1. Introduction

Hexapod is a new type of machine tool based on the parallel closed-chain kinematic structure. Compared to the conventional machine tool, parallel mechanism structure offers superior stiffness, lower mass and higher acceleration, resulting from the parallel structural arrangement of the motion systems. Moreover, hexapod has the potential to be highly modular and re-configurable, with other advantages including higher dexterity, simpler and fewer fixtures, and multi-mode manufacturing capabilities.

Initially, hexapod was developed based on the Stewart platform, i.e. the prismatic type of parallel mechanism with the variable leg length. Commercial hexapods, such as VARIAX from Giddings & Lewis, Tornado from Hexel Corp., and Geodetic from Geodetic Technology Ltd., are all based on this structure. One of the disadvantages for the variable leg length structure is that the leg stiffness varies as the leg moves in and out. To overcome this problem, recently the constant leg length hexapod has been envisioned, for instance, HexaM from Toyada (Susuki et al., 1997). Hexaglibe form the Swiss Federal Institute of Technology (Honegger et al., 1997), and Linapod form University of Stuttgart (Pritschow & Wurst, 1997). Between these two types, the fixed-length leg is stiffer (Tlusty et al., 1999) and, here, becoming popular.

Dynamic modeling and analysis of the parallel mechanisms is an important part of hexapod design and control. Much work has been done in this area, resulting in a very rich literature (Fichter, 1986; Sugimoto, 1987; Do & Yang, 1988; Geng et al., 1992; Tsai, 2000; Hashimoto & Kimura, 1989; Fijany & Bejezy, 1991). However, the research work conducted so far on the inverse dynamics has been focused on the parallel mechanisms with extensible legs.

In this chapter, first, in the inverse dynamics of the new type six d.o.f. hexapods with fixed-length legs, shown in Fig. 1, is developed with consideration of the masses of the moving platform and the legs. (Xi & Sinatra, 2002) This system consists of a moving platform MP and six legs sliding along the guideways that are mounted on the support structure. Each leg is connected at one end to the guideway by a universal joint and at another end to the moving platform by a spherical joint. The natural orthogonal complement method (Angeles & Lee, 1988; Angeles & Lee, 1989) is applied, which provides an effective way of solving multi-body dynamics systems. This method has been applied to studying serial and parallel manipulators (Angeles & Ma, 1988; Zanganesh et al., 1997) automated vehicles (Saha & Angeles, 1991) and flexible mechanisms (Xi & Sinatra, 1997). In this development, the...
Newton-Euler formulation is used to model the dynamics of each individual body, including the moving platform and the legs. All individual dynamics equations are then assembled to form the global dynamics equations. Based on the complete kinematics model developed, an explicit expression is derived for the natural orthogonal complement which effectively eliminates the constraint forces in the global dynamics equations. This leads to the inverse dynamics equations of hexapods that can be used to compute required actuator forces for given motions.

Fig. 1. New hexapod design

Finally, for completeness of the dynamic study of the parallel manipulator with the fixed-length legs, the static balancing is studied (Xi et al., 2005).

A great deal of work has been carried out and reported in the literature for the static balancing problem. For example, in the case of serial manipulator, Nathan (Nathan, 1985) and Hervé (Hervé, 1986) applied the counterweight for gravity compensations. Streit et al. (Streit & Gilmore, 1991), (Walsh et al., 19) proposed an approach to static balanced rotary bodies and two degrees of freedom of the revolute links using springs. Streit and Shin presented a general approach for the static balancing of planar linkages using springs (Streit & Shin, 1980). Ulrich and Kumar presented a method of passive mechanical gravity compensation using appropriate pulley profiles (Ulrich & Kumar, 1991). Kazerooni and Kim presented a method for statically-balanced direct drive arm (Kazerooni & Kim, 1990).

For the parallel manipulator much work was done by Gosselin et al. Research reported in (Gosselin & Wang, 1998) was focused on the design of gravity-compensated of a six-degree-of-freedom parallel manipulator with revolute joints. Each leg with two links is connected by an actuated revolute joint to the base platform and by a spherical joints the moving platform. Two methods are used, one approach using the counterweight and the other using springs. In the former method, if the centre of mass of a mechanism can be made stationary, the static balancing is obtained in any direction of the Cartesian space. In the second approach, if the total energy is kept constant, the mechanism is statically balanced only in the direction of gravity vector. The static balancing conditions are derived for the three-degree-of-freedom spatial parallel manipulator (Wang & Gosselin, 1998) and in similar
conditions are obtained for spatial four-degree-of-freedom parallel manipulator using two common methods, namely, counterweights and springs (Wang & Gosselin, 2000). In this chapter, following the same approach presented by Gosselin, the static balancing of the six d.o.f. platform type parallel manipulator with the fixed-length legs shown is studied. The mechanism can be balanced using the counterweight with a smart design of pantograph. The mechanism can be balanced using the method, i.e., the counterweight with a smart design of pantograph. By this design a constant global center of mass for any configurations of the manipulator is obtained.

Finally, the leg masses become important for hexapods operating at high speeds, such as high-speed machining; then in the future research and development the effect of leg inertia on hexapod dynamics considering high-speed applications will be investigated.

2. Kinematic modeling

2.1 Notation

As shown in Figure 2, this hexapod system consists of a moving platform MP to which a tool is attached, and six legs sliding along the guideways that are mounted on the support structure including the base platform BP. Each leg is connected at one end to the guideway by a universal joint and at another end to the moving platform by a spherical joint.

Fig. 2. Kinematic notation of the i-th leg

The coordinate systems used are a fixed coordinate system O-xyz is attached to the base and a local coordinate system O_{i}-x_{i}y_{i}z_{i} attached to the moving platform. Vector b_{i}, s_{i}, and l_{i} are directed from O to B_{i}, from B_{i} to U_{i}, and from U_{i} to S_{i} respectively. B_{i} indicates the position of one end of the i-th guideway attached to the base, U_{i} indicates the position of the i-th
universal joint, and $S_i$ indicates the position of the $i$th spherical joint. Six legs are numbered from 1 to 6. Furthermore, a local coordinate frame $O_i$-$x_i$$y_i$$z_i$ is defined for each leg, with its origin located at the center of the $i$th universal joint. Two unit vectors are used. Unit vector $u_i^l$ is along the leg length representing the direction of the $i$th leg, and unit vector $u_i^s$ is along the guideway representing the direction of the $i$th guideway. The orientation of the $i$th coordinate frame with respect to the base can therefore be defined by a $3 \times 3$ rotation matrix, for $i = 1, \ldots, 6$, as

$$Q_i = \begin{bmatrix} u_i^a & u_i^l \times u_i^a & u_i^s \end{bmatrix}$$

where $u_i^a$ is expressed as

$$u_i^a = \frac{u_i^s \times u_i^l}{\|u_i^s \times u_i^l\|}$$

Note that vector $u_i^l$ is configuration-dependent and determined for the given location of the moving platform; vector $u_i^s$ is constant and defined by the geometry of the hexapod.

For the purpose of carrying out the inverse dynamics analysis of the hexapod, the following symbols are defined. As shown in Figure 2, $C_i$ is the center of mass of the $i$th leg, $C_p$ is the center of mass of the moving platform, $c_i$, $\dot{c}_i$ and $\ddot{c}_i$ are the position, velocity and acceleration vectors, respectively, of $C_i$ with respect to the fixed coordinate frame, $\rho$ is the vector pointing from $O_t$ to $C_p$ with respect to the local coordinate frame $O_t$-$x_t$$y_t$$z_t$.

### 2.2 Kinematics

Consider one branch of the leg-guideway system, as shown in Figure 2, the following loop equation for $i = 1, \ldots, 6$, holds,

$$h + R \vec{p}_i - \vec{b}_i - s_i - \vec{l}_i = 0$$

where $h$ and $R$ are the vector and rotation matrix that define the position and orientation of the moving platform relative to the base, respectively, $\vec{p}_i$ is the vector representing the position of the $i$th spherical joint on the moving platform in the local coordinates. Since the leg always moves along the guideway, $s_i$ can be expressed as

$$s_i = s_i u_i^s$$

where $s_i$ is a scalar representing the displacement of the $i$th actuator along the guideway. Likewise, leg vector $\vec{l}_i$ can be expressed as

$$\vec{l}_i = l_i u_i^l$$

where $l_i$ is a scalar representing the fixed length of the $i$th leg. As mentioned in Section 2.1, the leg axis is parallel to the $z_i$ axis of the local coordinate frame $O_i$-$x_i$$y_i$$z_i$. In the light of eq. (1), $u_i^l$ can be expressed as
Substituting eqs. (4 & 5) into eq. (3) and rearranging it yields the following kinematics equations for the fixed-length leg hexapod, for $i = 1, \ldots, 6$,

$$
\mathbf{s}_i \mathbf{u}_i = \mathbf{h} + \mathbf{R} \mathbf{p}_i - \mathbf{b}_i - l_i \mathbf{u}_i^T
$$

To obtain the velocity of the moving platform, taking the time derivative of eq. (7) yields

$$
\dot{\mathbf{s}}_i \mathbf{u}_i = \mathbf{v} + (\omega \times \mathbf{R} \mathbf{p}_i) - (\omega_i \times \mathbf{l}_i)
$$

where $\mathbf{v}$ and $\mathbf{\omega}$ are the vectors representing the velocity and angular velocity of the moving platform, respectively, and $\mathbf{\omega}$ is the vector representing the angular velocity of the $i$th leg.

Furthermore, by taking dot product on both sides of eq. (8) by $\mathbf{l}_i$ it leads to

$$
\dot{\mathbf{s}}_i (\mathbf{u}_i^T \cdot \mathbf{l}_i) = [\mathbf{v} + (\omega \times \mathbf{R} \mathbf{p}_i)] \cdot \mathbf{l}_i
$$

It is well known that the kinematic analysis of parallel manipulator leads to two Jacobian matrices, namely, the forward and the inverse Jacobian (Gosselin & Angeles, 1990). To find the Jacobians for the hexapod under study, rearranging eq. (9) yields the following form

$$
\dot{\mathbf{s}}_i (\mathbf{u}_i^T \cdot \mathbf{l}_i) = [\mathbf{l}_i^T, (\mathbf{R} \mathbf{p}_i \times \mathbf{l}_i)^T] \mathbf{t}_p
$$

where $\mathbf{t} = [\mathbf{v}, \mathbf{\omega}]$ is the $6 \times 1$ twist vector of the moving platform. Consider all six legs it leads to the following expression

$$
\mathbf{B} \mathbf{s} = \mathbf{A} \mathbf{t}_p
$$

where $\mathbf{s} = [\dot{s}_1, \ldots, \dot{s}_6]^T$ is the $6 \times 1$ vector of the actuator speeds, and $\mathbf{A}$ and $\mathbf{B}$ are the $6 \times 6$ matrices representing the inverse and forward Jacobian of the hexapod and they are defined as

$$
\mathbf{A} = \begin{bmatrix}
\mathbf{l}_1^T & (\mathbf{R} \mathbf{p}_1 \times \mathbf{l}_1)^T \\
\vdots & \vdots \\
\mathbf{l}_6^T & (\mathbf{R} \mathbf{p}_6 \times \mathbf{l}_6)^T
\end{bmatrix}
$$

$$
\mathbf{B} = \text{diag}(\mathbf{u}_1^T \cdot \mathbf{l}_1, \ldots, \mathbf{u}_6^T \cdot \mathbf{l}_6)
$$

Eq. (11) defines the differential relationship between the actuator speeds $\mathbf{s}$ and the twist of the moving platform $\mathbf{t}_p$. Rewriting eq. (11) gives

$$
\dot{\mathbf{s}} = \mathbf{J}_p \mathbf{t}_p
$$

Provided that $\mathbf{B}$ is invertible, the Jacobian matrix of the moving platform $\mathbf{J}_p$ can be given as

$$
\mathbf{J}_p = \mathbf{B}^{-1} \mathbf{A} = \begin{bmatrix}
\mathbf{J}_{p1}^T, \ldots, \mathbf{J}_{p6}^T
\end{bmatrix}^T
$$
where

$$J_{pi} = \begin{bmatrix} I_i^T & (R\mathbf{\bar{P}}_i \times \mathbf{l}_i)^T \\ I_i^T \mathbf{u}_i & I_i^T \mathbf{u}_i^2 \end{bmatrix}$$  \hspace{1cm} (16)$$

for \(i = 1, \ldots, 6\). From eq.(14), \(\mathbf{t}_p\) can be expressed in terms of \(\mathbf{s}\) as,

$$\mathbf{t}_p = T_p \mathbf{s}$$  \hspace{1cm} (17)

where \(T_p = J_p^{-1}\).

To obtain the acceleration of the moving platform, taking the time derivative of eq. (14) yields

$$\ddot{s} = J_p \mathbf{t}_p + J_p \dot{J}_p$$  \hspace{1cm} (18)

where \(\ddot{s} = [\ddot{s}_1, \ldots, \ddot{s}_6]^T\) is the \(6 \times 1\) vector of the actuator accelerations, \(\dot{\mathbf{t}}_p = [\ddot{\mathbf{a}}, \ddot{\mathbf{\omega}}]^T\) is the time derivative of the twist of the moving platform, \(\dot{\mathbf{J}}_p\) is the time derivative of the Jacobian matrix of the moving platform obtained by differentiating \(\mathbf{J}_p\) with respect to time, that is

$$\dot{\mathbf{J}}_p = \mathbf{B}^{-1}(\dot{\mathbf{A}} - \mathbf{B} \mathbf{B}^{-1} \mathbf{A})$$  \hspace{1cm} (19)

where \(\dot{\mathbf{A}}\) and \(\mathbf{B}\) given as

$$\dot{\mathbf{A}} = \begin{bmatrix} (\mathbf{\omega}_1 \times \mathbf{l}_1)^T & ((\mathbf{\omega} \times R\mathbf{\bar{P}}_1) \times \mathbf{l}_1 + R\mathbf{\bar{P}}_1 \times (\mathbf{\omega}_1 \times \mathbf{l}_1))^T \\ \vdots & \vdots \\ (\mathbf{\omega}_6 \times \mathbf{l}_6)^T & ((\mathbf{\omega} \times R\mathbf{\bar{P}}_6) \times \mathbf{l}_6 + R\mathbf{\bar{P}}_6 \times (\mathbf{\omega}_6 \times \mathbf{l}_6))^T \end{bmatrix}$$  \hspace{1cm} (20)

$$\mathbf{B} = \text{diag}(\mathbf{u}_1 \cdot (\mathbf{\omega}_1 \times \mathbf{l}_1), \ldots, \mathbf{u}_6 \cdot (\mathbf{\omega}_6 \times \mathbf{l}_6))$$  \hspace{1cm} (21)

If the mass of the leg is uniformly distributed, then the center of mass is in its middle. The velocity of the center of mass can be given as

$$\dot{\mathbf{c}}_i = \dot{s}_i + \mathbf{\omega}_i \times \frac{\mathbf{l}_i}{2}$$  \hspace{1cm} (22)

Upon differentiating eq.(22), the acceleration of the center of mass can be given as

$$\ddot{\mathbf{c}}_i = \ddot{s}_i + \dot{\mathbf{\omega}}_i \times \frac{\mathbf{l}_i}{2} + \mathbf{\omega}_i \times (\mathbf{\omega}_i \times \frac{\mathbf{l}_i}{2})$$  \hspace{1cm} (23)

To obtain the leg angular velocity and acceleration, denote by \(\mathbf{E}_i\) the \(3 \times 3\) cross-product matrix associated with vector \(\mathbf{u}_i\), then eq.(9) may be re-written as

$$\mathbf{E}_i \mathbf{\omega}_i = \frac{1}{l_i} \left[ \mathbf{v} + \mathbf{\omega} \times R\mathbf{\bar{P}}_i - \dot{s}_i \mathbf{u}_i \right]$$  \hspace{1cm} (24)
Consider all six legs, it forms a set of linear equations containing the unknowns of the leg angular velocity. There are three components of $\omega_i$ for each leg. Because matrix $E_i$ is a skew symmetric and singular, it is impossible to directly solve eq.(24). However, since the leg does not spin about its longitudinal axis, this indicates (Tsai, 2000)

$$\omega_i \cdot l_i = 0 \quad (25)$$

In the light of eq.(25), eq. (24) may be rewritten as

$$A_i \omega_i = e_i \quad (26)$$

where $A_i$ is a $4 \times 3$ matrix and $e_i$ is a 4-dimensional vector, and they are defined as

$$A_i = \begin{bmatrix} E_i \\ I_i^T \end{bmatrix} \quad (27)$$

$$e_i = \frac{1}{l_i} \begin{bmatrix} v + \omega \times \mathbf{R} \mathbf{p}_i - \dot{s}_i \mathbf{u}_i^* \\ 0 \end{bmatrix} \quad (28)$$

Solving eq. (26) leads to the expression for the leg angular velocity

$$\omega_i = \frac{1}{l_i^2} \times \begin{bmatrix} v + \omega \times \mathbf{R} \mathbf{p}_i - \dot{s}_i \mathbf{u}_i^* \end{bmatrix} \quad (29)$$

Now eq.(29) is substituted back into eq.(22), and the velocity becomes

$$\dot{c}_i = \frac{1}{2} \begin{bmatrix} v + \omega \times \mathbf{R} \mathbf{p}_i - \dot{s}_i \mathbf{u}_i^* \end{bmatrix} \quad (30)$$

By examining eqs.(29 & 30), it may be noted that the two terms in the brackets are identical. The first term may be expressed as

$$v + \omega \times \mathbf{R} \mathbf{p}_i = [1, E_{pi}] t_p \quad (31)$$

where $E_{pi}$ is the cross-product matrix of $\mathbf{R} \mathbf{p}_i$. In the light of eq.(17), eq.(31) may be related to $\dot{s}$ as

$$[1, E_{pi}] t_p = T_{1i} \dot{s} \quad (32)$$

where $1$ is the $3 \times 3$ identity matrix and $T_{1i}$ is the $3 \times 6$ matrix pertaining to the first term defined as

$$T_{1i} = [1, E_{pi}] T_p \quad (33)$$

The second term in eqs.(29) and (30) can also be expressed in terms of $\dot{s}$

$$\dot{s}_i \mathbf{u}_i^* = T_{2i} \dot{s} \quad (34)$$
where $T_{2i}$ is the $3 \times 6$ matrix pertaining to the second term defined as

$$T_{2i} = [0_3, \ldots, u_i^*, \ldots, 0_3]$$

In eq.(35), $0_3$ is the 3-dimensional null vector. The twist of the $i$th leg can be expressed in terms of $\mathbf{s}$ as

$$t_i = T_i \dot{s}$$

where $t_i$ is the twist of the $i$th leg, i.e. $t_i = \begin{bmatrix} c_i^T, \omega_i^T \end{bmatrix}^T$, and the $6 \times 6$ matrix $T_i$ is given as

$$T_i = \begin{bmatrix} T_{1i}^T, T_{2i}^T \end{bmatrix}^T$$

Furthermore, the leg angular acceleration can be obtained by differentiating eq.(26) with respect to time, that is

$$\mathbf{A}_i \ddot{\omega}_i = \mathbf{e}_i - \mathbf{\dot{A}}_i \omega_i$$

where

$$\mathbf{A}_i \omega_i = \begin{bmatrix} \omega_i \times u_i^* \end{bmatrix} \times \omega_i$$

$$\mathbf{e}_i = \frac{1}{l_i} \begin{bmatrix} a + \omega \times R \mathbf{P}_i + \omega \times (\omega \times R \mathbf{P}_i) - \dot{s}_i u_i^* \\ 0 \end{bmatrix}$$

From eq.(38), vector $\dot{\omega}_i$ representing the angular acceleration of the $i$th leg is given as

$$\dot{\omega}_i = \frac{1}{l_i} \begin{bmatrix} (\omega_i \times l_i) \times (v + \omega \times R \mathbf{P}_i - \dot{s}_i u_i^*) + l_i \times \left( a + \omega \times R \mathbf{P}_i + \omega \times (\omega \times R \mathbf{P}_i) - \dot{s}_i u_i^* \right) \end{bmatrix}$$

3. Dynamic modeling

3.1 The natural orthogonal complement method

Prior to performing dynamic modeling of the hexapod, a brief review of the natural orthogonal complement method (Angeles & Lee, 1988) is provided. Consider a system composed of $p$ rigid bodies under holonomic constraints, the Newton-Euler equations for each individual body can be written, for $i = 1, \ldots, p$, as

$$M_i \ddot{t}_i = -W_i M_i \dot{t}_i + \mathbf{w}_i$$

where $t_i$ is the twist of the $i$th body, $\mathbf{w}_i = \begin{bmatrix} n_i^T, f_i^T \end{bmatrix}^T$ represent the wrench acting on the $i$th body, $n_i$ and $f_i$ are the resultant moment and the resultant force acting at the center of mass. In general $\mathbf{w}_i$ can be decomposed into working wrench $\mathbf{w}_i^w$ and non-working wrench $\mathbf{w}_i^N$. The former can further be decomposed as
\[ w_i^a = w_i^g + w_i^d \] (43)

where \( w_i^a, w_i^g \) and \( w_i^d \) are the actuator, gravity and dissipate wrenches, respectively.

In eq. (42), the \( 6 \times 6 \) angular velocity matrix \( \Omega_i \) and the \( 6 \times 6 \) inertia matrix \( M_i \) are defined as

\[
\Omega_i = \begin{bmatrix} \Omega_i & O \\ O & O \end{bmatrix}, \quad M_i = \begin{bmatrix} I_i & O \\ O & m_i O \end{bmatrix}
\] (44)

with

\[
\Omega_i = \frac{\partial (\omega_i \times e)}{\partial e}
\] (45)

where \( I_i \) is the \( 3 \times 3 \) matrix of the moment of inertia of the \( i \)th body, \( m_i \) is the body mass, \( O \) denotes the \( 3 \times 3 \) null matrix, and \( e \) is an arbitrary vector.

If consider all \( p \) bodies, the assembled system dynamics equations are given as

\[
M \dot{t} = -W M \dot{t} + W^W + W^N
\] (46)

where the \( 6p \times 6p \) generalized mass matrix \( M \) and generalized angular velocity matrix \( W \) are defined as

\[
M = \text{diag}(M_1, \ldots, M_p),
\] (47)

\[
W = \text{diag}(W_1, \ldots, W_p)
\] (48)

and the \( 6p \)-dimensional generalized twist \( t \), generalized working wrench \( W^W \) and generalized non-working wrench \( W^N \) are defined as

\[
t = \begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix}, \quad W^W = \begin{bmatrix} w_1^W \\ \vdots \\ w_p^W \end{bmatrix}, \quad W^N = \begin{bmatrix} w_1^N \\ \vdots \\ w_p^N \end{bmatrix}
\] (49)

It can be shown that the kinematic constraints hold the following relation with the generalized twist

\[
Kt = 0_{6p}
\] (50)

where \( 0_{6p} \) is the \( 6p \)-dimensional null vector, \( K \) is the \( 6p \times 6p \) velocity constraint matrix with a rank of \( m \) which is equal to the number of independent holonomic constraints. The number of degrees of freedom of the system, i.e. independent variables, is determined as \( n = 6p - m \). Denote the independent variables by \( s \), they can be related to the twist as

\[
t = Ts
\] (51)
\[ \mathbf{\dot{t}} = \mathbf{T}\mathbf{s} + \dot{\mathbf{T}}\mathbf{s} \]  

(52)

where \( \mathbf{T} \) is a \( 6p \times n \) twist-mapping matrix.

By substituting eq.(51) into eq.(50), the following relation can be obtained

\[ \mathbf{KT} = 0_{6p} \]  

(53)

where \( \mathbf{T} \) is the natural orthogonal complement of \( \mathbf{K} \). As shown in (Angeles & Lee, 1988, 1989) the non-working vector \( \mathbf{w}^N \) lies in the null space of the transpose of \( \mathbf{T} \). Thus, if both sides of eq. (46) are multiplied by \( \mathbf{T}^T \), in the aid of eqs. (51 & 52), the system dynamics equations can be obtained as

\[ \mathbf{I}\mathbf{s} + \mathbf{C}\mathbf{s} = \mathbf{T}^T (\mathbf{w}^d + \mathbf{w}^g + \mathbf{w}^d) \]  

(54)

where the \( n \times n \) generalized inertia matrix \( \mathbf{I} \) and coupling matrix \( \mathbf{C} \) are defined as

\[ \mathbf{I} \equiv \mathbf{T}^T \mathbf{M} \mathbf{T}, \quad \mathbf{C} \equiv \mathbf{T}^T (\dot{\mathbf{M}} \mathbf{T} + \mathbf{WMT}) \]  

(55)

Furthermore, by defining the following generalized forces

\[ \tau^a = \mathbf{T}^T \mathbf{w}^a, \quad \tau^g = \mathbf{T}^T \mathbf{w}^g, \quad \tau^d = \mathbf{T}^T \mathbf{w}^d, \quad \tau^I = \mathbf{I}\mathbf{s} + \mathbf{C}\mathbf{s} \]  

(56)

the inverse dynamics of the system can be given as

\[ \tau^I = \tau^I - \tau^g - \tau^d \]  

(57)

where \( \tau^I \) is the vector representing the applied actuator forces.

### 3.2 Inverse dynamics

The key in applying the natural orthogonal complement method is to derive the expression for the twist-mapping matrix \( \mathbf{T} \), which relates the speeds of the independent variables to the generalized twist. For the hexapod under study, the independent variable \( \mathbf{s} \) is the vector representing the actuator displacement, with the total number of six, as defined before. The generalized twist is expressed as

\[ \mathbf{t} = \begin{bmatrix} \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_6 \\ \mathbf{t}_{pc} \end{bmatrix} \]  

(58)

Note that \( \mathbf{t}_1 \) to \( \mathbf{t}_6 \) are the twists for the six legs. Since the twist in eq.(36) is defined at the center of mass of the leg, \( \mathbf{T}_i \) represents the twist-mapping for the legs. For the moving platform, \( \mathbf{t}_{pc} \) is defined as the center of mass which may be expressed as

\[ \mathbf{c}_p = \mathbf{h} + \mathbf{R}\bar{\mathbf{p}} \]  

(59)

Differentiating eq.(59) gives
\( \dot{p} = v + \omega \times R \rho \)  
(60)

In the light of eq.(60), the following relation can be obtained

\[ t_{pc} = H_p t_p \]  
(61)

where \( H_p \) is the \( 6 \times 6 \) matrix defined as

\[
H_p = \begin{bmatrix}
1 & E_\rho \\
0 & 1
\end{bmatrix}
\]  
(62)

In eq.(62), \( E_\rho \) is the cross-product matrix of \( R \rho \). Note that when \( \rho \) is zero, i.e. the center of mass coincides with the coordinate origin, \( H_p \) becomes an identity matrix, and \( t_{pc} = t_p \).

The twist-mapping matrix \( T \) for the hexapod under study can be given in the light of eqs.(17), (36) and (62) as

\[
T = \begin{bmatrix}
T_1 \\
\vdots \\
T_6 \\
H_p T_p
\end{bmatrix}
\]  
(63)

where \( T \) is a \( 42 \times 6 \) matrix. With \( T \), the generalized forces can be defined according to eq.(56), and the applied actuator forces can be determined according to eq.(57).

4. Simulation

4.1 Geometric and inertial parameters

The geometry of the base and the moving platform is shown in Figure 3.

![Figure 3. Geometry of the base and the moving platform](image_url)

Accordingly, the coordinates of vector \( b \), with respect to the fixed frame are given as
where $L_b$ and $l_b$ are the long and short side of the base hexagon, $c_s = \cos(30^\circ)$, and $y_b = \left(L_b^2 / 2 + l_b \right) \tan(30^\circ)$. Likewise, the coordinates of vector $\vec{p}_i$ with respect to the local frame are given as

\begin{align*}
\vec{p}_1 &= \left[ L_p / 2, -y_p, 0 \right] ^T \\
\vec{p}_2 &= \left[ (L_p + l_p) / 2, -l_p c_s, 0 \right] ^T \\
\vec{p}_3 &= \left[ l_p / 2, (L_p + l_p) c_s - y_p, 0 \right] ^T \\
\vec{p}_4 &= \left[ -l_p / 2, (L_p + l_p) c_s - y_p, 0 \right] ^T \\
\vec{p}_5 &= \left[ (L_p + l_p) / 2, l_p c_s - y_p, 0 \right] ^T \\
\vec{p}_6 &= \left[ -L_p / 2, -y_p, 0 \right] ^T
\end{align*}

where $L_p$ and $l_p$ are the long and short side of the moving platform hexagon, and $y_p = \left(L_p^2 / 2 + l_p \right) \tan(30^\circ)$. The geometric parameters and inertial parameters are given in Tables 1 and 2, respectively. In Table 1, $S$ is the guideway length, $\gamma$ is the guideway angle between the guideway and the vertical direction, and $l$ is the length of the leg. These three parameters are the same for all the guideways and legs. Parameters $L_b$, $l_b$, $L_p$, and $l_p$ are defined in Figure 3. In Table 2, $m$ is the mass, and $I_{xx}$, $I_{yy}$, and $I_{zz}$ are the moments of inertia.

<table>
<thead>
<tr>
<th>S Guideway length</th>
<th>S Guideway angle</th>
<th>L Leg length</th>
<th>Lb Long side BP</th>
<th>lb Short side BP</th>
<th>Lp Long side MP</th>
<th>lp Short side MP</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.60 m</td>
<td>45°</td>
<td>0.50 m</td>
<td>1.00 m</td>
<td>0.09 m</td>
<td>0.50 m</td>
<td>0.09 m</td>
</tr>
</tbody>
</table>

Table 1. Geometric parameters

<table>
<thead>
<tr>
<th></th>
<th>$m$ (kg)</th>
<th>$I_{xx} = I_{yy}$ (kg·m²)</th>
<th>$I_{zz}$ (kg·m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Platform</td>
<td>3.983</td>
<td>0.068</td>
<td>0.136</td>
</tr>
<tr>
<td>Leg</td>
<td>0.398</td>
<td>0.0474</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2. Inertial parameters
4.2 Numerical example

A simulation program has been developed using Matlab based on the method described in the previous sections. In terms of computation, as can be seen from eq. (54), the inverse dynamics of the hexapods mainly involves the twist-mapping matrix $T$ and its derivative, which could be computed numerically for each time interval. This way, it is computationally more efficient. To further speed up computation, parallel computation techniques could be used. As shown in Figure 4, the motion part including actuator speeds and accelerations could be computed in parallel to the inertia part including mass matrix $I$ and coupling matrix $C$. Since the program is done in Matlab, parallel computation is not realized. However, this strategy can certainly be applied to model-based control using the dynamic equations.

In terms of singularity, as can be shown from eq. (63), the twist-mapping matrix $T$ becomes degenerate when the moving platform Jacobian $J_p (T_p = J_p^{-1})$ is singular. The movement of the moving platform is defined in terms of 3-4-5 polynomials that guarantee zero velocities and zero accelerations at the beginning and at the end. The selection of a smooth motion profile is very important for the hexapod as it is operated under high speeds. The conventional machine tools are run at a maximum velocity of 30 m/min with a maximum acceleration of 0.3 g. Hexapods can run at a maximum velocity of 100 m/min with a maximum acceleration over 1 g.

The first simulation is for high speed, with a maximum velocity of 102 m/min. The second simulation is for low speed with a maximum velocity of 30 m/min. In both cases, the hexapod moves a distance of 0.1 m along the z axis. The initial position of the moving platform is at $x_o = 0$, $y_o = 0$ and $z_o = 0.7$ m. Figure 4 shows the velocity profiles of the moving platform. Figures 5, 6 and 7 show the displacements, velocities and accelerations of the six actuators, respectively. Figure 8 shows the computed actuator forces. The simulations show that high speed motions result in large actuator forces.

![Fig. 4. Motion Profile of moving platform](www.intechopen.com)
Fig. 5. Actuator displacements

Fig. 6. Actuator velocities

Fig. 7. Actuator accelerations
5. Static balancing of the hexapod

The static and dynamic balancing is a classic problem in the theory of machines and mechanisms. In particular, when a mechanism is not statically balanced, the weight of linkage produces force or torque at actuators under static conditions and actuators have to contribute to support the weight of the moving links for any configurations. The problem becomes more serious for the parallel manipulator applied as flight simulator where the weight of the moving platform is very large with respect to the masses of the links. Static balancing also called gravity compensation is important. If the forces/torques exerted by joint actuators are reduced, the full potential of machine will be improved.

In this paragraph, following the same approach presented by Gosselin, the static balancing of the hexapod with the fixed-length legs is studied.

5.1 Static balancing using counterweight

The static balancing of the parallel manipulator under study is investigated using counterweights. The base coordinate frame Oxyz frame, is fixed to the base with Z-axis pointing vertically upward and the moving coordinate frame O’x’y’z’ is attached to the moving platform. The Cartesian coordinates used to describe the pose of the platform are as shown in Fig. 9 given by the position of O’ with respect to the fixed frame and the orientation of the platform represented by the rotation matrix \( Q \)

\[
Q = \begin{bmatrix}
q_{11} & q_{12} & q_{13} \\
q_{21} & q_{22} & q_{23} \\
q_{31} & q_{32} & q_{33}
\end{bmatrix}
\] (66)

Using the counterweights, static balancing is obtained if the global center mass of the mechanism is kept stationary at any values of the independent variables. To choose an suitable constant, namely

\[
Mr = 0 
\] (67)
Fig. 9 Kinematic mode

where \( r \) is the position vector of the global mass center, and \( M \) is:

\[
M = m_p + \sum_{i=1}^{6} m_i
\]  
(68)

where \( m_p \) is the mass of the platform, \( m_i \) is the mass of the leg. The global centre of the mass of the manipulator is written as

\[
Mr = m_p r_p + \sum_{i=1}^{6} m_i r_i
\]  
(69)

where \( r_p \) is the platform center of the mass, \( r_i \) is the leg center of the mass. From Fig. 9, vectors \( r_p \), \( r_i \) can be derived, and substituted into eq.(69), yielding

\[
Mr = m_p (h + Q \cdot g) + \sum_{i=1}^{6} m_i \left[ (h + Q \cdot p_i) - (h + Q \cdot p_i - b_i - s_i) \frac{l_i}{l_i} \right]
\]  
(70)

where \( g \) is the vector center of mass of the moving platform with respect to the frame \( O'x'y'z' \), \( h \) is the position of \( O' \) with respect to the fixed frame, \( p_i \) is the position of the spherical joint with respect to the moving coordinate frame, \( b_i \) is the position of the lower end of the guideway with respect to the fixed frame, \( l_i \) is the length of the leg, \( s_i \) can be written, for \( i=1,\ldots,6 \), as

\[
s_i = \rho_i \cdot \hat{s}_i
\]  
(71)

where \( \hat{s}_i \) is the unit vector of guideway, \( \rho_i \) is the independent variable of the prismatic joint. In concise form, eq. (70) is expressed as
\[ Mr = A_1 h + QB + \sum_{i=1}^{6} A_{si} s_i + A_0 \]

where

\[ A_1 = m_p + \sum_{i=1}^{6} m_i \left( 1 - \frac{l_{gi}}{l_i} \right) \]

\[ B = m_p g + \sum_{i=1}^{6} m_i p_i \left( 1 - \frac{l_{gi}}{l_i} \right) \]

\[ A_{si} = m_i \frac{l_{gi}}{l_i}, \text{ for } i=1,..,6 \]

\[ A_0 = \sum_{i=1}^{6} A_{si} b_i \]

The conditions for static balancing can be given for \( i=1,..,6 \), as follows:

\[ A_1 = 0, \ B = 0, \ A_{si} = 0, \ A_0 = 0 \]

From conditions \( A_{si} = 0, \ i=1,..,6 \), one can obtain

\[ l_{gi} = 0 \]

By condition \( A_1 = 0 \), one can obtain

\[ m_p = -\sum_{i=1}^{6} m_i \]

Eq.(79) shows that the balancing by counterweight is impossible. If it was substituted in the condition \( B = 0 \),

\[ m_p g + \sum_{i=1}^{6} m_i p_i = 0 \]

then one can obtain

\[ g = \frac{\sum_{i=1}^{6} m_i p_i}{\sum_{i=1}^{6} m_i} \]

From eq. (81), it shows that the manipulator could be balanced by a device that provide a force that is
1. equal to the weight of the links and the platform;
2. in opposite direction of the weight.

5.2 Static balancing with a pantograph counterweight

Since it is shown that the static balancing of the examined mechanism is impossible with the help of counterweights, we propose a method to add a pantograph connecting the moving platform \( O' \) to the fixed platform \( O \), as shown in Fig. 10. The pantograph is a device that allows to keep two end points on the same line and keep their distance at the centre with a constant ratio. In this application it is possible to use a pantographs with two or more mesh as shown in Fig. 10 and Fig. 11, respectively. In both cases the manipulator is balanced. The pantograph is fixed to the moving platform on the point \( O' \) by a spherical joint and fixed to the point \( O \) by an universal joint. The leg counterweight is shown in Fig. 12.

In this case, the mass \( M \) becomes,

\[
M = m_p + m_p^* + m_d + m_d^* + \sum_{i=1}^{6} m_i + \sum_{i=1}^{6} m_i^*
\]  

(82)
where \( m_p \) and \( m'_p \) are the mass of the platform and the mass of the platform counterweight, \( m_i \) and \( m'_i \) are the mass of the legs and the mass of the legs counterweights, \( m_a \) and \( m'_a \) are the mass of the pantograph and the mass of the pantograph counterweight. In this case, the global center of the mass of the manipulator is written as

\[
Mr = m_p r_p + m'_p r'_p + m_a r_a + m'_a r'_a + \sum_{i=1}^{6} m_i r_i + \sum_{i=1}^{6} m'_i r'_i
\]  

(83)

where \( r_p \) and \( r'_p \) are the platform center of the mass and the platform counterweight position, \( r_i \) and \( r'_i \) are the legs center of the mass and the legs counterweight position, \( r_a \) and \( r'_a \) are the pantograph center of the mass and the pantograph counterweight position.

From Figs. 9-10, vectors \( r_p, r'_p, r_a, r'_a, r_i \) and \( r'_i \) can be derived and substituted into eq.(83), yielding

\[
Mr = m_p (h + Q \cdot g) + m'_p (h + Q \cdot g') + m_a \left( h \frac{l_{p1}}{|h|} \right) + m'_a \left( h \frac{l_{p1}}{|h|} \right) + \\
+ \sum_{i=1}^{6} m_i \left[ (h + Q \cdot p_i) - (h + Q \cdot p_i - b_i - s_i) \frac{l_{gi}}{l_i} \right] + \\
+ \sum_{i=1}^{6} m'_i \left[ (h + Q \cdot p_i) - (h + Q \cdot p_i - b_i - s_i) \frac{l'_{gi}}{l_i} \right]
\]

(84)

where, \( l_a \) is the center of mass of the pantograph with respect to the fixed frame, \( l'_a \) is the pantograph counterweight position with respect to the fixed frame, \( l_{gi} \) is the length of the leg counterweight link, \( l_i \) is the length of the leg, \( s_i \) can be written, for \( i=1,..,6 \), as

\[
s_i = \rho_i \cdot \hat{s}_i
\]

(85)

In concise form, eq.(84) can be expressed as

\[
Mr = A_3 h + QB + \sum_{i=1}^{6} A_{5i} s_i + A_0
\]

(86)

where

\[
A_3 = m_p + m'_p + m_a \frac{l_{p1}}{|h|} + m'_a \frac{l_{p1}}{|h|} + \sum_{i=1}^{6} m_i \left( 1 - \frac{l_{ia}}{l_i} \right) + \sum_{i=1}^{6} m'_i \left( 1 - \frac{l'_{ia}}{l_i} \right)
\]

(87)

\[
B = m_p g + m'_p g' + \sum_{i=1}^{6} m_i p_i \left( 1 - \frac{l_{gi}}{l_i} \right) + \sum_{i=1}^{6} m'_i p_i \left( 1 - \frac{l'_{gi}}{l_i} \right)
\]

(88)

\[
A_{5i} = m_i \frac{l_{gi}}{l_i} + m'_i \frac{l'_{gi}}{l_i}, \ i=1,..,6
\]

(89)
\[ \mathbf{A}_0 = \sum_{i=1}^{6} A_{5i} \mathbf{b}_i \quad (90) \]

The conditions for static balancing can be given, for \( i = 1, \ldots, 6 \), as follows

\[ A_i = 0, \quad B = 0, \quad A_{5i} = 0, \quad \mathbf{A}_0 = 0 \quad (91) \]

From conditions \( A_{5i} = 0 \), for \( i = 1, \ldots, 6 \), one can obtain

\[ m_i l_{gi}^* + m_i^* l_{gi}^* = 0 \quad (92) \]

From eq. (92), for \( i = 1, \ldots, 6 \), the following is obtained

\[ l_{gi}^* = -\frac{m_i l_{gi}}{m_i^*} \quad (93) \]

By condition \( A_i = 0 \), i.e.,

\[ m_p + m_p^* + \sum_{i=1}^{6} (m_i + m_i^*) + m_a \frac{l_a}{|\mathbf{h}|} + m_a^* \frac{l_a}{|\mathbf{h}|} = 0 \quad (94) \]

one can obtain

\[ l_a^* = -\frac{|\mathbf{h}|}{m_a^*} \left( m_p + m_p^* + \sum_{i=1}^{6} (m_i + m_i^*) + m_a \frac{l_a}{|\mathbf{h}|} \right) \quad (95) \]

Finally, condition \( B = 0 \) leads to the following

\[ m_p \mathbf{g} + m_p^* \mathbf{g}^* + \sum_{i=1}^{6} (m_i + m_i^*) \mathbf{p}_i = 0 \quad (96) \]

Eq. (96) shows that the static balancing can be achieved by fixing the global center of the mass of the moving platform, that of the legs and their counterweights at the same position, \( \mathbf{O}' \). In order to obtain it, the platform counterweight should be placed in the position:

\[ \mathbf{g}^* = \frac{m_p \mathbf{g} + \sum_{i=1}^{6} (m_i + m_i^*) \mathbf{p}_i}{m_p^*} \quad (97) \]

Simulation is carried out to demonstrate the proposed method. The results are shown in Figs. 13-14, from which it can be seen that the centre of mass of the robot is non-stationary for non-balanced case, while it is fixed for the balanced case. After static balancing the global mass of the device increases by

\[ \Delta M = m_p^* + m_a + m_a^* + \sum_{i=1}^{6} m_i^* \quad (98) \]
The negative effect for the dynamic performance by the increasing global mass can be reduced by optimum design of the pantograph. A graph can be arranged to provide such help. Fig. 15 shows the ratios,

$$\frac{M + \Delta M}{M}, \frac{I_i + I_i^*}{I_i}, \frac{I_a + I_a^*}{I_a},$$  \hspace{1cm} (99)

which vary respect to the ratio $r_a^*/h$ and $l_{gi}^*/l_{gi}$ and where $I_i$ is the moment of inertia of the leg, $I_i^*$ is the moment of inertia of the leg counterweight with respect to $P$, $I_a$ is the moment of inertia of the moving platform and $I_a^*$ is the moment of inertia of the pantograph counterweight with respect to $O$. It should be noted that with a suitable design it is possible to reduce $\Delta M$ at the same time, it may increase $l_i$ and $I_a$. The effect of gravity compensation on the dynamic performances was studied in detail in (Xi, 1999).

Fig. 13. Mobile center of mass Hexapod

Fig. 14. Fixed center of mass of Balanced Hexapod
Figure 15. Graph for optimum design

<table>
<thead>
<tr>
<th>Input</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
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<td>Mobile platform</td>
<td></td>
<td></td>
</tr>
<tr>
<td>mass [kg]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>short side [mm]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>long side [mm]</td>
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<td>8</td>
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<td>800</td>
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<tr>
<td>Fixed platform 1</td>
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<tr>
<td>mass [kg]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>short side [mm]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>long side [mm]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>/</td>
<td>100</td>
<td>400</td>
</tr>
<tr>
<td>Fixed platform 2</td>
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</tr>
<tr>
<td>mass [kg]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>short side [mm]</td>
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<tr>
<td>/</td>
<td>250</td>
<td>1000</td>
</tr>
<tr>
<td>leg</td>
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<td></td>
</tr>
<tr>
<td>mass [kg]</td>
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<td></td>
</tr>
<tr>
<td>$l_i$ [mm]</td>
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<td>$l_{gi}$ [mm]</td>
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<td></td>
</tr>
<tr>
<td>0.5</td>
<td>750</td>
<td>375</td>
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<tr>
<td>Pantograph</td>
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<tr>
<td>mass [kg]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>side length [mm]</td>
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<td></td>
</tr>
<tr>
<td>$r_a$ [mm]</td>
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<td></td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3. Geometric and inertial parameters
7. Conclusion

In this chapter, the inverse dynamics of hexapods with fixed-length legs is analyzed using the natural orthogonal complement method, with considering the mass of the moving platform and those of the legs. A complete kinematics model is developed, which leads to an explicit expression for the twist-mapping matrix. Based on that, the inverse dynamics equations are derived that can be used to compute the required applied actuator forces for the given movement of the moving platform. The developed method has been implemented and demonstrated by simulation.

Successively, the static balancing of hexapods is addressed. The expression of the global center of mass is derived, based on which a set of static balancing equations has been obtained. It is shown that this type of parallel mechanism cannot be statically balanced by counterweights because prismatic joints do not have a fixed point to pivot as revolute joints. A new design is proposed to connect the centre of the moving platform to that of the fixed platform by a pantograph. The conditions for static balancing are derived. This mechanism is able to release the actuated joints from the weight of the moving legs for any configurations of the robot.

In the future research the leg inertia will be include for modeling the dynamics of the hexapod for high-speed applications.

8. References


In recent years, parallel kinematics mechanisms have attracted a lot of attention from the academic and industrial communities due to potential applications not only as robot manipulators but also as machine tools. Generally, the criteria used to compare the performance of traditional serial robots and parallel robots are the workspace, the ratio between the payload and the robot mass, accuracy, and dynamic behaviour. In addition to the reduced coupling effect between joints, parallel robots bring the benefits of much higher payload-robot mass ratios, superior accuracy and greater stiffness; qualities which lead to better dynamic performance. The main drawback with parallel robots is the relatively small workspace. A great deal of research on parallel robots has been carried out worldwide, and a large number of parallel mechanism systems have been built for various applications, such as remote handling, machine tools, medical robots, simulators, micro-robots, and humanoid robots. This book opens a window to exceptional research and development work on parallel mechanisms contributed by authors from around the world. Through this window the reader can get a good view of current parallel robot research and applications.

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