_1}\langle A\rangle_{\widehat{1}}\sum_i a_{i\widehat{1}} \left|\langle a_{i\widehat{2}}|\psi_{\widehat{2}}\rangle_{\widehat{2}}\right|^2$$
$$+ \mathbf{e_2}\sum_i a_{i\widehat{2}}^2 \left|\langle a_{i\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\right|^2 + \mathbf{e_2}\langle A\rangle_{\widehat{2}}^2 - 2\mathbf{e_2}\langle A\rangle_{\widehat{2}}\sum_i a_{i\widehat{2}} \left|\langle a_{i\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\right|^2\Big\}. \tag{110}$$

With the help of (97) and (98), we find

$$\langle A'^2\rangle = \langle A^2\rangle - \langle A\rangle^2 = (\Delta A)^2, \tag{111}$$

and clearly, $\langle A'^2\rangle_{\widehat{k}} = (\Delta A)_{\widehat{k}}^2$.
By doing the same with the operator B', that is $B'_{\widehat{k}} := B_{\widehat{k}} - \langle B\rangle_{\widehat{k}} I$, we find the same equation as for A. Moreover, it is not hard to verify that those definitions leads to $M'_{\widehat{k}} = M_{\widehat{k}}$. From this, (105) becomes

$$(\Delta A)_{\widehat{1}}^2 (\Delta B)_{\widehat{1}}^2 \geq \frac{1}{4}\left|\langle\psi_{\widehat{1}}|M_{\widehat{1}}|\psi_{\widehat{1}}\rangle_{\widehat{1}}\right|^2. \tag{112}$$

Because (99) is symmetrical in $\mathbf{e_k}$, the term $(\Delta A)_{\widehat{2}}^2 (\Delta B)_{\widehat{2}}^2$ will be identical at $(\Delta A)_{\widehat{1}}^2 (\Delta B)_{\widehat{1}}^2$ but with all the index 1 replaced by 2.
It is tempting to simply build the term $(\Delta A)(\Delta B)$ from (112) and say that this is the bicomplex uncertainty principle. However, we must recall that an inequality can only stand on real number and the term $(\Delta A)(\Delta B)$ is hyperbolic. The simplest way, maybe not the only,

to express our result in term of a simple bicomplex equation is to consider the norm of $(\Delta A)(\Delta B)$. Then, from (29), we have

$$\begin{aligned}|(\Delta A)(\Delta B)| &= \frac{1}{\sqrt{2}}\sqrt{\left|(\Delta A)_{\hat{1}}(\Delta B)_{\hat{1}}\right|^2 + \left|(\Delta A)_{\hat{2}}(\Delta B)_{\hat{2}}\right|^2} \\ &\geq \frac{1}{\sqrt{2}}\sqrt{\left|\frac{1}{2}\left|\langle\psi_{\hat{1}}|M_{\hat{1}}|\psi_{\hat{1}}\rangle_{\hat{1}}\right|\right|^2 + \left|\frac{1}{2}\left|\langle\psi_{\hat{2}}|M_{\hat{2}}|\psi_{\hat{2}}\rangle_{\hat{2}}\right|\right|^2} \\ &= \frac{1}{\sqrt{2}}\sqrt{\frac{1}{4}\left|\langle\psi_{\hat{1}}|M_{\hat{1}}|\psi_{\hat{1}}\rangle_{\hat{1}}\right|^2 + \frac{1}{4}\left|\langle\psi_{\hat{2}}|M_{\hat{2}}|\psi_{\hat{2}}\rangle_{\hat{2}}\right|^2} \\ &= \frac{1}{2}\left|\langle\psi|M|\psi\rangle\right|,\end{aligned} \tag{113}$$

or, finally

$$|(\Delta A)(\Delta B)| \geq \frac{1}{4}\left|\langle\psi|[A,B]|\psi\rangle\right|. \tag{114}$$

This equation is the general bicomplex uncertainty principle of two non-commuting linear self-adjoint operator.

It can be remarked that (114) has the same form as the standard uncertainty principle, except that the $1/2$ factor replaced by $1/4$ here, and that it apply on $|(\Delta A)(\Delta B)|$ instead of $(\Delta A)(\Delta B)$. We would like to warn the reader that, according to (97), the right hand side of (114) **cannot** be written in the usual shorter form $\frac{1}{4}\left|\langle[A,B]\rangle\right|$.

5.1 Application: Position-momentum operators

We would now apply eq. (114) to the case of the position and momentum self-adjoint bicomplex linear operator X and P.

In section 4, we have seen that the commutator of X and P is given by

$$[X,P] = \mathbf{i_1}\hbar(\xi_{\hat{1}}\mathbf{e_1} + \xi_{\hat{2}}\mathbf{e_2})I, \tag{115}$$

with $\xi_{\hat{1}}, \xi_{\hat{2}} \in \mathbb{R}^+$. From this, we find that

$$|(\Delta X)(\Delta P)| \geq \frac{\left|\langle\psi|\mathbf{i_1}\hbar(\xi_{\hat{1}}\mathbf{e_1} + \xi_{\hat{2}}\mathbf{e_2})I|\psi\rangle\right|}{4} = \frac{\hbar\left|\mathbf{e_1}\xi_{\hat{1}} + \mathbf{e_2}\xi_{\hat{2}}\right|}{4} = \frac{\hbar\sqrt{\xi_{\hat{1}}^2 + \xi_{\hat{2}}^2}}{4\sqrt{2}} = \frac{\hbar|\xi|}{4}. \tag{116}$$

As the eigenfunctions of the harmonic oscillator, the bicomplex uncertainty principle is completely determined by the two parameters $\xi_{\hat{1}}$ and $\xi_{\hat{2}}$ of our model. As we do in section 4.1, we can decompose ξ in the basis $\{1, \mathbf{j}\}$ instead of $\{\mathbf{e_1}, \mathbf{e_2}\}$ by taking $\xi = \alpha + \beta\mathbf{j}$ and then $\xi_{\hat{1}} = \alpha + \beta$ and $\xi_{\hat{2}} = \alpha - \beta$. This leads to

$$|(\Delta X)(\Delta P)| \geq \frac{\hbar\sqrt{(\alpha+\beta)^2 + (\alpha-\beta)^2}}{4\sqrt{2}} = \frac{\hbar\sqrt{\alpha^2+\beta^2}}{4}. \tag{117}$$

It is interesting to note that if we restrict BQM to SQM by setting $\xi_{\hat{1}} = 1 = \xi_{\hat{2}}$ or $\alpha = 1, \beta = 0$ (and indirectly $X_{\hat{1}} = X_{\hat{2}}$, $P_{\hat{1}} = P_{\hat{2}}$ and $|\psi_{\hat{1}}\rangle = |\psi_{\hat{2}}\rangle$), we find

$$|(\Delta X)(\Delta P)|_{BQM\mapsto SQM} \geq \frac{\hbar}{4}, \tag{118}$$

that is $1/2$ times the standard result. Then, from bicomplex quantum mechanics, we generated a lower bound for the Heisenberg uncertainty principle that is in accord with the standard quantum mechanics. In fact, the $1/2$ factor comes from the three last terms of (104) that we neglected. Indeed, the terms that we neglected in (104) would have contributed for $\hbar/4$ to the uncertainty principle **but only** when we do the restriction BMQ$\rightarrow$SQM.
In other words, we can say that computing the **standard uncertainty principle** from BQM (in the SQM approximation) give a $1/2$ time poorer bound, compare with the complex (standard) way of computation. This, however, doesn't imply in any way that (114) is a poor approximation **in the BQM**.

6. Conclusion

With the results presented here, quantum mechanics was successfully extended to bicomplex numbers in two concrete problems, the harmonic oscillator and the Heisenberg uncertainty principle. We strongly believe that bicomplex quantum mechanics can be extended to other significant problems of standard quantum mechanics and such investigations are actually in progress. However, we think it is too early to try to give a physical interpretation to our results. We hope that this work will motivate the reader to consider generalizations of complex numbers in other significant problem of physics. We also believe that if those generalized theory do not end with some new predictions, they will at least give some crucial insight about the apparent requirement of complex numbers in physics.

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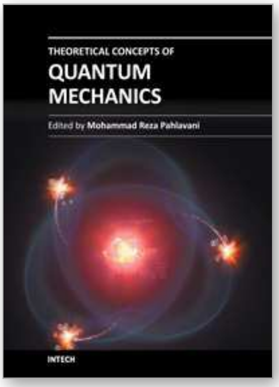

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