Chapter 0

A Game Theoretic Approach Based Adaptive Control Design for Sequentially Interconnected SISO Linear Systems

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Additional information is available at the end of the chapter

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1. Introduction

Adaptive control has attracted a lot of research attention in control theory for many decades. In the certainty equivalence based adaptive controller design [4, 5], the unknown parameters of the uncertainty system are substituted by their online estimates, which are generated through a variety of identifiers, as long as the estimates satisfy certain properties independent of the controller. This approach leads to structurally simple adaptive controllers and has been demonstrated its effectiveness for linear systems with or without stochastic disturbance inputs [10] when long term asymptotic performance is considered. Yet, the certainty equivalence approach is unsuccessful to generalize to systems with severe nonlinearities. Also, early designs based on this approach were shown to be nonrobust [13] when the system is subject to exogenous disturbance inputs and unmodeled dynamics. Then, the stability and the performance of the closed-loop system becomes an important issue. This has motivated the study of robust adaptive control in the 1980s and 1990s, and the study of nonlinear adaptive control in the 1990s.

The topic of adaptive control design for nonlinear systems was studied intensely in the last decade after the celebrated characterization of feedback linearizable or partially feedback linearizable systems [7]. A breakthrough is achieved when the integrator backstepping methodology [8] was introduced to design adaptive controllers for parametric strict-feedback and parametric pure-feedback nonlinear systems systematically. Since then, a lot of important contributions were motivated by this approach, and a complete list of references can be found in the book [9]. Moreover, this nonlinear design approach has been applied to linear systems to compare performance with the certainty equivalence approach. However, simple designs using this approach without taking into consideration the effect of exogenous disturbance inputs have also been shown to be nonrobust when the system is subject to exogenous disturbance inputs.

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The robustness of closed-loop adaptive systems has been an important research topic in late 1980s and early 1990s. Various adaptive controllers were modified to render the closed-loop systems robust [6]. Despite their successes, they still fell short of directly addressing the disturbance attenuation property of the closed-loop system.

The objectives of robust adaptive control are to improve transient response, to accommodate unmodeled dynamics, and to reject exogenous disturbance inputs, which are the same as the objectives to motivate the study of the $H^\infty$-optimal control problem. $H^\infty$-optimal control was proposed as a solution to the robust control problem, where these objectives are achieved by studying only the disturbance attenuation property for the closed-loop system. The game-theoretic approach to $H^\infty$-optimal control developed for the linear quadratic problems, offers the most promising tool to generalize the results to nonlinear systems [3]. Worst-case analysis based adaptive control design was proposed in late 1990s to address the disturbance attenuation property directly, and it is motivated by the success of the game-theoretic approach to $H^\infty$-optimal control problems [2]. In this approach, the robust adaptive control problem is formulated as a nonlinear $H^\infty$ control problem under imperfect state measurements. By cost-to-come function analysis, it is converted into an $H^\infty$ control problem with full information measurements. This full information measurements problem is then solved using nonlinear design tools for a suboptimal solution. This design scheme has been applied to worst-case parameter identification problems [11], which has led to new classes of parametrized identifiers for linear and nonlinear systems. It has also been applied to adaptive control problems [1, 12, 14, 15, 18, 19], and the convergence properties is studied in [20]. In [14], adaptive control for a strict-feedback nonlinear systems was considered with noiseless output measurements, and more general class of nonlinear systems was studied in [1]. In [12], single-input and single output (SISO) linear systems were considered with noisy output measurements. SISO linear systems with partly measured disturbance was studied in [18], which leads to a disturbance feed-forward structure in the adaptive controller. [19] generalizes the results of [12] to the adaptive control design for SISO linear systems with zero relative degree under noisy output measurements. In [17], adaptive control for a sequentially interconnected SISO linear system was considered, and a special class of unobservable systems was also studied using the proposed approach. More recently, [16] generalized the result of [17] to adaptive control design for a linear system under simultaneous driver, plant and actuation uncertainties.

In this Chapter, we study the adaptive control design for sequentially interconnected SISO linear systems, $S_1$ and $S_2$ (see Figure 1), under noisy output measurements and partly measured disturbance using the similar approaches as [12] and [17]. We assume that the linear systems satisfy the same assumption as [17], and the adaptive control design follows the same design method discussed above. The robust adaptive controller achieves asymptotic tracking of the reference trajectories when disturbance inputs are of finite energy. The closed-loop system is totally stable with respect to the disturbance inputs and the initial conditions. Furthermore, the closed-loop system admits a guaranteed disturbance attenuation level with respect to the exogenous disturbance inputs, where ultimate lower bound for the achievable attenuation performance level is equal to the noise intensity in the measurement channel of $S_1$. The results are as same as those in [17]. In addition, the controller achieves arbitrary positive distance attenuation level with respect to the measured disturbances by proper scaling. Moreover, if the measured disturbances satisfy the assumption 2 for $\hat{w}_{1,b}$ and $\hat{w}_{2,b}$, the
A Game Theoretic Approach Based Adaptive Control Design for Sequentially Interconnected SISO Linear Systems

proposed controller achieves disturbance attenuation level zero with respect to the measured disturbances, which further leads to a stronger asymptotic tracking property, namely, the tracking error converges to zero when the unmeasured disturbances are $L_2 \cap L_\infty$, and the measured disturbances are $L_\infty$ only.

The balance of this Chapter is organized as follows. In Section 2, we list the notations used in the Chapter. In Section 3, we present the formulation of the adaptive control problem and discuss the general solution methodology. In Section 4, we first obtain parameter identifier and state estimator using the cost-to-come function analysis in Subsection 4.1, then we derive the adaptive control law in Subsection 4.2. We present the main results on the robustness of the system in Section 5, and the example in Section 6. The Chapter ends with some concluding remarks in Section 7.

2. Notations

We denote $\mathbb{R}$ to be the real line; $\mathbb{R}_e$ to be the extended real line; $\mathbb{N}$ to be the set of natural numbers; $\mathbb{C}$ to be the set of complex numbers. For a function $f$, we say that it belongs to $C$ if it is continuous; we say that it belongs to $C_k$ if it is $k$-times continuously (partial) differentiable.

For any matrix $A$, $A'$ denotes its transpose. For any $b \in \mathbb{R}$, $\text{sgn}(b) = \begin{cases} -1 & b < 0 \\ 0 & b = 0 \\ 1 & b > 0 \end{cases}$. For any vector $z \in \mathbb{R}^n$, where $n \in \mathbb{N}$, $|z|$ denotes $(z'z)^{1/2}$. For any vector $z \in \mathbb{R}^n$, and any $n \times n$-dimensional symmetric matrix $M$, where $n \in \mathbb{N}$, $|z|_M^2 = z'Mz$. For any matrix $M$, the vector $\mathbf{\bar{M}}$ is formed by stacking up its column vectors. For any symmetric matrix $M$, $\mathbf{\bar{M}}$ denotes the vector formed by stacking up the column vector of the lower triangular part of $M$. For $n \times n$-dimensional symmetric matrices $M_1$ and $M_2$, where $n \in \mathbb{N}$, we write $M_1 > M_2$ if $M_1 - M_2$ is positive definite; we write $M_1 \geq M_2$ if $M_1 - M_2$ is positive semi-definite. For $n \in \mathbb{N}$, the set of $n \times n$-dimensional positive definite matrices is denoted by $\mathcal{S}_+$. For $n \in \mathbb{N} \cup \{0\}$, $I_n$ denotes the $n \times n$-dimensional identity matrix. For any matrix $M$, $\|M\|_p$ denotes its $p$-induced norm, $1 \leq p \leq \infty$. $L_2$ denotes the set of square integrable functions and $L_\infty$ denotes the set of bounded functions. For any $n, m \in \mathbb{N} \cup \{0\}$, $0_{n \times m}$ denotes the $n \times m$-dimensional matrix whose elements are zeros. For any $n \in \mathbb{N}$ and $k \in \{1, \cdots, n\}$, $e_{n,k}$ denotes $\begin{bmatrix} 0_{1 \times (k-1)} & 1 & 0_{1 \times (n-k)} \end{bmatrix}'$.

3. Problem Formulation

We consider the robust adaptive control problem for the system which is described by the block diagram in Figure 1.

Figure 1. Diagram of two sequentially interconnected SISO linear systems.
We assume that the system dynamics for $S_1$ and $S_2$ are given by,

\[
\begin{align*}
\dot{x}_1 &= \dot{A}_1 x_1 + \dot{B}_1 y_2 + \dot{D}_1 \bar{w}_1; \\
y_1 &= \dot{C}_1 x_1 + \dot{E}_1 \bar{w}_1 \\
\dot{x}_2 &= \dot{A}_2 x_2 + \dot{B}_2 u + \dot{A}_{2,q} \bar{y}_2 + \dot{D}_2 \bar{w}_2; \\
y_2 &= \dot{C}_2 x_2 + \dot{E}_2 \bar{w}_2
\end{align*}
\]

where $\dot{x}_i$ is the $n_i$-dimensional state vectors with initial condition $\dot{x}_i(0) = \dot{x}_{i,0}$, $n_i \in \mathbb{N}$; $u$ is the scalar control input; $y_i$ is the scalar measurement output; $\bar{w}_i$ is $\sigma_i$-dimensional unmeasured disturbance input vector, $\sigma_i \in \mathbb{N}$; the elements of $\bar{w}_i$ are $[\bar{w}_{i,1} \cdots \bar{w}_{i,\sigma_i}]'$; $\bar{y}_2 = y_1$; the matrices $\dot{A}_i, \dot{A}_{i,q}, \dot{B}_i, \dot{C}_i, \dot{D}_i, \dot{D}_2$, and $\dot{E}_i$ are of the appropriate dimensions, generally unknown or partially unknown, $i = 1, 2$. For subsystem $S_1$, the transfer function from $y_2$ to $y_1$ is $H_1(s) = \dot{C}_1 (s I_{n_1} - \dot{A}_1)^{-1} \dot{B}_1$; for subsystem $S_2$, the transfer function from $u$ to $y_2$ is $H_2(s) = \dot{C}_2 (s I_{n_2} - \dot{A}_2)^{-1} \dot{B}_2$. All signals in the system are assumed to be continuous.

The subsystems $S_1$ and $S_2$ satisfy the following assumptions,

**Assumption 1.** For $i = 1, 2$, the pair $(\dot{A}_i, \dot{C}_i)$ is observable; the transfer function $H_i(s)$ is known to have relative degree $r_i \in \mathbb{N}$, and is strictly minimum phase. The uncontrollable part of $S_1$ (with respect to $y_2$) is stable in the sense of Lyapunov; any uncontrollable mode corresponding to an eigenvalue of the matrix $\dot{A}_1$ on the $j\omega$-axis is uncontrollable from $[\bar{w}_1' \bar{w}_2']'$. The uncontrollable part of $S_2$ (with respect to $u$) is stable in the sense of Lyapunov; any uncontrollable mode corresponding to an eigenvalue of the matrix $\dot{A}_2$ on the $j\omega$-axis is uncontrollable from $[\bar{w}_2' \bar{y}_2' \bar{w}_2']'$.

Based on Assumption 1, for $i = 1, 2$, there exists a state diffeomorphism: $x_i = T_i \dot{x}_i,$ and a disturbance transformation: $w_i = M_i \bar{w}_i,$ such that $S_i$ can be transformed into the following state space representation,

\[
\begin{align*}
\dot{x}_1 &= A_1 x_1 + (y_1 \bar{A}_{1,211} + y_2 \bar{A}_{1,212} + \sum_{j=1}^{q_1} \bar{w}_{1,j} \bar{A}_{1,213}) \theta_1 + B_1 y_2 + D_1 \bar{w}_1 + D_1 \bar{w}_1; \\
y_1 &= \dot{C}_1 x_1 + \dot{E}_1 \bar{w}_1 \\
\dot{x}_2 &= A_2 x_2 + (y_2 \bar{A}_{2,211} + u \bar{A}_{2,212} + \sum_{j=1}^{q_2} \bar{w}_{2,j} \bar{A}_{2,213} + \bar{y}_2 \bar{A}_{2,214}) \theta_2 + B_2 u + A_{2,q} \bar{y}_2 + D_2 \bar{w}_2 + \bar{D}_2 \bar{w}_2; \\
y_2 &= \dot{C}_2 x_2 + \dot{E}_2 \bar{w}_2
\end{align*}
\]

where $\theta_i$ is the $\sigma_i$-dimensional vector of unknown parameters for the subsystem $S_i$, $\sigma_i \in \mathbb{N}$; the matrices $\bar{A}_{i,j}, \bar{A}_{i,211}, \bar{A}_{i,212}, \bar{A}_{i,213}, \cdots, \bar{A}_{i,213} j, \bar{A}_{2,214}, A_{2,q}, B_i, D_i, \bar{D}_i,\bar{C}_i$, and $E_i$ are known and have the following structures, $A_i = (a_{i,jk})_{n_i \times n_i}; a_{i,jk}(j+1) = 1, a_{i,jk} = 0$, for $1 \leq j \leq r_i - 1$ and $j + 2 \leq k \leq n_i$; $\bar{A}_{i,212} = [0_{r_i \times (r_i - 1)} \bar{A}_{i,2120}^{T} \bar{A}_{i,212r}]'$, $C_i = [1_{1 \times (n_i - 1)}], A_{2,2120}$ is a row vector, $B_i = [0_{1 \times (r_i - 1)} b_{i,pj} \cdots b_{i,p(n_i - r_i)}]'$, $b_{i,pj} j = 0, 1, \cdots, n_i - r_i$ are constants.

We denote the elements of $x_1$ and $x_2$ by $[x_{1,1} \cdots x_{1,n_1}]'$ and $[x_{2,1} \cdots x_{2,n_2}]'$, with initial conditions $x_{1,0}$ and $x_{2,0}$, respectively.
Assumption 2. The measured disturbance \( \hat{w}_1 \) can be partitioned as: \( \hat{w}_1 = \begin{bmatrix} \hat{w}_{1,a} \hat{w}_{1,b} \end{bmatrix} \) where \( \hat{w}_{1,a} \) is \( q_{1,a} \)-dimensional, \( q_{1,a} \in \mathbb{N} \cup \{0\} \), and the transfer function from each element of \( \hat{w}_{1,a} \) to \( y_1 \) has relative degree less than \( r_1 + r_2 \); the measured disturbance \( \hat{w}_2 \) can be partitioned as: \( \hat{w}_2 = \begin{bmatrix} \hat{w}_{2,a} \hat{w}_{2,b} \end{bmatrix} \) where \( \hat{w}_{2,a} \) is \( q_{2,a} \)-dimensional, \( q_{2,a} \in \mathbb{N} \cup \{0\} \), and the transfer function from each element of \( \hat{w}_{2,a} \) to \( y_2 \) has relative degree less than \( r_2 \).

Based on Assumption 2, the matrix \( \hat{D}_i \) can be partitioned into \( \hat{D}_i = \begin{bmatrix} \hat{D}_{i,a} & \hat{D}_{i,b} \end{bmatrix} \), where \( \hat{D}_{i,a} \) and \( \hat{D}_{i,b} \) have \( n_i \times q_{i,a} \) and \( n_i \times q_{i,b} \)-dimensional, respectively; and \( \hat{D}_{i,b} \), \( \hat{A}_{i,213}(q_{i,a}+1), \cdots, \hat{A}_{i,213}q_i \) have the following structure

\[
\hat{D}_{i,b} = \begin{bmatrix} 0_{(r_i-1)\times q_{i,b}} & \hat{D}_{i,b0} & \hat{D}_{i,b1} \\ \hat{D}_{i,b1} & \hat{D}_{i,b2} & \hat{D}_{i,b3} \end{bmatrix} ; \quad \hat{A}_{i,213}j = \begin{bmatrix} 0_{(r_i-1)\times q_i} \\ \hat{A}_{i,213}j0 \\ \hat{A}_{i,213}jr_i \end{bmatrix}, \quad j = q_{i,a} + 1, \cdots, q_i
\]

where \( \hat{D}_{i,b0} \) and \( \hat{A}_{i,213}j0 \), \( j = q_{i,a} + 1, \cdots, q_{i,a} + q_{i,b} \), are row vectors, \( i = 1, 2 \).

Since we will base our design of adaptive controllers using the model (2), we call (2) the design model, and make the following two assumptions.

Assumption 3. For \( i = 1, 2 \), the matrices \( E_i \) are such that \( E_iE_i^T > 0 \).

Define \( \zeta_i := (E_iE_i^T)^{-\frac{1}{2}} \) and \( L_i := D_iE_i^T, i = 1, 2 \).

Due to the structures of \( A_i \), \( \hat{A}_{i,212} \) and \( B_i \), the high frequency gain of the transfer function \( H_i(s), b_{i,0} \), is equal to \( b_{i,0} + \hat{A}_{i,212}q_i \), \( i = 1, 2 \).

To guarantee the stability of the identified system, we make the following assumption on the parameter vectors \( \theta_1 \) and \( \theta_2 \).

Assumption 4. The sign of \( b_{i,0} \) is known; there exists a known smooth nonnegative radially-unbounded strictly convex function \( P_i : \mathbb{R}^q_i \rightarrow \mathbb{R} \), such that the true value \( \theta_i \in \Theta_i := \{\theta_i \in \mathbb{R}^{q_i} | P_i(\theta_i) \leq 1\} \); moreover, \( \forall \theta_i \in \Theta_i, \text{sgn}(b_{i,0})(b_{i,0} + \hat{A}_{i,212}q_i) > 0, i = 1, 2 \).

Assumption 4 delineates a priori convex compact sets where the parameter vectors \( \theta_1 \) and \( \theta_2 \) lie in, respectively. This will guarantee the stability of the closed-loop system and the boundedness of the estimate of \( \theta_1 \) and \( \theta_2 \).

We make the following assumption about the reference signal, \( y_d \).

Assumption 5. The reference trajectory, \( y_d \), is \( r_1 + r_2 \) times continuously differentiable. Define vector \( Y_d := [y_d(0), \cdots, y_d(r_1+r_2)] \), where \( y_d(0) = y_d \) and \( y_d(j) \) is the \( j \)-th order time derivative of \( y_d \), \( j = 1, \cdots, r_1 + r_2 \); define \( Y_{d0} := [y_d(0), \cdots, y_d(r_1+r_2)] \) is \( \mathbb{R}^{r_1+r_2} \). The signal \( Y_d \) is available for feedback.

The uncertainty of subsystem \( S_1 \) is \( \hat{w}_1 := (x_{1,0}, \theta_1, \hat{w}_{1,0}, \bar{w}_{1,0}, y_{d0}, y_{d}(r_1+r_2)) \subseteq \hat{W}_1 := \mathbb{R}^{q_1} \times \Theta_1 \times C \times C \times \mathbb{R}^{r_1+r_2} \times C \), which comprises the initial state \( x_{1,0} \), the true value of the parameters \( \theta_1 \), the unmeasured disturbance waveform \( \bar{w}_{1,0} \), the measured disturbance waveform \( \hat{w}_{1,0} \), the initial conditions of the reference trajectory \( y_{d0} \), and the waveform...
of the \((r_1 + r_2)\) th order derivative of the reference trajectory, \(y^{(r_1+r_2)}_{d(0,\infty)}\). The uncertainty for subsystem \(S_2\) is \(\hat{\omega}_2 := (x_{2,0}, \theta_2, \bar{w}_{2(0,\infty)}, \bar{w}_2^{2(0,\infty)}) \in \mathcal{W}_2 := \mathbb{R}^{n_2} \times \Theta_2 \times C \times C\), which comprises the initial state \(x_{2,0}\), the true value of the parameters \(\theta_2\), the unmeasured disturbance waveform \(\bar{w}_{2(0,\infty)}\), and the measured disturbance waveform \(\bar{w}_2^{2(0,\infty)}\).

Our objective is to derive a control law, which is generated by the following mapping,

\[
u(t) = \mu(y_{2[0,t]}, \bar{y}_2^{2[0,t]}, Y_d[0,t], \bar{w}_1, \bar{w}_2)
\]

where \(\mu : C \times C \times C \times C \times C \rightarrow \mathbb{R}\), such that \(x_{1,1}\) can asymptotically track the reference trajectory \(y_d\), while rejecting the uncertainty \((\hat{\omega}_1, \hat{\omega}_2) \in \mathcal{W}_1 \times \mathcal{W}_2\), and keeping the closed-loop signals bounded. The control law \(\mu\) must also satisfy that, \(\forall (\hat{\omega}_1, \hat{\omega}_2) \in \mathcal{W}_1 \times \mathcal{W}_2\), there exists a solution \(\hat{x}_{1[0,\infty]}\) and \(\hat{x}_2^{2[0,\infty]}\) to the system (1), which yields a continuous control signal \(u[0,\infty]\). We denote the class of these admissible controllers by \(M_\mu\).

For design purposes, instead of attenuating the effect of \(\bar{w}_1^{(1)} \bar{w}_1^{(2)} \bar{w}_2^{(2)}\), we design the adaptive controller to attenuate the effect of \(\bar{w}_1^{(1)} \bar{w}_1^{(2)} \bar{w}_2^{(2)}\). This is done to allow our design paradigm to be carried out. This will result in a guaranteed attenuation level with respect to \(\hat{\omega}_1\) and \(\hat{\omega}_2\). To simplify the notation, we take the uncertainty \(\omega_1 := (x_{1,0}, \theta_1, \bar{w}_{1[0,\infty]}, \bar{w}_1^{1[0,\infty]}, Y_{d[0,\infty]}y^{(r_1+r_2)}_{d(0,\infty)}) \in \mathcal{W}_1 := \mathbb{R}^{n_1} \times \Theta_1 \times C \times C \times \mathbb{R}^{r_1+r_2} \times C\), and \(\omega_2 := (x_{2,0}, \theta_2, \bar{w}_{2[0,\infty]}, \bar{w}_2^{2[0,\infty)}) \in \mathcal{W}_2 := \mathbb{R}^{n_2} \times \Theta_2 \times C \times C\).

We state the control objective precisely as follows,

**Definition 1.** A controller \(\mu \in M_\mu\) is said to achieve disturbance attenuation level \(\gamma\) with respect to \(w_1^{(1)} w_1^{(2)} w_2^{(2)}\), and disturbance attenuation level zero with respect to \(w_1^{(1)} w_1^{(2)} w_2^{(2)}\), if there exists functions \(l_1(t, \theta_1, x_{1[0,t]}, y_{1[0,t]}, \bar{w}_1^{1[0,t]}, \bar{w}_2^{2[0,t]}, Y_{d[0,t]}), l_2(t, \theta_2, x_{2[0,t]}, y_{2[0,t]}, \bar{w}_1^{1[0,t]}, \bar{w}_2^{2[0,t]}, Y_{d[0,t]}), \bar{w}_2^{2[0,t]}, Y_{d[0,t]}), l_0(\hat{x}_{1,0}, \hat{x}_2, \hat{\theta}_{1,0}, \hat{\theta}_{2,0})\), and a known nonnegative constant \(l_0(\hat{x}_{1,0}, \hat{x}_2, \hat{\theta}_{1,0}, \hat{\theta}_{2,0})\), such that

\[
\sup_{\omega_1 \in \mathcal{W}_1, \omega_2 \in \mathcal{W}_2} J_{\gamma t_f} \leq 0; \ \forall t_f \geq 0
\]

and \(l_1 \geq 0\) and \(l_2 \geq 0\) along the closed-loop trajectory, where

\[
J_{\gamma t_f} = J_{1,\gamma t_f} + J_{2,\gamma t_f}
\]

\[
J_{1,\gamma t_f} = \int_0^{t_f} \left( (C_1 x_1 - y_d)^2 + l_1 - \gamma^2 |w_1|^2 - \gamma^2 |\bar{w}_1^{1,0}|^2 \right) \, d\tau - \gamma^2 \left[ \theta_1 - \hat{\theta}_{1,0} x_{1,0} - \hat{x}_{1,0} \right]^2 Q_{i,0}
\]

\[
J_{2,\gamma t_f} = \int_0^{t_f} \left( l_2 - \gamma^2 |w_2|^2 - \gamma^2 |\bar{w}_2^{2,0}|^2 \right) \, d\tau - l_2 - \gamma^2 \left[ \theta_2 - \hat{\theta}_{2,0} x_{2,0} - \hat{x}_{2,0} \right]^2 Q_{i,2}
\]

\(\hat{\theta}_{i,0} \in \Theta_i\) is the initial guess of \(\theta_i\); \(\hat{x}_{i,0} \in \mathbb{R}^{n_i}\) is the initial guess of \(x_{i,0}\); \(\bar{Q}_{i,0} > 0\) is a \((n_i + \sigma_i) \times (n_i + \sigma_i)\)-dimensional weighting matrix, quantifying the level of confidence in the estimate \(\left[ \theta_{i,0}^{T} x_{i,0}^{T} \right]\); \(\bar{Q}_{i,0}^{-1}\) admits the structure

\[
\begin{bmatrix}
Q_{i,0}^{-1} & 0 \\
0 & \Phi_{i,0}^{-1} \Pi_{i,0} + \Phi_{i,0} Q_{i,0}^{-1} \Phi_{i,0}^T
\end{bmatrix},
\]

\(Q_{i,0}\) and \(\Pi_{i,0}\) are \(\sigma_i \times \sigma_i\) and \(n_i \times n_i\)-dimensional positive definite matrices, respectively, \(i = 1, 2\).
Clearly, when the inequality (4) is achieved, the squared $L_2$ norm of the output tracking error $C_1 x_1 - y_d$ is bounded by $\gamma^2$ times the squared $L_2$ norm of the transformed disturbance input $[\dot{w}_1, \dot{w}_1', \dot{w}_2, \dot{w}_2', \dot{w}_3]$, plus some constant. When the $L_2$ norm of $\dot{w}_1$, $\dot{w}_2$, and $\dot{w}_3$ are finite, the squared $L_2$ norm of $C_1 x_1 - y_d$ is also finite, which implies $\lim_{t \to \infty} (C_1 x_1(t) - y_d(t)) = 0$, under additional assumptions.

Let $\xi_i$ denote the expanded state vector $\xi_i = [\theta_i', x_i']'$, $i = 1, 2$, and note that $\dot{\theta}_j = 0$, we have the following expanded dynamics for system (2),

$$
\dot{\xi}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} y_2 + \begin{bmatrix} 0 \\ D_1 \end{bmatrix} \dot{w}_1
$$

$$
y_1 = \begin{bmatrix} 0 C_1 \end{bmatrix} \xi_1 + E_1 \dot{w}_1
$$

$$
\dot{\xi}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} w_2 + \begin{bmatrix} 0 \\ D_2 \end{bmatrix} \dot{w}_2
$$

$$
y_2 = \begin{bmatrix} 0 C_2 \end{bmatrix} \xi_2 + E_2 \dot{w}_2
$$

The worst-case optimization of the cost function (4) can be carried out in two steps as depicted in the following equations.

$$
sup_{\omega_m \in W_m} J_{\gamma_f} = sup_{\omega_m \in W_m} \sup_{\omega_m \in W_m} \sup_{\omega_m \in W_m} J_{\gamma_f}
$$

$$
\leq sup_{\omega_m \in W_m} \sup_{\omega_m \in W_m} J_{\gamma_f}
$$

$$
= sup \left( \sum_{i=1}^{2} \sup_{\omega_m \in W_m} J_{\gamma_f} \right)
$$

(6)

where $\omega_m$ is the measured signals of the system, and defined as

$$
\omega_m := (y_1, y_2, \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3, Y_d, y_d^{r_1 + r_2}) \in W_m := C \times C \times C \times R^{r_1 + r_2} \times C.
$$

The inner supremum operators will be carried out first. We maximize over $\omega_i$ given that the measurement $\omega_m$ is available for estimator design, $i = 1, 2$. In this step, the control input, $u$, is a function only depended on $\omega_m$, then $u$ is an open-loop time function and available for the optimization. Using cost-to-come function analysis, we derive the dynamics of the estimators for subsystem $S_1$ and $S_2$ independently.

The outer supremum operator will be carried out second. In this step, we use a backstepping procedure to design the controller $\mu$.

This completes the formulation of the robust adaptive control problem.
4. Adaptive control design

In this section, we present the adaptive control design, which involves estimation design and control design. First, we discuss estimation design.

4.1. Estimation design

In this subsection, we present the estimation design for the adaptive control problem formulated. First, we will derive the identifier of subsystem $S_1$. In this step, the measurement waveform $y_1$, $y_2$ and measured disturbance $\tilde{w}_1$ are assumed to be known. Then we can obtain the identifier of subsystem $S_1$ from a game-theoretic solution methodology – cost-to-come function analysis.

We first set function $l_1$ in the definition to be $|\zeta_1 - \hat{\xi}_1|^2 + 2(\hat{\xi}_1 - l_1)'l_1 + \bar{l}_1$, where $\hat{\xi}_1 = [\hat{\theta}_1', x_1']'$ is the worst-case estimate for the expanded state $\zeta_1$, $\hat{Q}_1(y_{1[0,\tau]}, Y_{d[0,\tau]}, \tilde{w}_{1[0,\tau]})$ is a matrix-valued weighting function, $l_{1,1}(y_{1[0,\tau]}, Y_{d[0,\tau]}, \tilde{w}_{1[0,\tau]}), l_{1,2}(y_{1[0,\tau]}, Y_{d[0,\tau]}, \tilde{w}_{1[0,\tau]})$ and $\bar{l}_1(y_{1[0,\tau]}, Y_{d[0,\tau]}, \tilde{w}_{1[0,\tau]})$ are three design functions to be introduced later, the cost function of subsystem $S_1$ is then of the a linear quadratic structure.

The robust adaptive problem for $S_1$ becomes an $H^\infty$ control of affine quadratic problem, and admits a finite dimensional solution. By cost-to-come function analysis, we obtain the dynamics of worst-case covariance matrix $\Sigma_1$, and state estimator $\xi_1$, which are given by

$$
\dot{\Sigma}_1 = (\tilde{A}_1 - \zeta_1^2 L_1 C_1)\Sigma_1 + \Sigma_1 (\tilde{A}_1 - \zeta_1^2 L_1 C_1)' - \Sigma_1 (2\zeta_1^2 \tilde{C}_1 \tilde{C}_1' - \zeta_1 \tilde{C}_1' \tilde{C}_1 - \tilde{Q}_1)\Sigma_1
+ \gamma^{-2} D_1 D_1' - \gamma^{-2} \zeta_1^2 L_1 L_1',
$$

$$
\xi_1 = (\tilde{A}_1 + \Sigma_1 (\tilde{C}_1' \tilde{C}_1 + \tilde{Q}_1))\xi_1 + \tilde{B}_1 y_2 + \zeta_1^2 (2\Sigma_1 \tilde{C}_1' + L_1)(y_1 - \tilde{C}_1 \xi_1)
- \Sigma_1 (\tilde{C}_1 y_d + \tilde{Q}_1 \xi_1' - l_{1,2}) + \tilde{D}_1 \tilde{w}_1; \quad \xi_1(0) = [\theta_{1,0}' x_{1,0}']'
$$

where $L_1$ is defined as $L_1 = [0_{1 \times \alpha_1} L_1']'$.

We partition $\Sigma_1$ as the same structure as

$$
\Sigma_1 = \begin{bmatrix}
\Sigma_{1,1} & \Sigma_{1,12} \\
\Sigma_{1,21} & \Sigma_{1,22}
\end{bmatrix} = \begin{bmatrix}
\Sigma_1 \Phi_1^{-1} \\
\Phi_1 \Sigma_1 \Phi_1^{-1} - \Pi_1 + \Phi_1 \Sigma_1 \Phi_1^{-1}
\end{bmatrix}
$$

where $\Phi_1(t) := \Sigma_{1,21}(t)(\Sigma_1(t))^{-1}$ and $\Pi_1(t) := \gamma^2 (\Sigma_{1,22}(t) - \Sigma_{1,21}(t)(\Sigma_1(t))^{-1})\Sigma_{1,12}(t), \forall t \in [0, t_f]$. Then the weighting matrix $\Sigma_1$ is positive definite if and only if $\Sigma_1$ and $\Pi_1$ are positive definite. To guarantee the boundedness of $\Sigma_1$, we choose weighing matrix $\tilde{Q}_1$ as follows,

$$
\tilde{Q}_1 = \Sigma_1^{-1} \begin{bmatrix}
0_{\alpha_1 \times \alpha_1} & 0_{\alpha_1 \times n_1} \\
0_{n_1 \times \alpha_1} & \Delta_1(t)
\end{bmatrix} \Sigma_1^{-1} + \begin{bmatrix}
\epsilon_1 \Phi_1' C_1' (\gamma^2 \zeta_1^2 - 1) C_1 \Phi_1 & 0_{\alpha_1 \times n_1} \\
0_{n_1 \times \alpha_1} & 0_{n_1 \times n_1}
\end{bmatrix}
$$

where $\Delta_1(t) = \gamma^{-2} \beta_{1,\Pi} \Pi_1(t) + \Delta_{1,1}$, with $\beta_{1,\Pi} \geq 0$ being a constant and $\Delta_{1,1}$ being an $n_1 \times n_1$-dimensional positive-definite matrix, and $\epsilon_1$ is a scalar function defined by

$$
\epsilon_1(t) := \operatorname{Tr}(\Sigma_1(t))^{-1}/K_{1,c} \quad \forall t \in [0, t_f]
$$

or

$$
\epsilon_1(t) := 1 \quad \forall t \in [0, t_f]
$$
Then the dynamics of $\Sigma_1$, $\Phi_1$, $\Pi_1$ are given as follows with initial conditions $\gamma^{-2}Q_{1,0}^{-1}$, $\Pi_{1,0}$, and $\Phi_{1,0}$ respectively,

$$
\dot{\Sigma}_1 = (\epsilon_1 - 1)\Sigma_1 \Phi_1' C_1' (\gamma^2 \zeta_1^2 - 1) C_1 \Phi_1 \Sigma_1 \\
\dot{\Pi}_1 = (A_1 - \zeta_1^2 L_1 C_1)\Pi_1 + \Pi_1 (A_1 - \zeta_1^2 L_1 C_1)' - \zeta_1^2 L_1 L_1' - \Pi_1 C_1' (\zeta_1^2 - \gamma^{-2}) C_1 \Pi_1 \\
+ D_1 D_1' + \gamma^2 \Delta_1
$$

(11a)

$$
\dot{\Phi}_1 = A_1, f \Phi_1 + y_1 \bar{A}_{1,211} + y_2 \bar{A}_{1,212} + \sum_{j=1}^{\delta_1} \tilde{A}_{1,213 j} \tilde{w}_{1,j}
$$

(11b)

where $A_{1, f} := A_1 - \zeta_1^2 L_1 C_1 - \Pi_1 C_1' (\zeta_1^2 - \gamma^{-2})$ is Hurwitz. By picking $\gamma \geq \zeta_1^{-1}$, we have the covariance matrix $\Sigma_1$ upper and lower bounded as summarized in the following Lemma [12].

**Lemma 1.** Consider the dynamic equation (11a) for the covariance matrix $\Sigma_1$. Let $K_{1,c} \geq \gamma^2 \text{Tr}(Q_{1,0})$, $Q_{1,0} > 0$, $\gamma \geq \zeta_1^{-1}$, and $\epsilon_1$ be given by either (10b) or (10b). Then, the matrix $\Sigma_1$ is upper and lower bounded as follows, whenever $\Phi_1$ is continuous on $[0, t_f]$,

$$
K_{1,c}^{-1} I_{1, c_1} \leq \Sigma_1(t) \leq \Sigma_1(0) = \gamma^{-2} Q_{1,0}^{-1};
$$

$$
\gamma^2 \text{Tr}(Q_{1,0}) \leq \text{Tr}(\Sigma_1(t))^{-1} \leq K_{1,c}; \quad \forall t \in [0, t_f]
$$

To avoid the calculation of $\Sigma_1^{-1}$ online, we define $s_{1, \Sigma} = \text{Tr}(\Sigma_1^{-1})$. Based on the structure of $\dot{Q}_1$, we have the following assumption to guarantee the boundedness of $\Sigma_1$ and $s_{1, \Sigma}$.

**Assumption 6.** If the matrix $A_1 - \zeta_1^2 L_1 C_1$ is Hurwitz, then the desired disturbance attenuation level $\gamma \geq \zeta_1^{-1}$. In case $\gamma = \zeta_1^{-1}$, choose $\beta_{1, \Delta} \geq 0$ such that $A_1 - \zeta_1^2 L_1 C_1 + \beta_{1, \Delta} / 2 I_{n_1}$ is Hurwitz. If the matrix $A_1 - \zeta_1^2 L_1 C_1$ is not Hurwitz, then the desired disturbance attenuation level $\gamma > \zeta_1^{-1}$. \hfill \Box

This assumption implies that the achievable disturbance attenuation level $\gamma$ is no smaller than $\zeta_1^{-1}$. Under this assumption, we initialize $\Pi_1$ as the unique positive definite solution of its Riccati Differential Equation (11b), which is summarized as the following assumption.

**Assumption 7.** The initial weighting matrix $\Pi_{1,0}$ is chosen as the unique positive definite solutions to the following algebraic Riccati equations:

$$
(A_1 - \zeta_1^2 L_1 C_1)\Pi_1 + \Pi_1 (A_1 - \zeta_1^2 L_1 C_1)' - \Pi_1 C_1' \zeta_1^2 C_1 \Pi_1 + D_1 D_1' - \zeta_1^2 L_1 L_1' + \gamma^2 \Delta_1 = 0_{n_1 \times n_1}
$$

(12)

To guarantee the estimates parameter to be bounded and the estimate of high frequency gain to be bounded away from zero, projection function scheme is applied to modify the dynamics of $\zeta_1$.

Define

$$
\rho_1 := \inf \{ P_1(\tilde{\theta}_1) \mid \tilde{\theta}_1 \in \mathbb{R}^{n_1}, b_{1,p_0} + \bar{A}_{1,2120} \tilde{w}_1 = 0 \}
$$

(13)

By Assumption 4 and Lemma 2 in [19] we have $1 < \rho_1 \leq \infty$. Fix any $\rho_{1, \rho} \in (1, \rho_1)$, and define the open set $\Theta_{1, \rho} := \{ \tilde{\theta}_1 \in \mathbb{R}^{n_1} \mid P_1(\tilde{\theta}) < \rho_{1, \rho} \}$. Our control design will guarantee that the
estimate $\tilde{\theta}_1$ lies in $\Theta_{1,o}$, which immediately implies $|b_{1,p_0} + A_{1,2120}\tilde{\theta}_1| > c_{1,0} > 0$, for some $c_{1,0} > 0$. Moreover, the convexity of $P_1$ implies the following inequality

$$\frac{\partial P_1}{\partial \tilde{\theta}_1}(\tilde{\theta}_1)(\tilde{\theta}_1 - \tilde{\theta}_1) < 0 \quad \forall \tilde{\theta}_1 \in \mathbb{R}^{c_1} \setminus \Theta_1$$

We set $l_{1,1} = \tilde{x}_1$, and $l_{1,2} = \left[ -(P_{1,r}(\tilde{\theta}_1))' 0_{1 \times n_1} \right]'$, where

$$P_{1,r}(\tilde{\theta}_1) := \begin{cases} \frac{e^{\frac{1}{T-P_1(\tilde{\theta}_1)}}}{P_{1,r}(\tilde{\theta}_1)} & \forall \tilde{\theta}_1 \in \Theta_{1,o} \setminus \Theta_1 \\ 0_{n_1 	imes 1} & \forall \tilde{\theta}_1 \in \Theta_1 \end{cases}$$

(14)

then, we obtain

$$\tilde{\xi}_1 = -\tilde{\Sigma}_1 \left[ (P_{1,r}(\tilde{\theta}_1))' 0_{1 \times n_1} \right] + A_1\tilde{x}_1 + \tilde{\Sigma}_1C_1'(y_d - C_1\tilde{x}_1) - \tilde{\Sigma}_1Q_1(\Phi_1,s_1,\Sigma)\tilde{\xi}_1 + B_1y_2 + \tilde{\Sigma}_1C_1'(\gamma^2\tilde{\xi}_1 + L_1)(y_1 - C_1\tilde{x}_1) + \tilde{D}_1\tilde{w}_1; \quad \tilde{\xi}_1(0) = \left[ \partial_{\tilde{\theta}_1}^{\tilde{\nu}_1} \tilde{\nu}_1 \right]' \quad (15)$$

where $\tilde{\xi}_{1,c} = \tilde{\xi}_1 - \tilde{\xi}_1$.

We summarize the equations for subsystem $S_1$ as follows,

$$0 = (A_1 - \xi_1^2L_1C_1)\Pi_1 + \Pi_1(A_1 - \xi_1^2L_1C_1)' - \Pi_1C_1'(\gamma^2\xi_1^2 - \gamma^{-2})C_1\Pi_1 + D_1D_1' - \xi_1^2L_1L_1' + \gamma^2\Delta_1$$

$$\hat{\Sigma}_1 = -(1 - \epsilon_1)\Sigma_1\Phi_1'\gamma_1 C_1'(\gamma^2\xi_1^2 - 1)C_1\Phi_1\Sigma_1$$

$$\hat{x}_1 = (\gamma^2\xi_1^2 - 1)(1 - \epsilon_1)C_1\Phi_1\Phi_1'\gamma_1$$

$$\epsilon_1 = K_{1,c}^{-1}\tilde{x}_1 \text{ or } 1$$

$$A_{1,f} = A_1 - \xi_1^2L_1C_1 - \Pi_1C_1'(\gamma^2\xi_1^2 - \gamma^{-2})$$

$$\Phi_1 = A_{1,f}\gamma_1 + y_1\tilde{A}_{1,211} + y_2\tilde{A}_{1,212} + \sum_{j=1}^{\tilde{n}_1} \tilde{A}_{1,213}j\tilde{w}_{1,j}$$

$$\hat{\theta}_1 = -\Sigma_1P_{1,r}(\tilde{\theta}_1) - \Sigma_1\Phi_1'\gamma_1 C_1'(y_d - C_1\tilde{x}_1) - \Sigma_1\Sigma_1\Phi_1' [ \tilde{Q}_1\tilde{\xi}_1 + \gamma^2\xi_1^2\Sigma_1\Phi_1'\gamma_1 (y_1 - C_1\tilde{x}_1) ]$$

$$\hat{x}_1 = -\Phi_1\Sigma_1P_{1,r}(\tilde{\theta}_1) + A_1\tilde{x}_1 - (\gamma^{-2}\Pi_1 + \Phi_1\Sigma_1\Phi_1')C_1'(y_d - C_1\tilde{x}_1) + B_1y_2 + \tilde{D}_1\tilde{w}_1$$

This completes the estimation design of $S_1$. 

\[ \tilde{g}_1 = -\tilde{\Sigma}_1 \left[ (P_{1,r}(\tilde{\theta}_1))' 0_{1 \times n_1} \right] + A_1\tilde{x}_1 + \tilde{\Sigma}_1C_1'(y_d - C_1\tilde{x}_1) - \tilde{\Sigma}_1Q_1(\Phi_1,s_1,\Sigma)\tilde{\xi}_1 + B_1y_2 + \tilde{\Sigma}_1C_1'(\gamma^2\tilde{\xi}_1 + L_1)(y_1 - C_1\tilde{x}_1) + \tilde{D}_1\tilde{w}_1; \quad \tilde{\xi}_1(0) = \left[ \partial_{\tilde{\theta}_1}^{\tilde{\nu}_1} \tilde{\nu}_1 \right]' \quad (15) \]
Next, we will derive the estimator for subsystem $S_2$. In this step, the measurements waveform $\omega_m$ is assumed to be known. Since the control input, $u$, is a causal function of $\omega_m$, then it is known. Again, we will apply the cost-to-come function methodology to derive the estimator. We briefly summarize the estimation design for $S_2$ as follows.

Set function $l_2$ in definition to be $|\xi_2 - \xi_2|^2 + 2(\xi_2 - \xi_2)'l_{2,2} + l_2$, where $\xi_2 = [\xi_2']'$ is the worst-case estimate for the expanded state $\xi_2$, $\xi_2$ is the estimate of $\xi_2$. $Q_2$ is a matrix-valued weighting function, $l_{2,2}$ and $\tilde{l}_2$ are two design functions to be introduced later, the cost function of subsystem $S_2$ is then of a linear quadratic structure. By cost-to-come function analysis, we obtain the dynamics of worst-case covariance matrix $\Sigma_2$, and state estimator $\hat{\xi}_2$. We partition $\Sigma_2$ as $\Sigma_2 = \begin{bmatrix} \Sigma_2 & \Sigma_{2,12} \\ \Sigma_{2,21} & \Sigma_{2,22} \end{bmatrix}$ and introduce $\Phi_2 := \Sigma_{2,21}\Sigma_{2,21}^{-1}$ and $\Pi_2 := \gamma^2(\Sigma_{2,22} - \Sigma_{2,21}\Sigma_{2,21}^{-1}\Sigma_{2,12})$, then the weighting matrix $\Sigma_2$ is positive definite if and only if $\Sigma_2$ and $\Pi_2$ are positive definite.

To guarantee the boundedness of $\Sigma_2$, we choose weighing matrix $\tilde{Q}_2$ as follows,

$$Q_2 = \begin{bmatrix} -\Phi_2' & 0 \\ \Pi_{n_2} & 0 \end{bmatrix} + \begin{bmatrix} -\Phi_2' & 0 \\ \Pi_{n_2} & 0 \end{bmatrix}' + \frac{e_22\Phi_2'C_2\gamma^2\xi_2^2C_2\Phi_2}{\tilde{Q}_2,n_2\sigma_{n_2}} 0_{n_2\times n_2} = \gamma^2 \bar{\Pi}_2(\gamma^2)$$

where $\Delta_2(t) = \gamma^2 - 2\Delta_2, \Pi_2(t) + \Delta_2, 2$, with $\Delta_2, 2 \geq 0$ being a constant and $\Delta_2, 2$ being an $n_2 \times n_2$-dimensional positive-definite matrix, and $e_2$ is a scalar function defined by $e_2 = K_2c\text{Tr}(\Sigma_2^{-1})$ or $e_2 = 1$. $K_2c \geq \gamma^2\text{Tr}(Q_2,0)$ is a design constant, $Q_2, 0$ is an $\sigma_{n_2} \times \sigma_{n_2}$-dimensional positive-definite matrix. Then the dynamics of $\Sigma_2, \Phi_2, \Pi_2$ are given as follows,

$$\dot{\Sigma}_2 = (e_2 - 1)\Sigma_2\Phi_2'C_2\gamma^2\xi_2^2C_2\Phi_2\Sigma_2; \quad \Sigma_2(0) = \frac{\gamma^2 - Q_2, 0}{Q_2, 0}$$

$$\dot{\Pi}_2 = (A_2 - \xi_2^2L_2C_2 + \beta_{2, l}/2l_{n_2})\Pi_2 + \Pi_2 (A_2 - \xi_2^2L_2C_2 + \beta_{2, l}/2l_{n_2})' - \Pi_2 C_2^2\xi_2^2C_2\Pi_2 + D_2D_2' \quad \Pi_2(0) = \Pi_2, 0$$

$$\dot{\Phi}_2 = A_2, \Pi_2 \gamma_2 \dot{A}_{2,211} + u \dot{A}_{2,212} + \sum_{j=1}^{\tilde{q}_2} \dot{A}_{2,213} |w_{2, j} + \gamma_2 \dot{A}_{2,214}; \quad \Phi_2(0) = \Phi_2, 0$$

where $A_2, \Pi_2 := A_2 - \xi_2^2L_2C_2 - \Pi_2 C_2^2\xi_2^2$ is Hurwitz. By Lemma [12], we have the covariance matrix $\Sigma_2$ upper and lower bounded as follows, $K_2c^{-1}I_{n_2} \leq \Sigma_2(t) \leq \Sigma_2(0) = \gamma^2 Q_2, 0', \gamma^2\text{Tr}(Q_2, 0) \leq \text{Tr}(\Sigma_2(t))^{-1} \leq K_2c, \text{whenever it exists on } [0, t_f] \text{ and } \Phi_2 \text{ is continuous on } [0, t_f]$.

To avoid the calculation of $\Sigma_2^{-1}$ online, we define $s_2, \Sigma = \text{Tr}(\Sigma_2^{-1})$.

To guarantee the estimates parameter to be bounded and the estimate of high frequency gain to be bounded away from zero without persistently exciting signals, we introduce the following soft projection design on the parameter estimate.

Define $\rho_2 := \inf\{P_2(\tilde{\theta}_2) \mid \tilde{\theta}_2 \in \mathbb{R}^c, b_2, p_0 + A_2, 2, 120 \tilde{\theta}_2 = 0\}$, we have $1 < \rho_2 \leq \infty$. Fix any $\rho_2, \in (1, \rho_2)$, we define the open set $\Theta_{2, o} := \{\tilde{\theta}_2 \mid P_2(\tilde{\theta}) < \rho_2, \sigma\}$. Our control design will guarantee that the estimate $\hat{\theta}_2$ lies in $\Theta_{2, o}$, which immediately implies $|b_2, p_0 + A_2, 2, 120 \hat{\theta}_2| > \epsilon_{2, 0} > 0$, for some $\epsilon_{2, 0} > 0$. Moreover, the convexity of $P_2$ implies the following inequality:

$$\frac{\partial P_2}{\partial \tilde{\theta}_2}(\tilde{\theta}_2) < 0 \forall \tilde{\theta}_2 \in \mathbb{R}^c \setminus \Theta_2.$$
we introduce \( l_{2,2} = \left[-(P_{2,r}( \hat{\theta}_2))' \right]_{0 \times n_2} \), where
\[
P_{2,r}( \hat{\theta}_2) := \begin{cases} \exp\left(\frac{\gamma_2( \hat{\theta}_2)}{\rho_2 - \rho_2( \hat{\theta}_2)}\right) \left(\frac{\partial P_2}{\partial \hat{\theta}_2}( \hat{\theta}_2)\right)' & \forall \theta_2 \in \Theta_2, \theta_2) \\end{cases} \quad \forall \theta_2 \in \Theta_2
\]
\[
= p_{2,r}( \hat{\theta}_2) \left(\frac{\partial P_2}{\partial \hat{\theta}_2}( \hat{\theta}_2)\right)'
\]
and the dynamics of \( \hat{\xi}_2 \) is then given as follows,
\[
\dot{\hat{\xi}}_2 = -\Sigma_2 \left[ (P_{2,r}( \hat{\theta}_2))' \right]_{0 \times n_2} + \tilde{A}_2 \hat{\xi}_2 + \tilde{B}_2 u + \xi^2_2 (\gamma^2 \Sigma_2 C_1' + \tilde{A}_2 y_2 + L_2) (y_2 - \hat{C}_2 \hat{\xi}_2)
\]
+ \tilde{D}_2 \tilde{w}_2 - \Sigma_2 \hat{Q}_2 (\hat{\xi}_2 - \tilde{\xi}_2)
where \( \xi^2_2 = [\hat{\theta}_2, \hat{\xi}_2]' \) with initial condition \( [\hat{\theta}_{2,0}, \hat{\xi}_{2,0}]' \), and \( L_2 \) is defined as \( L_2 = [0_{1 \times n_2} L_2]' \). This completes the estimation design of \( S_2 \).

Associated with the above identifier and estimator of subsystem \( S_i, i = 1, 2 \), we introduce the value function \( W_i : \mathbb{R}^{n_1 + \sigma_1} \times \mathbb{R}^{n_2 + \sigma_2} \times S_{+(n_1 + \sigma_1)} \rightarrow \mathbb{R} \) and the time derivative are as follows
\[
W_i(\xi_i, \hat{\xi}_i, \Sigma_i) = |\theta_i - \hat{\theta}_i|_{\Sigma_i^{-1}}^2 + \gamma^2|x_i - \hat{x}_i| - \Phi_1 (|\theta_i - \hat{\theta}_i|)^2_{\Pi_i^{-1}}
\]
\[
\dot{\tilde{W}}_1 = -|x_{1,1} - y_{1,1}|^2 - \gamma^4|x_1 - \hat{x}_1| - \Phi_1 (|\theta_1 - \hat{\theta}_1|)^2_{\Pi_1^{-1}} - |C_1 \hat{\xi}_1 - y_{1,1}|^2
\]
- \( \epsilon_1 (\gamma^2 \xi^2_1 - 1) - \theta_1 [\tilde{\Phi}_1 C_1 \Phi_1 - 2 \xi_1 \gamma^2 y_{1,1} - \xi_1 C_1 \hat{\xi}_1] - \gamma^2 |w_{1,1} - 2 |w_1 - w_{1,1}|^2
\]
+ \( 2 (\theta_1 - \hat{\theta}_1)' P_{1,r}( \hat{\theta}_1) + |\xi_1|^2_{Q_1} \)
\[
\dot{\tilde{W}}_2 = -\gamma^2 |x_{2,1} - \hat{x}_2| - \Phi_2 (|\theta_2 - \hat{\theta}_2|)^2_{\Pi_2^{-1}} - |C_2 \hat{\xi}_2| - \gamma^2 |w_{2,1} - 2 |w_2 - w_{2,1}|^2
\]
- \( \gamma^2 |y_{2,1} - C_2 \hat{\xi}_2| - \gamma^2 |w_{2,1} - 2 |w_2 - w_{2,1}|^2
\)
\[
= 2 \epsilon_2 (|\tilde{\Phi}_2 C_2 \Phi_2 - \theta_2 |^2_{\Pi_2^{-1}} - |\tilde{\Phi}_2 C_2 \Phi_2 + |\xi_2|^2_{Q_2} \)
\]
\[
\dot{\tilde{W}}_i = -|x_{i,1} - y_{i,1}|^2 - \gamma^2|x_i - \hat{x}_i| - \Phi_i (|\theta_i - \hat{\theta}_i|)^2_{\Pi_i^{-1}} - |C_i \hat{\xi}_i| - \gamma^2 |w_{i,1} - 2 |w_i - w_{i,1}|^2
\]
\[
- \gamma^2 |y_{i,1} - C_i \hat{\xi}_i| - \gamma^2 |w_{i,1} - 2 |w_i - w_{i,1}|^2
\]
\[
= 2 \epsilon_i (|\tilde{\Phi}_i C_i \Phi_i - \theta_i |^2_{\Pi_i^{-1}} - \gamma^2 \xi_1 \gamma^2 y_{1,1} - \xi_1 C_1 \hat{\xi}_1 - 2 \epsilon_i \gamma^2 y_{1,1} - \xi_1 C_1 \hat{\xi}_1 - 2 \epsilon_i \gamma^2 w_{1,1} - w_{1,1}^2 - 2 \epsilon_i \gamma^2 w_{1,1} - w_{1,1}^2
\]
\[
= 2 \epsilon_i E_i (y_{1,1} - C_1 \hat{\xi}_1) + \gamma^2 (I_{\xi_1} - \gamma^2 \xi_1 E_i E_i) \tilde{D}_i \Pi_i^{-1} (\xi_i - \hat{\xi}_i); \quad i = 1, 2
\]
We note that (18) holds when \( \Sigma_i > 0 \) and \( \theta_i \in \Theta_{i,0} \), and the last term in \( W_i \) is nonpositive, zero on the set \( \Theta_{i,0} \) and approaches \(-\infty \) as \( \hat{\theta}_i \) approaches the boundary of the set \( \Theta_{i,0} \), which guarantees the boundedness of \( \hat{\theta}_i, i = 1, 2 \).

Then (5) can be equivalently written as, \( i = 1, 2 \):
\[
\begin{align*}
J_{1,\gamma t} &= \int_{0}^{t_f} \left( C_1 \hat{\xi}_1 - y_{1,1}^2 + \xi_1 \epsilon_{1,1}^2_{Q_1} + \Pi_1 - \gamma^2 \xi_1^2 |y_{1,1} - C_1 \hat{\xi}_1|^2 - \gamma^2 |w_{1,1}|^2 - \gamma^2 |w_{1,1}|^2 \right) d\tau
\end{align*}
\]
\[
J_{2,\gamma t} = \int_{0}^{t_f} \left( \xi_2 \epsilon_{2,1}^2_{Q_2} + \Pi_2 - \gamma^2 \xi_2^2 |y_{2,1} - C_2 \hat{\xi}_2|^2 - \gamma^2 |w_{2,1}|^2 - \gamma^2 |w_{2,1}|^2 \right) d\tau
\]
This completes the identification design step.
4.2. Control design

In this section, we describe the controller design for the uncertain system under consideration. Note that, we ignored some terms in the cost function (5) in the identification step, since they are constant when \( y_1, y_2, \tilde{w}_1, \tilde{w}_2 \) and \( \hat{y}_2 \) are given. In the control design step, we will include such terms. Then, based on the cost function (5), the controller design is to guarantee that the following supremum is less than or equal to zero for all measurement waveforms,

\[
\sup_{\hat{w}_1 \in \mathcal{W}_1, \hat{w}_2 \in \mathcal{W}_2} J_{\bar{\gamma}t_f} \leq \sup_{\omega_m \in \mathcal{W}_m} \left( \sup_{\omega_1 \in \mathcal{W}_1, \omega_2 \in \mathcal{W}_2} J_{1,\bar{\gamma}t_f} + \sup_{\omega_2 \in \mathcal{W}_2, \omega_m \in \mathcal{W}_m} J_{2,\bar{\gamma}t_f} \right) \\
\leq \sup_{\omega_m \in \mathcal{W}_m} \left\{ \int_0^{t_f} \left( |C_1 \hat{x}_1 - y_d|^2 + \sum_{i=1}^{2} \left( |\hat{c}_{i,a}|^2 + |\hat{c}_{i,b}|^2 \right) |y_i - C_i \hat{x}_i|^2 - \gamma^2 |\hat{w}_{i,a}|^2 \right) \mathrm{d}\tau \right\} \tag{21}
\]

where function \( I_1(\tau, y_1[0,\tau], Y_d[0,\tau], \hat{w}_1) \) is part of the weighting function \( l_1(\tau, \theta_1, x_1, y_1[0,\tau], Y_d[0,\tau], \hat{w}_1) \), and \( I_2(\tau, y_2[0,\tau], Y_d[0,\tau], \hat{w}_2) \) is part of the weighting function \( l_2(\tau, \theta_2, x_2, y_2[0,\tau], Y_d[0,\tau], \hat{w}_2) \) to be designed, which are constants in the identifier design step and are therefore neglected.

By equation (21), we observe that the cost function is expressed in terms of the states of the estimator we derived, whose dynamics are driven by the measurement \( y_1, y_2, \tilde{w}_1, \tilde{w}_2, \hat{y}_2 \), the reference trajectory \( y_d \), the input \( u \), and the worst-case estimate for the expanded state vector \( \hat{x}_1 \) and \( \hat{x}_2 \), which are signals we either measure or can construct. This is then a nonlinear \( H^\infty \)-optimal control problem under full information measurements. Since \( \hat{y}_2 = y_1 \) in the adaptive system under consideration, we can equivalently deal with the following transformed variables instead of considering \( y_1, y_2, \tilde{w}_1, \tilde{w}_2, \) and \( \hat{y}_2 \) as the maximizing variable,

\[
v = \begin{bmatrix} \hat{c}_1 (y_1 - C_1 \hat{x}_1) \\ \hat{w}_{1,a} \\ \hat{w}_{1,b} \\ \hat{c}_2 (y_2 - C_2 \hat{x}_2) \\ \hat{w}_{2,a} \\ \hat{w}_{2,b} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}
\]

where \( v_i = \begin{bmatrix} \hat{c}_i (y_i - C_i \hat{x}_i) & \hat{w}_{i,a} & \hat{w}_{i,b} \end{bmatrix}' \), \( i = 1, 2 \).

By the special structure of the system, we define \( v_{i,a} = \begin{bmatrix} \hat{c}_i (y_i - C_i \hat{x}_i) & \hat{w}_{i,a} \end{bmatrix}' \), \( i = 1, 2 \), \( v_a = \begin{bmatrix} v_{1,a} & v_{2,a} \end{bmatrix}' \), and we will attenuate disturbance \( v_a \), and cancel the disturbance \( \hat{w}_{1,b} \) and \( \hat{w}_{2,b} \). In view of \( y_2 = \hat{x}_2 - 1 + \hat{q}_{2,a} \), \( 2 \hat{q}_{2,a} + \hat{v}_{2,a} \), we will treat \( \hat{x}_2 \) as the virtual control input of subsystem \( S_1 \), where \( \hat{q}_{2,a} = \hat{q}_{1,a} + \hat{q}_{2,a} \).

For \( i = 1, 2 \), we introduce the matrix \( M_{i,f} := \begin{bmatrix} A_{i,f}^{-1} p_{i,n_i} \cdots A_{i,f} p_{i,n_i} \end{bmatrix} \), where \( p_{i,n_i} \) is a \( n_i \)-dimensional vector such that the pair \( (A_{i,f}, p_{i,n_i}) \) is controllable. We note that \( \hat{y}_2 = y_1 \), then
the following $3n_1 + 4n_2 + \dot{q}_1 + \dot{q}_2$-dimensional prefiltering system for $y_1, y_2, u, \bar{w}_1, \bar{w}_2, \ddot{y}_2$ generates the $\Phi_1$ and $\Phi_2$ online:

$$\begin{align*}
\dot{\eta}_1 &= A_{1,f}\eta_1 + p_{1,m_1}y_1; \\
\dot{\eta}_{\bar{w}_{1,i}} &= A_{1,f}\eta_{\bar{w}_{1,i}} + p_{1,m_1}\bar{w}_{1,i}; \quad \eta_{\bar{w}_{1,i}}(0) = \eta_{\bar{w}_{1,i}}0, i = 1, \ldots, q_1 \\
\dot{\lambda}_1 &= A_{1,f}\lambda_1 + p_{1,m_1}y_2; \quad \lambda_1(0) = \lambda_{1,0} \\
\Phi_1 &= \left[ A_{1,f}^{-1}\eta_1 \cdots A_{1,f}\eta_1 \right] M_{1,f}^{-1}A_{1,211} + \left[ A_{1,f}^{-1}\lambda_1 \cdots A_{1,f}\lambda_1 \right] M_{1,f}^{-1}A_{1,212} \\
&\quad + \sum_{j=1}^{q_2} \left[ A_{1,f}^{-1}\eta_{\bar{w}_{1,i}} \cdots A_{1,f}\eta_{\bar{w}_{1,i}} \right] M_{1,f}^{-1}A_{1,213j} \\
\dot{\eta}_2 &= A_{2,f}\eta_2 + p_{2,m_2}y_2; \\
\dot{\eta}_{\bar{w}_{2,j}} &= A_{2,f}\eta_{\bar{w}_{2,j}} + p_{2,m_2}\bar{w}_{2,j}; \quad \eta_{\bar{w}_{2,j}}(0) = \eta_{\bar{w}_{2,j}}0, j = 1, \ldots, q_2 \\
\dot{\lambda}_2 &= A_{2,f}\lambda_2 + p_{2,m_2}u; \quad \lambda_2(0) = \lambda_{1,0} \\
\dot{\eta}_{2,y} &= A_{2,f}\eta_{2,y} + p_{2,m_2}\bar{y}_2; \eta_{2,y}(0) = \eta_{2,y0} \\
\Phi_2 &= \left[ A_{2,f}^{-1}\eta_1 \cdots A_{2,f}\eta_2 \right] M_{2,f}^{-1}A_{2,211} + \left[ A_{2,f}^{-1}\lambda_2 \cdots A_{2,f}\lambda_2 \right] M_{2,f}^{-1}A_{2,212} \\
&\quad + \sum_{j=1}^{q_2} \left[ A_{2,f}^{-1}\eta_{\bar{w}_{2,j}} \cdots A_{2,f}\eta_{\bar{w}_{2,j}} \right] M_{2,f}^{-1}A_{2,213j} \\
&\quad + \left[ A_{2,f}^{-1}\eta_{2,y} \cdots A_{2,f}\eta_{2,y} \right] M_{2,f}^{-1}A_{2,214}
\end{align*}$$

The variables to be designed at this stage include $\dot{x}_{2,1}$, $u$, $\xi_{1,c}$, and $\xi_{2,c}$. Note that the structures of $A_1$ and $A_2$ in the dynamics is in strict-feedback form, we will use the backstepping methodology, see [9], to design the control input $u$, which will guarantee the global boundedness of the closed-loop system states and the asymptotic convergence of the tracking error. Since there are the nonnegative definite weighting on $\xi_{1,c}$ and $\xi_{2,c}$ in the cost function (21), we can not use integrator backstepping to design feedback law for $\xi_{1,c}$ and $\xi_{2,c}$. Hence, we set $\xi_{1,c} = \xi_{2,c} = 0$ in the backstepping procedure. After the completion of the backstepping procedure, we will then optimize the choice of $\xi_{1,c}$ and $\xi_{2,c}$ based on the value function obtained. Note that $\Sigma_1, \Pi_1, \bar{\eta}_1, \Sigma_2, \Pi_2, \bar{\xi}_2, \bar{\xi}_2$, and $\bar{\eta}_2$ are always bounded by the design in Section 4.1. Since $\Phi_1$ is driven by control $y_2$, and $\Phi_2$ is explicitly driven by $u$, they can not be stabilized in conjunction with $\dot{x}_1$ and $\dot{x}_2$ in the backstepping design. We will assume they are bounded and prove later they are indeed so under the derived control law.

We carry out the backstepping design for subsystem $\mathbf{S}_1$ first, and treat $\dot{x}_{2,1}$ as the virtual control input of subsystem $\mathbf{S}_1$ in view of $y_2 = \zeta_2^{-1}v^f + \dot{q}_2\dot{\bar{y}} + \bar{y}_2 + \bar{x}_{2,1}$. To stabilize $\eta_1$, we introduce variable $\eta_{1,dr}$ which satisfies $\eta_{1,dr} = A_{1,f}\eta_{1,dr} + p_{1,m_1}y_d$ with initial condition $\eta_{1,dr}(0) = \eta_{1,dr0}$, and is the reference trajectory to track. Choosing value function $V_{1,0} := |\eta_1 - \eta_{1,dr}|^2$, where $Z_1$ is the solution to an algebraic Riccati equation. Treating $\dot{x}_{1,1}$ as the virtual control input, we complete the step 0 with the virtual control law $a_{1,0} = y_d$, which will guarantee the $\dot{V}_{1,0} \leq 0$ under $\dot{x}_{1,1} = a_{1,0}$. At step 1, we introduce $z_{1,1} := \dot{x}_{1,1} - y_d$, and choose value function $V_{1,1} = V_{1,0} + \frac{1}{2}z_{1,1}^2$. Treating $\dot{x}_{1,2}$ as the virtual control input, we end the
step 1 with the virtual control law $a_{1,1}$, which guarantees $V_{1,1} \leq 0$ under $\dot{x}_{1,2} = a_{1,1}$. Define the variable $z_{1,2} = \hat{x}_{1,2} - a_{1,1}$ for step 2. Repeating the backstepping procedure until step $r_1$, the virtual control input $\hat{x}_{2,1}$ will appear in the dynamic of $\dot{x}_{1,r_1}$. Using the similar procedure as previous steps, we can derive the robust adaptive controller $a_{1,r_1}$ such that $V_{1,r_1} \leq 0$ under $\dot{x}_{2,1} = a_{1,r_1}$. This completes the control design for subsystem $S_1$.

To stabilize $\eta_2$, we introduce variable $\eta_{2,d}$ as below,

$$\dot{\eta}_{2,d} = A_2 f \eta_{2,d} + p_{2,n_2} a_{1,r_1} + p_{2,n_2} \epsilon_{q,2} + 2 u_{1,r_1}; \eta_{2,d}(0) = \eta_{2,d,0}$$

and is the reference trajectory for $\eta_2$ to track, where $u_{1,r_1}$ is a function obtained after step $r_1$. Choosing value function $V_{2_0} := |\eta_2 - \eta_{2,d}|^2 + V_{1,r_1}$, where $Z_2$ is the solution to an algebraic Riccati equation. We complete the step $r_1 + 1$ with the virtual control law $a_2,0 = a_{1,r_1}$, which will guarantee the $V_{2_0} \leq 0$ under $\dot{x}_{2,1} = a_{2,0}$. Repeating the backstepping procedure until step $r_1 + r_2 + 1$, the virtual control input $u$ will appear in the dynamic of $\dot{x}_{2,r_2}$. Introduce $V_{2_2} = \sum_{j=1}^2 (|\eta_j|^2 + \sum_{k=1}^2 \frac{1}{2} z_{k,j}^2)$, we then can derive the robust adaptive controller $\mu$ such that $V_{2,r_2} \leq 0$ under $u = \mu$. Later, we will show that the control law $\mu$ will guarantee the boundedness of the closed-loop system states and the asymptotic convergence of tracking error.

For the closed-loop adaptive nonlinear system, we have the following value function, $U = W_1 + W_2 + V_2$, and its time derivative is given by

$$\dot{U} = -|x_{1,1} - y_d|^2 - \sum_{j=1}^2 \left( \gamma^2 |x_j - \hat{x}_j - \Phi_j(\theta_j - \hat{\theta}_j)|^2 + \epsilon_j (\gamma^2 \xi_j^2 - 1) |\theta_j - \hat{\theta}_j|^2 \Phi_j |\Phi_j| \right)$$

$$- 2 (\theta_j - \hat{\theta}_j)^T P_{j,r}(\hat{\theta}_j) + |\eta_j|^2 + \sum_{k=1}^{r_1} \beta_{j,k} z_{k,j}^2 - \gamma^2 |w_j|^2 + \gamma^2 |w_j - w_{j,OPT}|^2 - \gamma^2 |\tilde{w}_{j,a}|^2$$

$$+ \gamma^2 |\tilde{w}_{j,a} - \tilde{w}_{j,OPT}|^2 - 2 |\tilde{w}_{j,a} - \tilde{w}_{j,OPT}|^2 - \frac{1}{4} |\xi_{1,(r_1+r_2)}|_{Q_1}^2 - \frac{1}{4} |\xi_{2,r_2}|_{Q_2}^2$$

where $\xi_{1,(r_1+r_2)}$ and $\xi_{2,r_2}$ are functions obtained after step $r_1 + r_2 + 1$, $w_{1,OPT}$ and $w_{2,OPT}$ are the worst case disturbance with respect to the value function $U$, which are given by

$$w_{1,OPT} = \zeta_{1} E_{1}^{'} e_{1,1} v_{1,r_1} + \gamma^{-2} (I_{q_1} - \zeta_{1}^2 E_{1} C_1) D_{1}^{\top} \Sigma_{1}^{-1} (\zeta_{1} - \hat{\zeta}_{1}) + \zeta_{2} E_{2} C_1 (\hat{x}_1 - x_1)$$

$$w_{2,OPT} = \zeta_{1} E_{1}^{'} e_{2,2} v_{2,r_2} + \gamma^{-2} (I_{q_1} - \zeta_{2}^2 E_{2} C_2) D_{2}^{\top} \Sigma_{2}^{-1} (\zeta_{2} - \hat{\zeta}_{2}) + \zeta_{2} E_{2} C_2 (\hat{x}_2 - x_2)$$

$$\tilde{w}_{1,OPT} = \left[ 0 \times (2 + \hat{q}_{1,a+q_{2,a}}) e_{1,1} (2 + \hat{q}_{1,a+q_{2,a}},1) \ldots e_{1,1} (2 + \hat{q}_{1,a+q_{2,a}},1) \right] v_{1,r_1}$$

$$\tilde{w}_{2,OPT} = \left[ 0 \times (2 + \hat{q}_{1,a}) e_{2,2} (2 + \hat{q}_{1,a+q_{2,a}},1) \ldots e_{2,2} (2 + \hat{q}_{1,a+q_{2,a}},1) \right] v_{2,r_2}$$

where $v_{1,r_1}$ and $v_{2,r_2}$ are functions obtained after backstepping design.
Then the optimal choice for the variable $\xi_{1,c}$ and $\hat{\xi}_{i}, i = 1, 2$, are:

$$\xi_{1,c*} = \frac{1}{2}\xi_{1,r_{1}+r_{2}} \iff \hat{\xi}_{1} = \frac{1}{2}\xi_{1,r_{1}+r_{2}};$$

$$\xi_{2,c*} = -\frac{1}{2}\xi_{2,r_{2}} \iff \hat{\xi}_{2} = \frac{1}{2}\xi_{2,r_{2}}$$

which yields that the closed-loop system is dissipative with storage function $U$ and supply rate with optimal choice for $\hat{\xi}_{i}, i = 1, 2$:

$$-|x_{1,1} - y_{d}|^{2} + \gamma^{2}|w_{1}|^{2} + \gamma^{2}|w_{2}|^{2} + \gamma^{2}|\tilde{w}_{1,\alpha}|^{2} + \gamma^{2}|\tilde{w}_{2,\alpha}|^{2}$$

This completes the adaptive controller design step. We will discuss the robustness and tracking properties of the proposed adaptive control laws.

5. Main result

In this Section, we present the main result by stating two theorems.

For the adaptive control law, with the optimal choice of $\xi_{i,c*}$, the closed-loop system dynamics are:

$$\dot{X} = F(X, y_{d}^{(r_{1}+r_{2})}) + G(X) [w_{1}' w_{2}'] + G_{\tilde{w}}(X) [\tilde{w}_{1}' \tilde{w}_{2}']; X(0) = X_{0}$$

where $F$, $G$ and $G_{M}$ are smooth mapping of $\mathcal{D} \times \mathbb{R}$, $\mathcal{D}$ and $\mathcal{D}$, respectively; and the initial condition $X_{0} \in \mathcal{D}_{0} := \{X_{0} \in \mathcal{D} | \theta_{i} \in \Theta_{i}, \bar{\theta}_{i,0} \in \bar{\Theta}_{i}, \Sigma_{i}(0) = \gamma^{-2}Q_{i,0}^{-1} > 0, Tr((\Sigma_{i}(0))^{-1}) \leq K_{i,c_{i},\Sigma}(0) = \gamma^{2}Tr(Q_{i,0}); i = 1, 2\}$. And the value function $U$ satisfies an Hamilton-Jacobi-Isaacs equation, $\forall X \in \mathcal{D}, \forall y_{d}^{(r_{1}+r_{2})} \in \mathbb{R}$.

$$\frac{\partial U}{\partial X} F(X, y_{d}^{(r_{1}+r_{2})}) + \frac{1}{4\gamma^{2}} \frac{\partial^{2} U}{\partial X^{2}} (X) \left[ G(X) G_{\tilde{w}}(X) \right] \left[ G(X)' G_{\tilde{w}}(X)' \right]' \left( \frac{\partial U}{\partial X} (X) \right)' + Q(X, y_{d}^{(r_{1}+r_{2})}) = 0;$$

where $Q : \mathcal{D} \times \mathbb{R} \rightarrow \mathbb{R}$ is smooth and given by

$$Q(X, y_{d}^{(r_{1}+r_{2})}) = |x_{1,1} - y_{d}|^{2} + \sum_{j=1}^{2} \left( \gamma^{4}|x_{j} - \tilde{x}_{j} - \Phi_{j}(\theta_{j} - \bar{\theta})|^{2}_{\Pi_{j}^{-1} \Delta_{i} \Pi_{j}^{-1}} + \epsilon_{j} (\gamma^{2}q_{j}^{2} - 1)|\theta_{j} - \bar{\theta}_{j}|^{2}_{\Phi_{j}C_{j}C_{j}\Phi_{j}} - 2(\theta_{j} - \bar{\theta}_{j})'P_{j,r}(\bar{\theta}_{j}) + |\bar{\theta}_{j}|^{2} + \sum_{k=1}^{r_{j}} \beta_{j,k}^{2} \tilde{x}_{j,k}^{2} \right) + \frac{1}{4} |\xi_{1,(r_{1}+r_{2})}|^{2}_{\Omega_{1}} + \frac{1}{4} |\xi_{2,r_{2}}|^{2}_{\Omega_{2}} + \epsilon_{2} |\bar{\theta}_{2} - \tilde{\theta}_{2}| \Phi_{2}C_{2}C_{2}\Phi_{2}$$

The closed-loop adaptive system possesses a strong stability property, which will be stated precisely in the following theorem.

**Theorem 1.** Consider the robust adaptive control problem formulated and assumptions in Section 3. The robust adaptive controller $\mu$ with the optimal choice of $\xi_{i,c*}$ achieves the following strong robustness properties for the closed-loop system.
1. Given \( c_w \geq 0 \), and \( c_d \geq 0 \), there exists a constant \( c_c \geq 0 \) and compact sets \( \Theta_{1,c} \subset \Theta_{1,0} \), and \( \Theta_{2,c} \subset \Theta_{2,0} \) such that for any uncertainty \( (x_{1,0}, \theta_1, \hat{w}_{1,[0,\infty)}, \hat{w}_{1,[0,\infty)}, Y_{d0}, Y_{d_d,0,\infty}) \in V_1 \) and \((x_{2,0}, \theta_2, \hat{w}_{2,[0,\infty)}, \hat{w}_{2,[0,\infty),}) \in V_2 \) with \( |x_{1,0}| \leq c_w; |x_{2,0}| \leq c_w; |\hat{w}_1(t)| \leq c_w; |\hat{w}_2(t)| \leq c_w; |Y_d(t)| \leq c_d; \forall t \in [0,\infty) \) all closed-loop state variables \( x_{1,1}, \hat{\theta}_1, \hat{\Sigma}_1, \eta_1, \eta_{1,d}, \Phi_{1,u,\theta_1, x_2, \hat{\theta}_2, \hat{\Sigma}_2, s_2, \Sigma_2, \eta_2, \eta_{2,d}, \Phi_{2,u} \) are bounded as follows, \( \forall t \in [0,\infty) \),

\[ |x_i(t)| \leq c_{ci}; |\hat{x}_i(t)| \leq c_{c_i}; \hat{\theta}_i(t) \leq c_{c_i}; |\eta_i(t)| \leq c_{c_i}; |\eta_{i,d}(t)| \leq c_{c_i}; \]

\[ |\eta_{1,d}(t)| \leq c_{c_i}; |\Phi_{1,u}(t)| \leq c_{c_i}; K_{i,c}^{-1} \leq |\Sigma_i(t)| \leq \gamma^{-2}Q_i^{-1}; \gamma^2\text{Tr}(Q_i,0) \leq s_i, \Sigma_i(t) \subseteq K_{i,c}; \quad i = 1, 2 \]

The inputs are also bounded \( |u(t)| \leq c_u, \forall t \in [0,\infty), \) for some constant \( c_u \geq 0 \). Furthermore, there exists constant \( c_{\lambda} \geq 0 \) such that \( |\lambda_{i,0}(t)| \leq c_{\lambda}, |\lambda_i(t)| \leq c_{\lambda}, \) \( i = 1, 2, \) and \( |\eta_{2,d}(t)| \leq c_{\lambda}, \forall t \geq 0. \)

2. For any uncertainty \( (x_{1,0}, \theta_1, \hat{w}_{1,[0,\infty)}, \hat{w}_{1,[0,\infty)}, Y_{d0}, Y_{d_d,0,\infty}) \in V_1 \), and \((x_{2,0}, \theta_2, \hat{w}_{2,[0,\infty)}, \hat{w}_{2,[0,\infty),}) \in V_2 \) the controller \( \mu \in M \) achieves disturbance attenuation level \( \gamma \) with respect to \( w_1 \) and \( w_2 \) arbitrary disturbance attenuation level \( \gamma^2 \) with respect to \( \hat{w}_1 \) and \( \hat{w}_2 \), and disturbance attenuation level zero with respect to \( \hat{w}_{1,b} \) and \( \hat{w}_{2,b} \).

3. For any uncertainty \( (x_{1,0}, \theta_1, \hat{w}_{1,[0,\infty)}, \hat{w}_{1,[0,\infty)}, Y_{d0}, Y_{d_d,0,\infty}) \in V_1 \), and \((x_{2,0}, \theta_2, \hat{w}_{2,[0,\infty)}, \hat{w}_{2,[0,\infty),}) \in V_2 \) \( w_{2,0,\infty}) \in V_2 \) with \( \hat{w}_{1,[0,\infty)} \in L_2 \cap L_{\infty}, \hat{w}_{2,[0,\infty)} \in L_2 \cap L_{\infty}, \hat{w}_{1,0,\infty} \in L_2 \cap L_{\infty}, \hat{w}_{2,0,\infty} \in L_2 \cap L_{\infty}, \) and \( Y_{d_d,0,\infty} \in L_{\infty}, \) the noiseless output of the system, \( x_{1,1,} \) asymptotically tracks the reference trajectory, \( y_d, \)

\[ \lim_{t \to \infty} (x_{1,1}(t) - y_d(t)) = 0 \]

4. The ultimate lower bound on the achievable performance level is only relevant to the Subsystem \( S_1 \), i.e., \( \gamma \geq \zeta_1^{-1} \) or \( \gamma > \zeta_1^{-1}. \)

**Proof** For the first statement, fix \( c_w \geq 0 \), and \( c_d \geq 0 \) consider any uncertainty \( (x_{1,0}, x_{2,0}, \theta_1, \theta_2, \hat{w}_{1,[0,\infty)}, \hat{w}_{2,[0,\infty)}, \hat{w}_{2,[0,\infty),}) \) that satisfies:

\[ |x_{1,0}| \leq c_w; |x_{2,0}| \leq c_w; |\hat{w}_1(t)| \leq c_w; |\hat{w}_2(t)| \leq c_w; |\hat{w}_1(t)| \leq c_w; |\hat{w}_2(t)| \leq c_w; |Y_d(t)| \leq c_{d,d}, \forall t \in [0,\infty) \]

We define \( [0, T_f) \) to be the maximal length interval on which the closed system (22) has a solution that lies in \( D \). Note that we have \( \Sigma_1, \Sigma_2, s_{1,\Sigma} \) and \( s_{2,\Sigma} \) are uniformly upper bounded and uniformly bounded away from 0 as desired by Section 4.

Introduce the vector of variables

\[ X_e := [\hat{\theta}_1^T \hat{\theta}_2^T (\hat{x}_1 - \Phi_1 \hat{\theta}_1)^T (\hat{x}_2 - \Phi_2 \hat{\theta}_2)^T \hat{\eta}_1^T \hat{\eta}_2^T z_{1,1} \cdots z_{1,r_1} z_{2,1} \cdots z_{2,r_2}]^T \]

and two nonnegative and continuous functions defined on \( R^{2n+2n_2+c_1+c_2+r_1+r_2} \)

\[ U_M(X_e) := \sum_{i=1}^{2} K_{i,c} |\hat{\theta}_i|^2 + \sum_{i=1}^{2} \gamma^2 |\hat{x}_i - \Phi_1 \hat{\theta}_1|^2_{1,1} + \sum_{i=1}^{2} |\hat{\eta}_i|^2_{Z_{i}} + \sum_{j=1}^{r_1} \gamma_{1,j} z_{1,j}^2 + \sum_{j=1}^{r_2} \gamma_{2,j} z_{2,j}^2 \]

\[ U_m(X_e) := \sum_{i=1}^{2} |\hat{\theta}_i|^2_{Q_{i,c}} + \sum_{i=1}^{2} \gamma^2 |\hat{x}_i - \Phi_1 \hat{\theta}_1|^2_{1,1} + \sum_{i=1}^{2} |\hat{\eta}_i|^2_{Z_{i}} + \sum_{j=1}^{r_1} \gamma_{1,j} z_{1,j}^2 + \sum_{j=1}^{r_2} \gamma_{2,j} z_{2,j}^2 \]
then, we have

$$U_m(X_e) \leq U(t, X_e) \leq U_M(X_e), \quad \forall (t, X_e) \in [0, T_f) \times \mathbb{R}^{2(n_1 + n_2) + r_1 + r_2 + r_2}$$

Since $U_m(X_e)$ is continuous, nonnegative definite and radially unbounded, then $\forall a \in \mathbb{R}$, the set $S_{1a} := \{ X_e \in \mathbb{R}^{2(n_1 + n_2) + r_1 + r_2 + r_2} \mid U_m(X_e) \leq a \}$ is compact or empty. Since $|\dot{w}_1(t)| \leq c_{w1}$ and $|\dot{w}_2(t)| \leq c_{w2}$, $\forall t \in [0, \infty)$, there exists a constant $c > 0$ such that we have the following inequality for the derivative of $U$:

$$\dot{U} \leq -2 \sum_{i=1}^{2} \left( \frac{\gamma^2}{2} |x_i - \dot{x}_i - \Phi_i (\theta_i - \dot{\theta}_i)|^2_{\Pi_i^{-1}} - 2 (\theta_i - \dot{\theta}_i)' P_i(r_i(\theta_i)) + \sum_{j=1}^{r_1} c_i \beta_i z_i^2 \right) + c$$

Since $-\sum_{i=1}^{2} \left( \frac{\gamma^2}{2} |x_i - \dot{x}_i - \Phi_i (\theta_i - \dot{\theta}_i)|^2_{\Pi_i^{-1}} + |\dot{\eta}_i|^2_{\Pi_i} - 2 (\theta_i - \dot{\theta}_i)' P_i(r_i(\theta_i)) + \sum_{j=1}^{r_1} c_i \beta_i z_i^2 \right)$ will tend to $-\infty$ when $X_e$ approaches the boundary of $\Theta_{1,0} \times \Theta_{2,0} \times \mathbb{R}^{2(n_1 + n_2) + r_1 + r_2}$, then there exists a compact set $\Omega_1(c_{w1}) \subset \Theta_{1,0} \times \Theta_{2,0} \times \mathbb{R}^{2(n_1 + n_2) + r_1 + r_2}$, such that $\dot{U} < 0$ for $\forall X_e \in \Theta_{1,0} \times \Theta_{2,0} \times \mathbb{R}^{2(n_1 + n_2) + r_1 + r_2} \setminus \Omega_1$.

Then we have $U(t, X_e(t)) \leq c_1$, and $X_e(t)$ is in the compact set $S_{1c_1} \subset \mathbb{R}^{2(n_1 + n_2) + r_1 + r_2 + r_2}, \forall t \in [0, T_f)$. It follows that the signal $X_e$ is uniformly bounded, namely, $\dot{\theta}_1, \dot{\theta}_2, \dot{x}_1 - \Phi \dot{\theta}_1, \dot{x}_2 - \Phi \dot{\theta}_2, \dot{\eta}_1, \dot{z}_{1,1}, \ldots, \dot{z}_{1,r_1}$ and $\dot{z}_{2,1}, \ldots, \dot{z}_{2,r_2}$ are uniformly bounded.

Based on the dynamics of $\eta_{1,d}$, we have $\eta_{1,d}$ is uniformly bounded. Since $\dot{\eta}_1 = \eta_1 - \eta_{1,d}$ is uniformly bounded, then $\eta_1$ is also uniformly bounded. Furthermore, there is a particular linear combination of the components of $\eta_1$, denoted by $\eta_{1,L}$,

$$\dot{\eta}_1 = A_{1,f} \eta_1 + p_{1,n_1} y_1$$

$$\eta_{1,L} = T_{1,L} \eta_1$$

which is strictly minimum phase and has relative degree 1 with respect to $y_1$. Then the signal $\eta_{1,L}$ has relative degree $r_1 + 1$ with respect to the input $y_2$, and is uniformly bounded. The composite system of $\eta_1$ and $\dot{x}_1$ with input $\dot{w}_1$ and $y_2$ and output $\eta_{1,L}$ may serve as a reference system in the application of bounding Lemma [12].

Note $\Phi_1 = \Phi_{1,y} + \Phi_{1,u}$ and $\Phi_{1,y}$ is uniformly bounded. To prove $\Phi_1$ is bounded, we need to prove $\Phi_{1,u}$ is uniformly bounded. Define the following equations to separate $\Phi_{1,u}$ into two part:

$$\Phi_{1,u} = \Phi_{1,u,s} + \lambda_{1,b} \tilde{A}_{1,2120}$$

$$\dot{\lambda}_{1,b} = A_{1,f} \lambda_{1,b} + e_{n_1} y_2; \quad \lambda_{1,b}(0) = 0_{n_1 \times 1}$$

$$\Phi_{1,u,s} = A_{1,f} \Phi_{1,u,s} + y_2 \left[ \begin{array}{c} 0_{n_1 \times n_1} \\ A_{1,2120} \end{array} \right] ; \quad \Phi_{1,u,s}(0) = \Phi_{1,u,0}$$

We observe that the relative degree for each element of $\Phi_{1,u,1}$ is at least $r_1 + 1$ with respect to the input $y_2$, and is the output of a stable linear system. Take $\eta_{1,L}$ and $y_2$ as output and input of
the reference system, we conclude $\Phi_{1,u_1}$ is uniformly bounded by bounding Lemma. Because the first row element of $\dot{x}_1 - \Phi_1 \dot{\theta}_1$ is:

$$\dot{x}_{1,1} - \Phi_{1,u_1} \dot{\theta}_1 - \lambda_{1,b_1} \dot{A}_{1,2120} \dot{\theta}_1 - \eta'_{1,T_1,1} \dot{\theta}_1$$

we can conclude that $\dot{x}_{1,1} - \lambda_{1,b_1} \dot{A}_{1,2120} \dot{\theta}_1$ is uniformly bounded in view of the boundedness of $\dot{x}_1 - \Phi_1 \dot{\theta}_1$, $\dot{\theta}_1$, $\Phi_{1,u_1}$, and $\eta_1$. Since $z_{1,1}$ is $\dot{x}_{1,1} - y_d$, and $z_{1,1}$, $y_d$ are both uniformly bounded, we have that $\dot{x}_{1,1}$ is also uniformly bounded.

Notice that $A_{1,f} = A_1 - \zeta_1^2 L_1 C_1 - \Pi_1 C'_1 C_1 (\zeta_1^2 - \gamma^{-2})$, we generated the signal $x_{1,1} - b_{1,0} \lambda_{1,b_1}$ by:

$$\dot{x}_1 - b_{1,0} \dot{\lambda}_{1,b} = A_{1,f} (x_1 - b_{1,0} \lambda_{1,b}) + \left[ \begin{array}{c} 0_{r_1 \times 1} \\ \bar{A}_{1,2121} \end{array} \right] y_2 + \bar{A}_{1,2111} \dot{\theta}_1 y_1 + D_1 \bar{M}_1 \dot{w}_1 + \left( \bar{\zeta}_1^2 L_1 + \Pi_1 C'_1 (\zeta_1^2 - \frac{1}{\gamma^2}) \right) (y_1 - E_1 \bar{M}_1 \dot{w}_1) + \left[ b_{1,p_1} \cdots b_{1,p_{n_1} - 1} \right] y_2$$

$$+ \sum_{j=1}^{\tilde{d}_1} A_{1,2131} \tilde{w}_1 \dot{\theta}_1 + \bar{D}_1 \dot{w}_1$$

$$x_{1,1} - b_{1,0} \lambda_{1,b_1} = C_1 (x_1 - b_{1,0} \lambda_{1,b})$$

Now we will separate the above dynamics into $y_1$ dependent and $y_2$ dependent parts by the linearity of the system, $x_{1,1} - b_{1,0} \lambda_{1,b_1} := x_{1,u_1} + x_{1,y_1}$, which are respectively given by,

$$\dot{x}_{1,u} = A_{1,f} x_{1,u} + \left[ \begin{array}{c} 0_{r_1 \times 1} \\ \bar{A}_{1,2121} \end{array} \right] y_2 + \left[ \begin{array}{c} b_{1,p_1} \cdots b_{1,p_{n_1} - 1} \end{array} \right] y_2$$

$$x_{1,u} = C_1 x_{1,u}$$

$$\dot{x}_{1,y} = A_{1,f} x_{1,y} + (\zeta_1^2 L_1 + \Pi_1 C'_1 (\zeta_1^2 - \frac{1}{\gamma^2}) \right) (y_1 - E_1 \bar{M}_1 \dot{w}_1) + \bar{A}_{1,2111} \dot{\theta}_1 y_1 + D_1 \bar{M}_1 \dot{w}_1$$

$$+ \sum_{j=1}^{\tilde{d}_1} A_{1,2131} \tilde{w}_1 \dot{\theta}_1 + \bar{D}_1 \dot{w}_1$$

$$x_{1,y_1} = C_1 x_{1,y}$$

We observe that the signal $x_{1,u_1}$ has relative degree at least $r_1 + 1$ with respect to $y_2$, take $\eta_{1,L}$ and $y_2$ as output and input of the reference system, we conclude $x_{1,u_1}$ is uniformly bounded by bounding Lemma. Since $x_{1,y_1}$ has relative degree at least 1 with respect to $y_1$, take $\eta_{1,L}$ and $y_1$ as output and input of the reference system, we conclude $x_{1,y_1}$ is uniformly bounded by bounding Lemma. Then, $x_{1,1} - b_{1,0} \lambda_{1,b_1}$ is uniformly bounded. It follows that $\dot{x}_{1,1} - \lambda_{1,b_1} (b_{1,p_0} + \bar{A}_{1,2120} \dot{\theta}_1)$ is also uniformly bounded. Since $\dot{x}_{1,1}$ is uniformly bounded and $\dot{\theta}_1$ is uniformly bounded away from 0, we have $\lambda_{1,b_1}$ is uniformly bounded. That further imply $\Phi_{1,1}$, i.e., $C_1 \Phi_{1,1}$, is uniformly bounded. Furthermore, since $x_{1,1} - b_{1,0} \lambda_{1,b_1}$ and $\dot{w}_1$ are bounded, we have that the signals of $x_{1,1}$ and $y_1$ are uniformly bounded.

Next, we need to prove the existence of a compact set $\Theta_{1,e} \subset \Theta_{1,o}$ such that $\dot{\theta}_1 (t) \in \Theta_{1,e}$, $\forall t \in [0, T_f)$. First introduce the function

$$Y_1 := U + (\rho_{1,o} - P_1 (\dot{\theta}_1))^{-1} P_1 (\dot{\theta}_1)$$
We notice that, when \( \tilde{\theta}_1 \) approaches the boundary of \( \Theta_{1,o} \), \( P_1(\tilde{\theta}_1) \) approaches \( \rho_{1,o} \). Then \( Y_1 \) approaches \( \infty \) as \( X_e \) approaches the boundary of \( \Theta_{1,o} \times \Theta_{2,o} \times \mathbb{R}^{2(n_1+n_2)+r_1+r_2} \). There exist some constant \( c > 0 \) such that the following inequalities hold.

\[
\gamma \left( \frac{\partial P_1(\tilde{\theta}_1)}{\partial \tilde{\theta}_1}(\tilde{\theta}_1) \right)^2 + (\rho_{1,o} - P_1(\tilde{\theta}_1))^{-4} (\rho_{1,o} - P_1(\tilde{\theta}_1))^2 - c + c
\]

Since \( \dot{Y}_1 \) will tend to \( -\infty \) when \( X_e \) approaches the boundary of \( \Theta_{1,o} \times \Theta_{2,o} \times \mathbb{R}^{2(n_1+n_2)+r_1+r_2} \), then there exists a compact set \( \Omega_{1,2}(c_o) \subset \Theta_{1,o} \times \Theta_{2,o} \times \mathbb{R}^{2(n_1+n_2)+r_1+r_2} \), such that \( \forall X_e \in \Theta_{1,o} \times \Theta_{2,o} \times \mathbb{R}^{2(n_1+n_2)+r_1+r_2} \backslash \Omega_{1,2}, Y_1(X_e) < 0 \).

Then there exists a compact set \( \Theta_{1,c} \subset \Theta_{1,o} \), such that \( \tilde{\theta}_1(t) \in \Theta_{1,c}, \forall t \in [0, T_f] \). Moreover, \( Y_1(t, X_e(t)) \leq c_2 \), and \( X_e(t) \) is in the compact set \( S_{1,c_2} \subset \Theta_{1,o} \times \Theta_{2,o} \times \mathbb{R}^{2(n_1+n_2)+r_1+r_2}, \forall t \in [0, T_f] \).

To derive the uniformly boundedness of the closed-loop system states, we separate the relative degree, \( r_1 \), into two cases: \( r_1 = 1 \), and \( r_1 \geq 2 \). First, we consider the case 1: \( r_1 = 1 \).

Taking \( x_{1,1} \) and \( y_2 \) as the output and input of the reference system, we note that \( x_{1,1} \) is strictly minimum phase and has relative degree \( r_1 \) with respect to input \( y_2 \). Since the state \( x_1 \) can be viewed as stably filtered output signals of \( y_2 \) and \( y_1 \), it is uniformly bounded. Since \( x_{1,1} \) is also some stably filtered signals of \( y_1 \) and \( y_2 \), it is uniformly bounded. It further implies \( \Phi_1 \) is uniformly bounded. Then we can conclude \( \dot{x}_1 \) is uniformly bounded from the boundedness of \( \dot{x}_1 - \Phi_1 \dot{\theta}_1 \). This further implies that the inputs \( \dot{x}_{2,1} \) and \( \dot{\theta}_1 \) are uniformly bounded.

Case 2: \( r_1 \geq 2 \). Considering the canonical form (78) in [12] for the true system (1), we denote the elements of \( \ddot{x} \) by \( [\ddot{x}_{1,1} \cdots \ddot{x}_{1,r_1}]^T \). We will use the mathematical induction to derive the boundedness of \( \Phi_{1,iu,i}, \ddot{x}_{1,i} - \lambda_{1,bi} \dot{A}_{1,2120} \dot{\theta}_1, \ddot{x}_{1,i}, x_{1,i} - b_{1,0} \lambda_{1,bi}, \lambda_{1,bi}, \Phi_{1,iu}, x_{1,i}, \ddot{x}_{1,1}, \forall i = \{1, \cdots, r_1\} \). For the boundedness of \( \ddot{x}_{1,i} \), we will show that \( \ddot{x}_{1,i} \) is a linear combination of \( x_{1,1}, \cdots, x_{1,i}, \ddot{x}_3, \) and \( \ddot{x}_4 \), i.e.,

\[
\ddot{x}_{1,i} = \tilde{a}_{1,i} x_{1,1} + \cdots + \tilde{a}_{1,i-1} x_{1,i-1} + x_{1,i} + \tilde{T}_{1,i3} \ddot{x}_3 + \tilde{T}_{1,i4} \ddot{x}_4; \quad 1 \leq i \leq r_1 \tag{24}
\]

where \( \tilde{a}_{1,1}, \cdots, \tilde{a}_{1,i-1} \) are constants, \( \tilde{T}_{1,3}, \tilde{T}_{1,4} \) are constant matrices, and \( \ddot{x}_3 \) and \( \ddot{x}_4 \) are defined at (78) in [12].

1°: We have deduced that \( \eta_{1,1}, \eta_{1,i}, \Phi_{1,iu,i}, \ddot{x}_{1,1} - \lambda_{1,bi} \dot{A}_{1,2120} \dot{\theta}_1, \ddot{x}_{1,i}, x_{1,i} - b_{1,0} \lambda_{1,bi}, \lambda_{1,bi}, \Phi_{1,iu,i}, x_{1,i}, \ddot{x}_{1,1} \) and \( \ddot{x}_{1,i} \) are uniformly bounded in \([0, T_f]\). \( \ddot{x}_{1,1} \) is bounded in view of \( x_{1,1} - \hat{C}_3 \ddot{x}_3 - \hat{C}_4 \ddot{x}_4 \).

2°: We assume that \( \Phi_{1,iu,i}, \ddot{x}_{1,i} - \lambda_{1,bi} \dot{A}_{1,2120} \dot{\theta}_1, \ddot{x}_{1,i}, x_{1,i} - b_{1,0} \lambda_{1,bi}, \lambda_{1,bi}, \Phi_{1,iu}, x_{1,i}, \ddot{x}_{1,i} \) and \( \ddot{x}_{1,i} \) are bounded, and

\[
\ddot{x}_{1,1} = \hat{a}_{1,1} x_{1,1} + \cdots + \hat{a}_{1,i-1} x_{1,i-1} + x_{1,i} + \hat{T}_{1,i3} \ddot{x}_3 + \hat{T}_{1,i4} \ddot{x}_4; \quad \forall i \in \{1, \cdots k\} \tag{25}
\]

where \( 1 \leq k < r_1 \).
3°: First, we need to show that \( \Phi_{1,u,k+1}, \dot{x}_{1,k+1} - \lambda_{1,b,k+1} A_{1,2120} \dot{\theta}_1, \dot{x}_{1,k+1}, x_{1,k+1} - b_{1,0} \lambda_{1,b,k+1}, \lambda_{1,b,k+1}, \Phi_{1,u,k+1}, x_{1,k+1}, \) and \( \ddot{x}_{1,k+1} \) are bounded.

From equation (23c), we note that every element of \( \Phi_{1,u,k+1} \) has relative degree of at least \( r_1 - k + 1 \) with respect to \( y_2 \), and is the output of a stable linear system. Since the boundedness of \( \ddot{x}_{11}, \cdots, \ddot{x}_{1k} \), we conclude \( \Phi_{1,u,k+1} \) is uniformly bounded by Lemma 11 in [12], where the reference system has input \( y_2 \) and output \( y_1 \).

Note that \( k + 1 \)st row element of \( \dot{x}_1 - \Phi_1 \dot{\theta}_1 \) is

\[
\dot{x}_{1,k+1} - \Phi_{1,u,k+1} \dot{\theta}_1 - \lambda_{1,b,k+1} A_{1,2120} \dot{\theta}_1 - \eta_1 T_{1,k+1} \dot{\theta}_1
\]

We can conclude that \( \dot{x}_{1,k+1} - \lambda_{1,b,k+1} A_{1,2120} \dot{\theta}_1 \) is uniformly bounded in view of the boundedness of \( \dot{x}_1 - \Phi_1 \dot{\theta}_1, \dot{\theta}_1, \Phi_{1,u,k+1}, \) and \( \eta_1 \). Since the boundedness of \( y_d, s_1, \Sigma, \eta_1, \eta_1, \Sigma_1, \dot{x}_{1,1}, y_d^{(1)}, \Phi_{1,u,1}, \cdots, \dot{x}_{1,k}, y_d^{(k)}, \Phi_{1,u,k}, \) and \( \dot{\theta}_1(t) \in \Theta_{1,c}, \forall t \in [0,T_f], \alpha_{1,k} \) is bounded. Since \( z_{1,k+1} = x_{1,k+1} - a_{1,k} \), and \( z_{1,k+1} \) is uniformly bounded, we have that \( \ddot{x}_{1,k+1} \) is also uniformly bounded.

The signal \( x_{1,k+1} - b_{1,0} \lambda_{1,b,k+1} \) is generated by:

\[
\dot{x}_1 - b_{1,0} \lambda_{1,b} = A_{1,f} (x_1 - b_{1,0} \lambda_{1,b}) + \left[ \begin{array}{c} (x_{1,2,1}) \\ \vdots \\ (x_{1,2,1}) \end{array} \right] y_2 + \left[ \begin{array}{c} \lambda_{1,b,k+1} A_{1,2120} \end{array} \right] \dot{\theta}_1 + D_1 M_1 \ddot{w}_1 + \left( \begin{array}{c} \sigma_1^2 L_1 \\ \vdots \\ \sigma_1^2 L_1 \end{array} \right) + \Pi_1 \left( \begin{array}{c} \lambda_{1,b,k+1} A_{1,2120} \end{array} \right) \dot{\theta}_1 + D_1 \ddot{w}_1 \]

\[
x_{1,k+1} - b_{1,0} \lambda_{1,b,k+1} = \epsilon_{x_{1,k+1}} (x_1 - b_{1,0} \lambda_{1,b})
\]

Now we will separate the above dynamics into \( y_1 \) dependent and \( y_2 \) dependent parts by the linearity of the system, \( x_{1,k+1} - b_{1,0} \lambda_{1,b,k+1} := x_{1,u,k+1} + x_{1,y,k+1} \), which are respectively given by,

\[
\dot{x}_{1,u} = A_{1,f} x_{1,u} + \left[ \begin{array}{c} \lambda_{1,b,k+1} A_{1,2120} \end{array} \right] \dot{\theta}_1 + D_1 \ddot{w}_1 + \left( \begin{array}{c} \sigma_1^2 L_1 + \Pi_1 \left( \begin{array}{c} \lambda_{1,b,k+1} A_{1,2120} \end{array} \right) \dot{\theta}_1 + D_1 \ddot{w}_1 \end{array} \right)
\]

\[
x_{1,y,k+1} = e'_{x_{1,y,k+1}} x_{1,y}
\]

We observe that the signal \( x_{1,u,k+1} \) has relative degree at least \( r_1 - k + 1 \) with respect to \( y_2 \). Since \( \ddot{x}_{11}, \cdots, \ddot{x}_{1k} \) are uniformly bounded, we conclude \( x_{1,u,k+1} \) is uniformly bounded by Lemma 11 in [12], where the reference system has input \( y_2 \) and output \( y_1 \). We conclude \( x_{1,y,k+1} \) is uniformly bounded since \( y_1 \) is bounded. Then, \( x_{1,k+1} - b_{1,0} \lambda_{1,b,k+1} \) is uniformly bounded. It follows that \( \dot{x}_{1,k+1} - \lambda_{1,b,k+1} (b_{1,p} + A_{1,2120} \dot{\theta}_1) \) is also uniformly bounded. Since
\( \tilde{x}_{1,k+1} \) is uniformly bounded and \( b_{1,p0} + \tilde{A}_{1,212,0} \tilde{\theta}_1 \) is uniformly bounded away from 0, we have \( \lambda_{1,b_{1,k+1}} \) is uniformly bounded. That further imply \( \Phi_{1,u_{1,k+1}} \) is uniformly bounded. Furthermore, since \( x_{1,k+1} - b_{1,0} \lambda_{1,b_{1,k+1}} \) and \( \lambda_{1,b_{1,k+1}} \) are bounded, we have that the signals of \( x_{1,k+1} \) is uniformly bounded.

Next, we need to show \( \tilde{x}_{1,1,k+1} \) is satisfied equation (24). Comparing the design model (2) and the canonical form (78) in [12], we have \( \tilde{C}_1 \tilde{x} = C_1 x_1 \). It further implies

\[
\tilde{C}_1 \tilde{x}^{k+1} = C_1 (A_1 + \tilde{A}_{1,211} \tilde{\theta}_1 C_1)^k x_1
\]

Hence, we have

\[
\tilde{x}_{1,k+1} = \tilde{a}_{1,1,k+1} x_1 + \cdots + \tilde{a}_{1,k+1,k} x_{1,k} + x_{1,k+1} + \tilde{T}_{1,k+13} \tilde{x}_3 + \tilde{T}_{1,k+14} \tilde{x}_4
\]

(26)

where \( \tilde{a}_{1,1,k+1}, \cdots, \tilde{a}_{1,k+1,k} \) are constants, and \( \tilde{T}_{1,k+13}, \tilde{T}_{1,k+14} \) are constant matrices.

Then, we have the boundedness of \( \tilde{x}_{1,k+1} \). Thus, we can conclude the boundedness of \( \Phi_{1,u_{1,i}}, \tilde{x}_{1,j} - \lambda_{1,b_i} \tilde{A}_{1,212,0} \tilde{\theta}_1, \tilde{x}_{1,j}, x_{1,i} - b_{1,0} \lambda_{1,b_i} \lambda_{1,b_i}, \Phi_{1,u_{1,i}}, x_{1,i}, \tilde{x}_{1,i}, \forall i \in \{1, \cdots r_1\} \).

Since the state \( x_1 \) can be viewed as stably filtered output signals of \( y_2 \) and \( y_1 \), it is uniformly bounded. Also, \( \eta_{1i}, \lambda_1 \) are some stably filtered signals of \( y_2 \) and \( y_1 \), they are uniformly bounded. It further implies \( \Phi_{1} \) is uniformly bounded. Then we can conclude \( \tilde{x}_1 \) is uniformly bounded from the boundedness of \( \tilde{x}_1 - \Phi_1 \tilde{\theta}_1 \). This further implies that the control input \( \tilde{x}_{2,1} \) is uniformly bounded. Therefore, it follows \( T_f = \infty \) and the complete system states are uniformly bounded on \([0, \infty)\).

The boundedness of closed-loop state variables of \( S_2 \) can be proven with the similar line of reasoning above. Thus, we have established statement 1 in all cases.

We define \( l_0 = l_{1,0} + l_{2,0} = V_{2,r_2}(X_1(0), X_2(0)) \), and

\[
l_1 + l_2 := \sum_{i=1}^{2} \left( \gamma^4 |x_{i} - \tilde{x}_{i} - \Phi_i (\theta_i - \hat{\theta}_i)|^2 + |\tilde{\eta}_{i}|^2 + \left( \Theta_i C_i \right)^2 \right) + \sum_{j=1}^{r_1} \beta_{ij} z_{i,j}^2 \]

\[
+ \left( \frac{1}{4} \right) \left[ \gamma_i \right]_{\gamma_i} \left( \frac{1}{4} \right) \left[ \gamma_i \right]_{\gamma_i} + \varepsilon_1 (\gamma^2 \varepsilon_{1}^2 - 1) |\theta_{i} - \hat{\theta}_i|^2 \right] \left( \Phi_{i} C_i \right)^2 + \varepsilon_2 |\varepsilon_i|^2 \left( \Phi_{i} C_i \right)^2 \Phi_{i}
\]

\[
\sup_{\tilde{w}_1 \in W_1, \tilde{w}_2 \in W_2} \left\{ \int_{0}^{t_f} \left( (x_{1,1} - y_d)^2 + l_1 + l_2 - \sum_{i=1}^{2} \gamma_i |w_i|^2 - \sum_{i=1}^{2} \gamma_i |w_i|^2 \right) \right. \]

\[
- \sum_{i=1}^{2} \gamma_i^2 \left( |\theta_{i} - \hat{\theta}_{i,0} x'_{i,0} - \hat{x}'_{i,0}|^2 \right) \left. \right|_{Q_{i,0}}^2 - l_0 \}
\]

\[
\leq \sup_{\tilde{w}_1 \in W_1, \tilde{w}_2 \in W_2} \left\{ \int_{0}^{t_f} \left( (x_{1,1} - y_d)^2 + l_1 + l_2 - \sum_{i=1}^{2} \gamma_i |w_i|^2 - \sum_{i=1}^{2} \gamma_i |w_i|^2 \right) \right. \]

\[
- \gamma_2 \sum_{i=1}^{2} \left( |\theta_{i} - \hat{\theta}_{i,0} x'_{i,0} - \hat{x}'_{i,0}|^2 \right) \left. \right|_{Q_{i,0}}^2 + \int_{0}^{t_f} \hat{U} d \tau - U(t) + U(0) \}
\]

\[
\leq -U(t) \leq 0
\]
then, we establish the second statement.

For the third statement, we consider the following inequality,

\[ \int_0^\infty \dot{U} \, d\tau \leq \int_0^\infty \left( -|x_1 - y_d|^2 + \gamma^2 |\dot{w}_{1,a}|^2 + \gamma^2 |\dot{w}_{2,a}|^2 + \gamma^2 |\dot{M}_1 \dot{w}_1|^2 + \gamma^2 |\dot{M}_2 \dot{w}_2|^2 \right) d\tau \]

it follows that

\[ \int_0^\infty |x_1 - y_d|^2 d\tau \leq \int_0^\infty \left( \gamma^2 |\dot{w}_{1,a}|^2 + \gamma^2 |\dot{M}_1 \dot{w}_1|^2 + \gamma^2 |\dot{w}_{2,a}|^2 + \gamma^2 |\dot{M}_2 \dot{w}_2|^2 \right) d\tau + U(0) < +\infty \]

By the first statement, we notice that

\[ \sup_{0 \leq t < \infty} |\dot{x}_1 - y_d| < \infty. \]

Then, we have

\[ \lim_{t \to \infty} |x_1(t) - y_d(t)| = 0 \]

For the last statement, it’s easy to establish by Section 4.

This complete the proof of the theorem.

6. Example

In this section, we present one example to illustrate the main results of this Chapter. The designs were carried out using MATLAB symbolic computation tools, and the closed-loop systems were simulated using SIMULINK.

Consider the following linear systems with zeros initial conditions:

\[ \dot{x}_1 = x_1 + x_2 + x_3 + 0.1 \dot{w}_1; \quad (27a) \]
\[ \dot{x}_2 = (1 + \theta_1) x_3 + (1 + \theta_2) \dot{w}_1 \quad (27b) \]
\[ \dot{x}_3 = -x_1 - x_3 + x_4 + u + \dot{w}_2 \quad (27c) \]
\[ \dot{x}_4 = x_1 + (2 + \theta_3) u + 0.1 \dot{w}_2 + \dot{w}_2 \quad (27d) \]
\[ y = x_1 + 0.1 \dot{w}_1 \quad (27e) \]

where \( \theta_1, \theta_2 \) and \( \theta_3 \) are three unknown parameters with true value 0s. The coefficient terms, 0.1 and 1, reflect the \textit{a priori} knowledge that the disturbances \( \dot{w}_1 \) and \( \dot{w}_2 \) are weak in power relative to that of the disturbance \( \dot{w}_1 \) and \( \dot{w}_2 \). We note that (27) is an unobservable system. We can decompose (27) into the following two SISO linear systems, \( S_1 \) and \( S_2 \), sequentially interconnected with additional output measurement,

\[ \dot{x}_{11} = x_{11} + x_{12} + y_2 + w_{11}; \quad (28a) \]
\[ \dot{x}_{12} = (1 + \theta_1) y_2 + (1 + \theta_2) \dot{w}_1 + w_{12}; \quad (28b) \]
\[ y_1 = x_{11} + w_{13}; \]  
\[ \dot{x}_{21} = -x_{21} + x_{22} + u - y_1 + \bar{\omega}_2 + w_{21}; \]  
\[ \dot{x}_{22} = (2 + \theta_3)u + y_1 + \eta; \]  
\[ y_2 = x_{21} + w_{23}; \]

where

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  w_1
\end{bmatrix} = \begin{bmatrix}
  x_{11} \\
  x_{12} \\
  w_{11} \\
  w_{12} \\
  w_{13}
\end{bmatrix}, \quad
\begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2 \\
  \dot{w}_1 \end{bmatrix} = \begin{bmatrix}
  \dot{x}_{21} \\
  \dot{x}_{22} \\
  \dot{w}_{21} \\
  \dot{w}_{22} \\
  \dot{w}_{23}
\end{bmatrix},
\]

\[ \gamma \]

Here \( \bar{\omega}_3 \) is the measurement disturbance of the state \( \dot{x}_3 \). It is easy to check that \( S_1 \) and \( S_2 \) in (28) satisfied the assumptions 1-5.

For the adaptive control design, we set the desired disturbance attenuation level \( \gamma = 10 \). We select the true value of the parameters in subsystem \( S_1 \) and subsystem \( S_2 \) are zeros, and belong to the interval \([-1, 1]\). The projection function \( P_1(\theta_1) \) and \( P_2(\theta_2) \) are chosen as \( P_1(\theta_1) = 0.5(\theta_1^2 + \theta_2^2) \), \( P_2(\theta_2) = \theta_2^2 \). The reference trajectory, \( y_{d,0} \), is generated by the following linear system \( \dot{x}_{d,1} = -x_{d,2}, \dot{x}_{d,2} = x_{d,1} - x_{d,2} + d, y_{d} = x_{d,1} \) with zeros initial condition, where \( d \) is the command input signal. The objective is to achieve asymptotic tracking of \( \dot{x}_1 \) to the reference trajectory \( y_d \).

For design and simulation parameters of \( S_i, (i = 1, 2) \), we select

\[
\begin{align*}
\dot{x}_{1,0} &= \begin{bmatrix} 0.2 \ 0 \end{bmatrix}; \quad \dot{x}_{2,0} = \begin{bmatrix} 0.1 \ 0 \end{bmatrix}; \quad \dot{\theta}_{1,0} = \begin{bmatrix} 0.5 \ -0.5 \end{bmatrix}; \quad \dot{\theta}_{2,0} = -1/2; \quad Q_{i,0} = 0.001 I_2; \\
K_{i,c} &= 0.2; \quad \Delta_i = I_2; \quad p_{i,n_i} = e_{2,2}; \quad \Phi_{i,0} = 0_{2 \times 1}; \quad \rho_{i,0} = 2; \quad \beta_{i,\Delta} = 0; \quad \epsilon_i = K_{i,c}^{-1} \Xi_i; \quad \lambda_{i,0} = 0_{2 \times 1}; \\
\beta_{i,1} &= 0.5; \quad \eta_{i,0} = 0_{2 \times 1}; \quad Z_1 = \begin{bmatrix} 0.0893 & -0.0081 \\
-0.0081 & 0.0097 \end{bmatrix}; \quad Z_2 = \begin{bmatrix} 0.1094 & -0.0099 \\
-0.0099 & 0.0099 \end{bmatrix}.
\end{align*}
\]

We present one set of simulation results in this example to illustrate the regulatory behavior of the adaptive controller. We set \( d(t) = 0.4 \sin(0.1t) + \sin(0.6t), \bar{\omega}_1(t) = 0, \bar{\omega}_2(t) = 0, \bar{\omega}_3(t) = 0, \bar{\omega}_1(t) = 0.1 \sin(12t + \frac{\pi}{4}) + 0.8 \sin(3t), \) and \( \bar{\omega}_2(t) = 3 \sin(3t + \frac{\pi}{4}) \). The results are shown in Figure 2(a)-(f). To illustrate that the proposed controller can improve the system performance by incorporating the measurements and/or the estimation of the significant external disturbances into the control design, the simulation results based on [17] are presented in Figure 2(c)-(d), where the measured disturbances \( \bar{\omega}_1 \) and \( \bar{\omega}_2 \) are treated as arbitrary disturbances and \( \theta_3 \) is treated as constant in control design. We observe that the output tracking error asymptotically converges to zero and the parameter estimates asymptotically converge to its true value 0 in (a) and (b) even if there is a non-zero measured disturbance in the system. But the parameter estimates doesn’t converge to the true value,
Figure 2. System response for Example under command input $d(t) = 0.4 \sin(0.1t) + \sin(0.6t)$, $\dot{w}_1 = 0$, $\dot{w}_2 = 0$, $\dot{w}_3 = 0$, $\dot{w}_4(t) = \sin(12t + \frac{\pi}{2}) + 0.8 \sin(3t)$, and $\dot{w}_5(t) = 3 \sin(3t + \frac{\pi}{2})$. (a) Parameter estimate; (b) Tracking error; (c) Parameter estimate (based on [17]); (d) Tracking error (based on [17]); (e) control input; (f) State estimation error;
and the tracking error doesn’t converge to zero in (c) and (d). State estimation error, \(x_1 - \hat{x}_1\) and \(x_2 - \hat{x}_2\), converge to zero in (f), and the transient performance behaves well as in (e).

7. Conclusions

In this Chapter, we present the game-theoretical approach based adaptive control design for a special class of MIMO linear systems, which is composed of two sequentially interconnected SISO linear systems, \(S_1\) and \(S_2\). We assume the subsystem under studied subject to noisy output measurements, unknown initial state conditions, linear unknown parametric uncertainties, measured and unmeasured additive exogenous disturbance input uncertainties. Our design objective is to address the asymptotical tracking, the transient response and robustness of the closed-loop system, which are the same as the objectives to motivate the study of the \(H^\infty\)-optimal control problem. In view of the similar solution between \(H^\infty\) optimal control design and zero sum differential game, we convert the original adaptive control design problem into a zero-sum game with soft constraints on the disturbance input uncertainties and the unknown initial state uncertainties, which incorporates the measures of transient response, disturbance attenuation, and asymptotic tracking into a single game-theoretic cost function and formulates the design problem as a nonlinear \(H^\infty\) control problem under imperfect state measurements. A game-theoretical approach, cost-to-come function analysis, is then applied to obtain the finite dimensional estimators of \(S_1\) and \(S_2\) independently, which is also converted the control design as an \(H^\infty\) control problem with full information measurements. The integrator backstepping methodology is finally applied on this full information measurements problem to obtain a suboptimal solution. The controller achieves the same result as [17], namely the total stability of the closed-loop system, the desired disturbance attenuation level, and asymptotic tracking of the reference trajectory when the disturbance is of finite energy and uniformly bounded. In addition, the proposed controller may achieve arbitrary positive disturbance attenuation level with respect to the measured disturbances by proper scaling. The contribution of the measurements of part of the disturbance inputs is that we can design an adaptive controller with disturbance feedforward structure with respect to \(\hat{w}_{1,b}\) and \(\hat{w}_{2,b}\) to eliminate their effect on the squared \(L_2\) norm of the tracking error. Moreover, the asymptotic tracking is achieved even if the measured disturbances are only uniformly bounded without requiring them to be of finite energy.

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8. References


