1

That IS-IN Isn't IS-A: A Further Analysis of Taxonomic Links in Conceptual Modelling

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1. Introduction

Ronald J. Brachman, in his basic article: "What IS-A Is and Isn't: An Analysis of Taxonomic Links in Semantic Networks", (1983), has analysed and catalogued different interpretations of inheritance link, which is called "IS-A", and which is used in different kind of knowledge-representation systems. This IS-A link is seen by Brachman as a relation "between the representational objects," which forms a "taxonomic hierarchy, a tree or a lattice-like structures for categorizing classes of things in the world being represented", (ibid., 30). This very opening phrase in Brachman's article reveals, and which the further analysis of his article confirms as it is done in this Chapter, that he is considering the IS-A relation and the different interpretations given to it as an *extensional* relation. Accordingly, in this Chapter we are considering an *intensional* IS-IN relation which also forms a taxonomic hierarchy and a lattice-like structure. In addition, we can consider the hierarchy provided by an IS-IN relation as a semantic network as well. On the other hand, this IS-IN relation, unlike IS-A relation, is a conceptual relation between concepts, and it is basically intensional in its character.

The purpose of this Chapter is to maintain that the IS-IN relation is not equal to the IS-A relation; more specifically, that Brachman's analysis of an extensional IS-A relation did not include an intensional IS-IN relation. However, we are not maintaining that Brachman's analysis of IS-A relation is wrong, or that there are some flaws in it, but that the IS-IN relation requires a different analysis than the IS-A relation as is done, for example, by Brachman.

This Chapter is composed as follows. Firstly, we are considering the different meanings for the IS-A relation, and, especially, how they are analysed by Brachman in (1983), and to which, in turn, we shall further analyse. Secondly, we are turning our attention to that of the IS-IN relation. We start our analysis by considering what the different senses of "in" are, and to do this we are turning first to Aristotle's and then to Leibniz's account of it. After that, thirdly, we are proceeding towards the basic relations between terms, concepts, classes (or sets), and things in order to propose a more proper use of the IS-IN relation and its relation to the IS-A relation. Lastly, as kind of a conclusion, we are considering some advances and some difficulties related to the intensional versus extensional approaches to a conceptual modelling.

2. The different meanings for the IS-A relation

The idea of IS-A relation seems to follow from the English sentences such as "Socrates is a man" and "a cat is a mammal", which provides two basic forms of using the IS-A relation. That is, a *predication*, where an individual (Socrates) is said to have a predicate (a man), and that one predicate (a cat) is said to be a *subtype* of the other predicate (a mammal). This second form is commonly expressed by the universally quantified conditional as follows: "for all entities x, if x is a cat, then x is a mammal". However, this formalization of the second use of the IS-A relation reveals, that it combines two commonly used expressions using the IS-A relation. Firstly, in the expressions of the form "x is a cat" and "x is a mammal" the IS-A relation is used as a predication, and secondly, by means of the universal quantifier and implication, the IS-A relation is used not as a predication, but as a connection between two predicates.

Accordingly, we can divide the use of the IS-A relation in to two major subtypes: one relating an individual to a species, and the other relating two species. When analysing the different meanings for the IS-A relation Brachman uses this division by calling them *generic/individual* and *generic/generic* relations, (Brachman 1983, 32).

2.1 Generic/individual relations

Brachman gives four different meanings for the IS-A relation connecting an individual and a generic, which we shall list and analyse as follows, (ibid.):

1. *A set membership relation*, for example, "Socrates is a man", where "Socrates" is an individual and "a man" is a set, and Socrates is a member of a set of man. Accordingly, the IS-A is an $\in$ -relation.
2. *A predication*, for example, a predicate "man" is predicated to an individual "Socrates", and we may say that a predicate and an individual is combined by a copula expressing a kind of function-argument relation. Brachman does not mention a copula in his article, but according to this view the IS-A is a copula.
3. *A conceptual containment relation*, for which Brachman gives the following example, "a king" and "the king of France", where the generic "king" is used to construct the individual description. In this view Brachman's explanation and example is confusing. Firstly, "France" is an individual, and we could say that the predicate "a king" is predicated to "France", when the IS-A relation is a copula. Secondly, we could say that the concept of "king" applies to "France" when the IS-A relation is an application relation. Thirdly, the phrase "the king of France" is a definite description, when we could say that the king of France is a definite member of the set of kings, *i.e.*, the IS-A relation is a converse of $\in$ -relation.
4. *An abstraction*, for example, when from the particular man "Socrates" we abstract the general predicate "a man". Hence we could say that "Socrates" falls under the concept of "man", *i.e.*, the IS-A is a falls under –relation, or we could say that "Socrates" is a member of the set of "man", *i.e.*, the IS-A is an $\in$ -relation.

We may notice in the above analysis of different meanings of the IS-A relations between individuals and generic given by Brachman, that three out of four of them we were able to interpret the IS-A relation by means of $\in$ -relation. And, of course, the copula expressing a function-argument relation is possible to express by $\in$ -relation. Moreover, in our analysis of

3. and 4. we used a term "concept" which Brachman didn't use. Instead, he seems to use a term "concept" synonymously with an expression "a structured description", which, according to us, they are not. In any case, what Brachman calls here a conceptual containment relation is not the conceptual containment relation as we shall use it, see Section 4 below.

2.2 Generic/generic relations

Brachman gives six different meanings for the IS-A relation connecting two generics, which we shall list and analyse as follows, (ibid.):

1. *A subset/superset*, for example, "a cat is a mammal", where "a cat" is a set of cats, "a mammal" is a set of mammals, and a set of cats is a subset of a set of mammals, and a set of mammals is a superset of a set of cats. Accordingly, the IS-A relation is a $\subseteq$ -relation.
2. *A generalisation/specialization*, for example, "a cat is a mammal" means that "for all entities x, if x is a cat, then x is a mammal". Now we have two possibilities: The first is that we interpret "x is a cat" and "x is a mammal" as a predication by means of copula, and the relation between them is a formal implication, where the predicate "cat" is a specialization of the predicate "mammal", and the predicate "mammal" is a generalization of the predicate "cat". Thus we can say that the IS-A relation is a formal implication $(\forall x)$ $(P(x) \rightarrow Q(x))$. The second is that since we can interpret "x is a cat" and "x is a mammal" by mean of $\in$ -relation, and then by means of a formal implication we can define a $\subseteq$ -relation, from which we get that the IS-A relation is a $\subseteq$ -relation.
3. *An AKO*, meaning "a kind of", for example, "a cat is a mammal", where "a cat" is a kind of "mammal". As Brachman points out, (ibid.), AKO has much common with generalization, but it implies "kind" status for the terms of it connects, whereas generalization relates arbitrary predicates. That is, to be a kind is to have an essential property (or set of properties) that makes it the kind that it is. Hence, being "a cat" it is necessary to be "a mammal" as well. This leads us to the natural kind inferences: if anything of a kind A has an essential property ϕ, then every A has ϕ. Thus we are turned to the Aristotelian essentialism and to a quantified modal logic, in which the IS-A relation is interpreted as a necessary formal implication $\Box$ $(\forall x)$ $(P(x) \rightarrow Q(x))$. However, it is to be noted, that there are two relations connected with the AKO relation. The first one is the relation between an essential property and the kind, and the second one is the relation between kinds. Brachman does not make this difference in his article, and he does not consider the second one. Provided there are such things as kinds, in our view they would be connected with the IS-IN relation, which we shall consider in the Section 4 below.
4. *A conceptual containment*, for example, and following Brachman, (ibid.), instead of reading "a cat is a mammal" as a simple generalization, it is to be read as "to be a cat is to be a mammal". This, according to him, is the IS-A of lambda-abstraction, wherein one predicate is used in defining another, (ibid.). Unfortunately, it is not clear what Brachman means by "the IS-A of lambda-abstraction, wherein one predicate is used in defining another". If it means that the predicates occurring in the *definiens* are among the predicates occurring in the *definiendum*, there are three possibilities to interpret it: The first one is by means of the IS-A relation as a $\subseteq$ -relation between predicates, *i.e.*, the

predicate of "mammal" is among the predicate of "cat". The second one is that the IS-A is a $=_{df}$ -sign between *definiens* and *definiendum*, or, perhaps, that the IS-A is a lambda-abstraction of it, *i.e.*, $\lambda xy(x =_{df} y)$, although, of course, "a cat $=_{df}$ a mammal" is not a complete definition of a cat. The third possibility is that the IS-IN is a relation between *concepts*, *i.e.*, the concept of "mammal" is contained in the concept of "cat", see Section 4 below. – And it is argued in this Chapter that the IS-IN relation is not the IS-A relation.

5. *A role value restriction*, for example, "the car is a bus", where "the car" is a role and "a bus" is a value being itself a certain type. Thus, the IS-A is a copula.
6. *A set and its characteristic type*, for example, the set of all cats and the concept of "a cat". Then we could say that the IS-A is an extension relation between the concept and its extension, where an extension of a concept is a set of all those things falling under the concept in question. On the other hand, Brachman says also that it associates the characteristic function of a set with that set, (ibid.). That would mean that we have a characteristic function ϕ_{Cat} defined for elements $x \in X$ by $\phi_{\mathrm{Cat}}(x) = 1$, if $x \in$ Cat, and $\phi_{\mathrm{Cat}}(x) = 0$, if $x \notin$ Cat, where Cat is a set of cats, *i.e.*, Cat = $\{x \mid Cat(x)\}$, where $Cat(x)$ is a predicate of being a cat. Accordingly, Cat $\subseteq X$, $\phi_{\mathrm{Cat}}: X \rightarrow \{0, 1\}$, and, in particular, the IS-A is a relation between the characteristic function ϕ_{Cat} and the set Cat.

In the above analysis of the different meanings of the IS-A relations between two generics given by Brachman, concerning the relations of the AKO, the conceptual containment, and the relation between "set and its characteristic type", we were not able to interpret them by using only the set theoretical terms. Since set theory is extensional *par excellence*, the reason for that failure lies simply in the fact that in their adequate analysis some intensional elements are present. However, the AKO relation is based on a philosophical, *i.e.*, ontological, view that there are such things as kinds, and thus we shall not take it as a proper candidate for *the* IS-A relation. On the other hand, in both the conceptual containment relation and the relation between "set and its characteristic type" there occur as their terms "concepts", which are basically intensional entities. Accordingly we shall propose that their adequate analysis requires an intensional IS-IN relation, which differs from the most commonly used kinds of IS-A relations, whose analysis can be made set theoretically. Thus, we shall turn to the IS-IN relation.

3. The IS-IN relation

The idea of the IS-IN relation is close the IS-A relation, but distinction we want to draw between them is, as we shall propose, that the IS-A relation is analysable by means of set theory whereas the IS-IN relation is an intensional relation between concepts.

To analyse the IS-IN relation we are to concentrate on the word "in", which has a complex variety of meanings. First we may note that "in" is some kind of relational expression. Thus, we can put the matter of relation in formal terms as follows,

$$A \text{ is } in\ B.$$

Now we can consider what the different senses of "in" are, and what kinds of substitutions can we make for *A* and *B* that goes along with those different senses of "in". To do this we are to turn first to Aristotle, who discuss of the term "in" in his *Physics*, (210a, 15ff, 1930). He lists the following senses of "in" in which one thing is said to be "in" another:

1. The sense in which a physical part is *in* a physical whole to which it belongs. For example, as the finger is *in* the hand.
2. The sense in which a whole is *in* the parts that makes it up.
3. The sense in which a species is *in* its genus, as "man" is *in* "animal".
4. The sense in which a genus is *in* any of its species, or more generally, any feature of a species is *in* the definition of the species.
5. The sense in which form is *in* matter. For example, "health is *in* the hot and cold".
6. The sense in which events center *in* their primary motive agent. For example, "the affairs of Creece center *in* the king".
7. The sense in which the existence of a thing centers *in* its final cause, its end.
8. The sense in which a thing is *in* a place.

From this list of eight different senses of "in" it is possible to discern four groups:

i. That which has to do with the *part-whole* relation, (1) and (2). Either the relation between a part to the whole or its converse, the relation of a whole to its part.
ii. That which has to do with the *genus-species* relation, (3) and (4). Either *A* is the genus and *B* the species, or *A* is the species and *B* is the genus.
iii. That which has to do with a *causal* relation, (5), (6), and (7). There are, according to Aristotle, four kinds of causes: material, formal, efficient, and final. Thus, *A* may be the formal cause (form), and *B* the matter; or *A* may be the efficient cause ("motive agent"), and *B* the effect; or, given *A*, some particular thing or event *B* is its final cause (*telos*).
iv. That which has to do with a *spatial* relation, (8). This Aristotle recognizes as the "strictest sense of all". *A* is said to be *in B*, where *A* is one thing and *B* is another thing or a place. "Place", for Aristotle, is thought of as what is occupied by some body. A thing located in some body is also located in some place. Thus we may designate *A* as the contained and *B* as the container.

What concerns us here is the second group II, *i.e.,* that which has to do with the *genus-species* relation, and especially the sense of "in" in which a genus *is in* any of its species. What is most important, according to us, it is this place in Aristotle's text to which Leibniz refers, when he says that "Aristotle himself seems to have followed the way of ideas [*viam idealem*], for he says that animal is in man, namely a concept in a concept; for otherwise men would be among animals [*insint animalibus*], (Leibniz *after* 1690a, 120). In this sentence Leibniz points out the distinction between conceptual level and the level of individuals, which amounts also the set of individuals. This distinction is crucial, and our proposal for distinguishing the IS-IN relation from the IS-A relation is based on it. What follows, we shall call the IS-IN relation an intensional containment relation between concepts.

4. Conceptual structures

Although the IS-A relation seems to follow from the English sentences such as "Socrates is a man" and "a cat is a mammal", the word "is" is logically speaking intolerably ambiguous, and a great care is needed not to confound its various meanings. For example, we have (1) the sense, in which it asserts Being, as in "*A* is"; (2) the sense of identity, as in "Cicero is Tullius"; (3) the sense of equality, as in "the sum of 6 and 8 is 14"; (4) the sense of predication, as in "the sky is blue"; (5) the sense of definition, as in "the *power set* of *A* is the set of all subsets of *A*"; etc. There are also less common uses, as "to be good is to be happy", where a relation of assertions is meant, and which gives rise to a formal implication. All this shows that the natural language is not precise enough to make clear the different meanings of the word "is", and hence of the words "is a", and "is in". Accordingly, to make differences between the IS-A relation and the IS-IN relation clear, we are to turn our attention to a logic.

4.1 Items connected to a concept

There are some basic items connected to a concept, and one possible way to locate them is as follows, see Fig. 1, (Palomäki 1994):

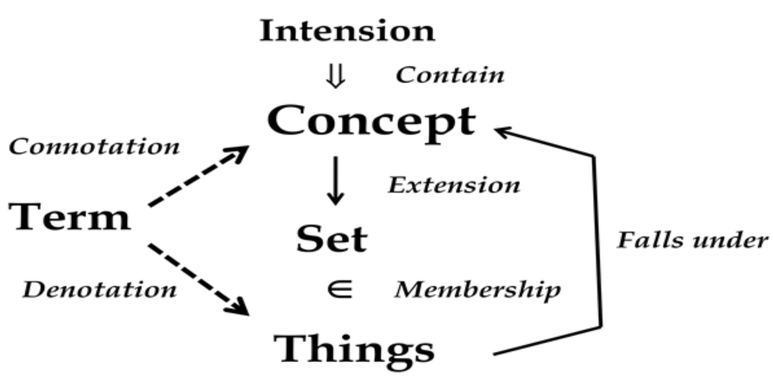

Fig. 1. Items connected to a concept

A *term* is a linguistic entity. It *denotes* things and *connotes* a concept. A concept, in turn, has an *extension* and an *intension*. The extension of a concept is a *set*, (or a *class*, being more exact), of all those things that *falls under* the concept. Now, there may be many different terms which denote the same things but connote different concepts. That is, these different concepts have the same extension but they differ in their intension. By an intension of a concept we mean something which we have to "understand" or "grasp" in order to use the concept in question correctly. Hence, we may say that the intension of concept is that knowledge content of it which is required in order to recognize a thing belonging to the extension of the concept in question, (Kangassalo, 1992/93, 2007).

Let $U = < V, C, F >$ be a universe of discourse, where i) V is a universe of (possible) individuals, ii) C is a universe of concepts, iii) $V \cap C \neq \{\,\}$, and iv) $F \subseteq V \times C$ is the falls under –relation. Now, if a is a concept, then for every (possible) individual i in V, either i falls under the concept a or it doesn't, *i.e*,

$$\text{if } a \in C, \text{ then } \forall i \in V : iFa \vee \sim iFa. \tag{1}$$

The extension-relation E between the set A and the concept a in V is defined as follows:

$$E_U(A,a) =_{\mathrm{df}} (\forall i)\ (i \in A \leftrightarrow i \in V \wedge iFa). \tag{2}$$

The extension of concept a may also be described as follows:

$$i \in E_U'(a) \leftrightarrow iFa, \tag{3}$$

where $E_U'(a)$ is the extension of concept a in V, *i.e.*, $E_U'(a) = \{ i \in V \mid iFa\}$.

4.2 An intensional containment relation

Now, the relations between concepts enable us to make conceptual structures. The basic relation between concepts is an intensional containment relation, (see Kauppi 1967, Kangassalo 1992/93, Palomäki 1994), and it is this intensional containment relation between concepts, which we are calling the IS-IN relation.

More formally, let there be given two concepts a and b. When a concept a contains intensionally a concept b, we may say that the intension of a concept a contains the intension of a concept b, or that the concept a intensionally entails the concept b, or that the intension of the concept a entails the intension of the concept b. This intensional containment relation is denoted as follows,

$$a \geq b. \tag{4}$$

Then, it was observed by Kauppi in (1967) that

$$a \geq b \to (\forall i)\ (iFa \to iFb), \tag{5}$$

that is, that the transition from intensions to extensions reverses the containment relation, *i.e.*, the intensional containment relation between concepts a and b is converse to the extensional set-theoretical subset-relation between their extensions. Thus, by (3),

$$a \geq b \to E_U'(a) \subseteq E_U'(b), \tag{6}$$

where "$\subseteq$" is the set-theoretical subset-relation, or the extensional inclusion relation between sets. Or, if we put $A = E_U'(a)$ and $B = E_U'(b)$, we will get,

$$a \geq b \to A \subseteq B\,[1] \tag{7}$$

For example, if the concept of a dog contains intensionally the concept of a quadruped, then the extension of the concept of the quadruped, *i.e.*, the set of four-footed animals, contains extensionally as a subset the extension of the concept of the dog, *i.e.*, the set of dogs. Observe, though, that we can deduce from concepts to their extensions, *i.e.*, sets, but not conversely, because for every set there may be many different concepts, whose extension that set is.

The above formula (6) is what was searched, without success, by Woods in (1991), where the intensional containment relation is called by him a structural, or an intensional subsumption relation.

4.3 An intensional concept theory

Based on the intensional containment relation between concepts the late Professor Raili Kauppi has presented her axiomatic intensional concept theory in Kauppi (1967), which is further studied in (Palomäki 1994). This axiomatic concept theory was inspired by Leibniz's

[1]In the set theory a subset-relation between sets A and B is defined by $\in$-relation between the elements of them as follows , $A \subseteq B =_{df} \forall x\ (x \in A \to x \in B)$. Unfortunately both $\subseteq$-relation and $\in$-relation are called IS-A relations, although they are different relations. On the other hand, we can take the intensional containment relation between concepts a and b, *i.e.*, $a \geq b$, to be the IS-IN relation.

logic, where the intensional containment relation between concepts formalises an "*inesse*"-relation[2] in Leibniz's logic.[3]

An intensional concept theory, denoted by *KC*, is presented in a first-order language *L* that contains individual variables *a*, *b*, *c*,..., which range over the *concepts*, and one non-logical 2-place *intensional containment relation*, denoted by "≥". We shall first present four basic relations between concepts defined by "≥", and then, briefly, the basic axioms of the theory. A more complete presentation of the theory, see Kauppi (1967), and Palomäki (1994).

Two concepts *a* and *b* are said to be *comparable*, denoted by $a \mathrm{H} b$, if there exists a concept *x* which is intensionally contained in both.

$$\mathrm{Df}_{\mathrm{H}} \qquad a\mathrm{H}b =_{\mathrm{df}} (\exists x)\ (a \geq x \wedge b \geq x).$$

If two concepts *a* and *b* are not comparable, they are *incomparable*, which is denoted by $a \mathrm{I} b$.

$$\mathrm{Df}_{\mathrm{I}} \qquad a\mathrm{I}b =_{\mathrm{df}} \sim a\mathrm{H}b.$$

Dually, two concepts *a* and *b* are said to be *compatible*, denoted by $a \perp b$, if there exists a concept *x* which contains intensionally both.

$$\mathrm{Df}_{\perp} \qquad a \perp b =_{\mathrm{df}} (\exists x)\ (x \geq a \wedge x \geq b)$$

If two concepts *a* and *b* are not compatible, they are *incompatible*, which is denoted by $a \mathrm{Y} b$.

$$\mathrm{Df}_{\mathrm{Y}} \qquad a\mathrm{Y}b =_{\mathrm{df}} \sim a \perp b.$$

The two first axioms of *KC* state that the intensional containment relation is a *reflexive* and *transitive* relation.

$$\mathrm{Ax}_{\mathrm{Refl}} \quad a \geq a.$$
$$\mathrm{Ax}_{\mathrm{Trans}} \quad a \geq b \wedge b \geq c \rightarrow a \geq c.$$

Two concepts *a* and *b* are said to be *intensionally identical*, denoted by $a \approx b$, if the concept *a* intensionally contains the concept *b*, and the concept *b* intensionally contains the concept *a*.

$$\mathrm{Df}_{\approx} \qquad a \approx b =_{\mathrm{df}} a \geq b \wedge b \geq a.$$

[2]Literally, "*inesse*" is "being-in", and this term was used by Scholastic translator of Aristotle to render the Greek "huparchei", *i.e.*, "belongs to", (Leibniz 1997, 18, 243).

[3]Cf. "*Definition 3*. That A 'is in' L, or, that L 'contains' A, is the same that L is assumed to be coincident with several terms taken together, among which is A", (Leibniz after 1690, 132). Also, e.g. in a letter to Arnauld 14 July 1786 Leibniz wrote, (Leibniz 1997, 62): "[I]n every affirmative true proposition, necessary or contingent, universal or singular, the notion of the predicate is contained in some way in that of the subject, *praedicatum inest subjecto* [the predicate is included in the subject]. Or else I do not know what truth is." This view may be called the conceptual containment theory of truth, (Adams 1994, 57), which is closely associated with Leibniz's preference for an "intensional" as opposed to an "extensional" interpretation of categorical propositions. Leibniz worked out a variety of both intensional and extensional treatments of the logic of predicates, *i.e.*, concepts, but preferring the intensional approach, (Kauppi 1960, 220, 251, 252).

The intensional identity is clearly a reflexive, symmetric and transitive relation, hence an equivalence relation.

A concept c is called an *intensional product* of two concepts a and b, if any concept x is intensionally contained in c if and only if it is intensionally contained in both a and b. If two concepts a and b have an intensional product, it is unique up to the intensional identity and we denote it then by $a \otimes b$.

$$\mathrm{Df}_{\otimes} \qquad c \approx a \otimes b =_{\mathrm{df}} (\forall x)\ (c \geq x \leftrightarrow a \geq x \wedge b \geq x).$$

The following axiom $\mathrm{Ax}_{\otimes}$ of *KC* states that if two concepts a and b are comparable, there exists a concept x which is their intensional product.

$$\mathrm{Ax}_{\otimes} \qquad a\mathrm{H}b \rightarrow (\exists x)\ (x \approx a \otimes b).$$

It is easy to show that the intensional product is idempotent, commutative, and associative.

A concept c is called an *intensional sum* of two concepts a and b, if the concept c is intensionally contained in any concept x if and only if it contains intensionally both a and b. If two concepts a and b have an intensional sum, it is unique up to the intensional identity and we denote it then by $a \oplus b$.

$$\mathrm{Df}_{\oplus} \qquad c \approx a \oplus b =_{\mathrm{df}} (\forall x)\ (x \geq c \leftrightarrow x \geq a \wedge x \geq b)\ ^{4}.$$

The following axiom $\mathrm{Ax}_{\oplus}$ of *KC* states that if two concepts a and b are compatible, there exists a concept x which is their intensional sum.

$$\mathrm{Ax}_{\oplus} \qquad a \perp b \rightarrow (\exists x)\ (x \approx a \oplus b)$$

The intensional sum is idempotent, commutative, and associative.

The intensional product of two concepts a and b is intensionally contained in their intensional sum whenever both sides are defined.

$$\mathrm{Th}\ 1 \qquad a \oplus b \geq a \otimes b.$$

Proof: If $a \otimes b$ exists, then by $\mathrm{Df}_{\otimes}$, $a \geq a \otimes b$ and $b \geq a \otimes b$. Similarly, if $a \oplus b$ exists, then by $\mathrm{Df}_{\oplus}$, $a \oplus b \geq a$ and $a \oplus b \geq b$. Hence, by $\mathrm{Ax}_{\mathrm{Trans}}$, the theorem follows.

A concept b is an *intensional negation* of a concept a, denoted by $\neg a$, if and only if it is intensionally contained in all those concepts x, which are intensionally incompatible with the concept a. When $\neg a$ exists, it is unique up to the intensional identity.

$$\mathrm{Df}_{\neg} \qquad b \approx \neg a =_{\mathrm{df}} (\forall x)\ (x \geq b \leftrightarrow x\mathrm{Y}a)\ .$$

The following axiom $\mathrm{Ax}_{\neg}$ of *KC* states that if there is a concept x which is incompatible with the concept a, there exists a concept y, which is the intensional negation of the concept a.

[4]Thus, $a \otimes b \leftrightarrow [a] \otimes [b]$ is a greatest lower bound in $C/\approx$, whereas $a \oplus b \leftrightarrow [a] \oplus [b]$ is a least upper bound in $C/\approx$.

$$\mathrm{Ax}_{\neg} \quad (\exists x)\ (x \mathrm{Y} a) \rightarrow (\exists y)\ (y \approx \neg a).$$

It can be proved that a concept a contains intensionally its intensional double negation provided that it exists.

$$\text{Th 2} \qquad a \geq \neg\neg a.$$ [5]

Proof: By $\mathrm{Df}_{\neg}$ the equivalence (1): $b \geq \neg a \leftrightarrow b \mathrm{\ Y\ } a$ holds. By substituting $\neg a$ for b to (1), we get $\neg a \geq \neg a \leftrightarrow \neg a \mathrm{\ Y\ } a$, and so, by $\mathrm{Ax}_{\mathrm{Refl}}$, we get (2): $\neg a \mathrm{\ Y\ } a$. Then, by substituting a for b and $\neg a$ for a to (1), we get $a \geq \neg\neg a \leftrightarrow a \mathrm{\ Y\ } \neg a$ and hence, by (2), the theorem follows.

Also, the following forms of the *De Morgan's formulas* can be proved whenever both sides are defined:

$$\text{Th 3} \quad \text{i)} \neg a \otimes \neg b \geq \neg(a \oplus b),$$
$$\text{ii)} \ \neg(a \otimes b) \approx \neg a \oplus \neg b.$$

Proof: First we are to proof the following important lemma:

Lemma 1 $\qquad a \geq b \rightarrow \neg b \geq \neg a.$

Proof: From $a \geq b$ follows $(\forall x)\ (x \mathrm{\ Y\ } b \rightarrow x \mathrm{\ Y\ } a)$, and thus by $\mathrm{Df}_{\neg}$ the Lemma 1 follows.

i. If $a \oplus b$ exists, then by $\mathrm{Df}_{\oplus}$, $a \oplus b \geq a$ and $a \oplus b \geq b$. By Lemma 1 we get $\neg a \geq \neg(a \oplus b)$ and $\neg b \geq \neg(a \oplus b)$. Then, by $\mathrm{Df}_{\otimes}$, Th 3 i) follows.
ii. This is proved in the four steps as follows:
1. $\neg(a \otimes b) \geq \neg a \oplus \neg b$. Since $a \geq a \otimes b$, it follows by Lemma 1 that $\neg(a \otimes b) \geq \neg a$. Thus, by $\mathrm{Df}_{\oplus}$, 1 holds.
2. $\neg(\neg\neg a \otimes \neg\neg b) \geq \neg(a \otimes b)$. Since $a \geq \neg\neg a$, by Th 2, it follows by $\mathrm{Df}_{\otimes}$ that $a \otimes b \geq \neg\neg a \otimes \neg\neg b$. Thus, by Lemma 1, 2 holds.
3. $(\neg\neg a \otimes \neg\neg b) \geq \neg(\neg a \oplus \neg b)$. Since $(a \oplus b) \geq a$, it follows by Lemma 1 that $\neg a \geq \neg(a \oplus b)$, and so, by $\mathrm{Df}_{\otimes}$, it follows $(\neg a \otimes \neg b) \geq \neg(a \oplus b)$. Thus, by substituting $\neg a$ for a and $\neg b$ for b to it, 3 holds.
4. $\neg a \oplus \neg b \geq \neg(a \otimes b)$. Since $\neg a \oplus \neg b \geq \neg\neg(\neg a \oplus \neg b)$, by Th 2, and from 3 it follows by Lemma 1 that $\neg\neg(\neg a \oplus \neg b) \geq \neg(\neg\neg a \otimes \neg\neg b)$, and by $\mathrm{Ax}_{\mathrm{Trans}}$ we get, $\neg a \oplus \neg b \geq \neg(\neg\neg a \otimes \neg\neg b)$. Thus, by 2 and by $\mathrm{Ax}_{\mathrm{Trans}}$, 4 holds.

From 1 and 4, by $\mathrm{Df}_{\approx}$, the Th 3 ii) follows.

If a concept a is intensionally contained in every concept x, the concept a is called a *general concept,* and it is denoted by G. The general concept is unique up to the intensional identity, and it is defined as follows:

$$\mathrm{Df}_{G} \qquad a \approx G =_{\mathrm{df}} (\forall x)\ (x \geq a).$$

The next axiom of *KC* states that there is a concept, which is intensionally contained in every concept.

[5]This relation does not hold conversely without stating a further axiom for intensional double negation, i.e., $\mathrm{Ax}_{\neg\neg}$: $b \mathrm{\ Y\ } \neg a \rightarrow b \geq a$. Thus, $\neg\neg a \geq a$, and hence by Th 2, $a \approx \neg\neg a$, holds only, if the concept a is intensionally contained in the every concept b, which is incompatible with the intensional negation of the concept a.

$$\mathrm{Ax}_G \qquad (\exists x)(\forall y)\ (y \geq x)\ .$$

Adopting the axiom of the general concept it follows that all concepts are to be comparable. Since the general concept is compatible with every concept, it has no intensional negation.

A *special concept* is a concept *a*, which is not intensionally contained in any other concept except for concepts intensionally identical to itself. Thus, there can be many special concepts.

$$\mathrm{Df}_S \qquad S(a) =_{\mathrm{df}} (\forall x)\ (x \geq a \rightarrow a \geq x)\,.$$

The last axiom of *KC* states that there is for any concept y a special concept x in which it is intensionally contained.

$$\mathrm{Ax}_S \qquad (\forall y)(\exists x)\ (S(x) \wedge x \geq y)\,.$$

Since the special concept s is either compatible or incompatible with every concept, the *law of excluded middle* holds for s so that for any concept x, which has an intensional negation, either the concept x or its intensional negation $\neg x$ is intensionally contained in it. Hence, we have

$$\mathrm{Th}\ 4 \qquad (\forall x)S(s) \rightarrow (s \geq x \vee s \geq -x).\cdot$$

A special concept, which corresponds Leibniz's complete concept of an individual, would contain one member of every pair of mutually incompatible concepts.

By *Completeness Theorem*, every consistent first-order theory has a model. Accordingly, in Palomäki (1994, 94-97) a model of *KC* + $\mathrm{Ax}_{\neg\neg}$ is found to be a *complete semilattice*, where every concept $a \in C$ defines a *Boolean algebra* B_a = $<\downarrow a,\otimes,\oplus,\neg,G,a>$, where $\downarrow a$ is an ideal, known as the *principal ideal generated by a, i.e.* $\downarrow a =_{\mathrm{df}} \{x \in C \mid a \geq x\}$, and the intensional negation of a concept $b \in \downarrow a$ is interpreted as a *relative complement of a.*

It should be emphasized that in *KC* concepts in generally don't form a lattice structure as, for example, they do in Formal Concept Analysis, (Ganter & Wille, 1998). Only in a very special case in *KC* concepts will form a lattice structure; that is, when all the concepts are both comparable and compatible, in which case there will be no incompatible concepts and, hence, no intensional negation of a concept either.[6]

5. That IS-IN Isn't IS-A

In current literature, the relations between concepts are mostly based on the set theoretical relations between the extensions of concepts. For example, in Nebel & Smolka (1990), the conceptual intersection of the concepts of "man" and "woman" is the empty-concept, and their conceptual union is the concept of "adult". However, intensionally the common concept which contains both the concepts of "man" and of "woman", and so is their intensional conceptual intersection, is the concept of 'adult', not the empty-concept, and the concept in which they both are contained, and so is their intensional conceptual union, is the concept of "androgyne", not the concept of "adult". Moreover, if the extension of the empty-

[6]How this intensional concept theory *KC* is used in the context of conceptual modelling, *i.e.*, when developing a conceptual schemata, see especially (Kangassalo 1992/93, 2007).

concept is an empty set, then it would follow that the concepts of "androgyne", "centaur", and "round-square" are all equivalent with the empty-concept, which is absurd. Thus, although Nebel and Smolka are talking about concepts, they are dealing with them only in terms of extensional set theory, not intensional concept theory.

There are several reasons to separate intensional concept theory from extensional set theory, (Palomäki 1994). For instance: i) intensions determine extensions, but not conversely, ii) whether a thing belongs to a set is decided primarily by intension, iii) a concept can be used meaningfully even when there is not yet, nor ever will be, any individuals belonging to the extension of the concept in question, iv) there can be many non-identical but co-extensional concepts, v) extension of a concept may vary according to context, and vi) from Gödel's two Incompleteness Theorems it follows that intensions cannot be wholly eliminated from set theory.

One difference between extensionality and intensionality is that in extensionality a collection is determined by its elements, whereas in intensionality a collection is determined by a concept, a property, an attribute, etc. That means, for example, when we are creating a semantical network or a conceptual model by using an extensional IS-A relation as its taxonomical link, the existence of objects to be modeled are presupposed, whereas by using an intensional IS-IN relation between the concepts the existence of objects falling under those concepts are not presupposed. This difference is crucial when we are designing an object, which does not yet exist, but we have plenty of conceptual information about it, and we are building a conceptual model of it. In the set theoretical IS-A approach to a taxonomy the Universe of Discourse consists of individuals, whereas in the intensional concept theoretical IS-IN approach to a taxonomy the Universe of Discourse consists of concepts. Thus, in extensional approach we are moving from objects towards concepts, whereas in intensional approach we moving from concepts towards objects.

However, it seems that from strictly extensional approach we are not able to reach concepts without intensionality. The principle of extensionality in the set theory is given by a first-order formula as follows,

$$\forall A \forall B (\forall x (x \in A \leftrightarrow x \in B) \rightarrow A = B) .$$

That is, if two *sets* have exactly the same members, then they are equal. Now, what is a set? - There are two ways to form a set: i) extensionally by listing all the elements of a set, for example, $A = \{a, b, c\}$, or ii) intensionally by giving the defining property $P(x)$, in which the elements of a set is to satisfy in order to belong to the set, for example, $B = \{x | \, blue(x)\}$, where the set B is the set of all blue things.[7] Moreover, if we then write "$x \in B$", we use the symbol

[7]In pure mathematics there are only sets, and a "definite" property, which appears for example in the axiom schemata of separation and replacement in the Zermelo-Fraenkel set theory, is one that could be formulated as a first order theory whose atomic formulas were limited to set membership and identity. However, the set theory is of no practical use in itself, but is used to other things as well. We assume a theory T, and we shall call the objects in the domain of interpretation of T *individuals, (*or *atoms*, or *Urelements*). To include the individuals, we introduce a predicate $U(x)$ to mean that x is an individual, and then we relativize all the axioms of T to U. That is, we replace every universal quantifier "$\forall x$" in an axiom of T with "$\forall x \, (U(x) \rightarrow ...)$ and every existential quantifier "$\exists x$" with "$\exists x \, (U(x) \wedge ...)$, and for every constant "a" in the language of T we add $U(a)$ as new axiom.

$\in$ to denote the membership. It abbreviates the Greek word *ἐστί*, which means "is", and it asserts that *x is* blue. Now, the intensionality is implicitly present when we are selecting the members of a set by some definite property $P(x)$, *i.e.*, we have to understand the property of being *blue*, for instance, in order to select the possible members of the set of all blue things (from the given Universe of Discourse).

An extensional view of concepts indeed is untenable. The fundamental property that makes extensions extensional is that concepts have the same extensions in case they have the same instances. Accordingly, if we use $\{x|\ a(x)\}$ and $\{x|\ b(x)\}$ to denote the extensions of the concepts a and b, respectively, we can express extensionality by means of the second-order principle,

$$\forall a \forall b (\forall x (a \cong b) \leftrightarrow (x \mid a(x)\} \ = \ \{x \mid b(x)\}) \,. \tag{*}$$

However, by accepting that principle some very implausible consequences will follow. For example, according to physiologists any creature with a heart also has a kidney, and *vice versa*. So the concepts of "heart" and "kidney" are co-extensional concepts, and then, by the principle (∗), the concepts of "heart" and "kidney" are 'identical' or interchangeable concepts. On the other hand, to distinguish between the concepts of "heart" and "kidney" is very relevant for instance in the case when someone has a heart-attack, and the surgeon, who is a passionate extensionalist, prefers to operate his kidney instead of the heart.

5.1 Intensionality in possible worlds semantic approach

Intensional notions (e.g. concepts) are not strictly formal notions, and it would be misleading to take these as subjects of study for logic only, since logic is concerned with the forms of propositions as distinct from their contents. Perhaps only part of the theory of intensionality which can be called formal is pure modal logic and its possible worlds semantic. However, in concept theories based on possible worlds semantic, (see e.g. Hintikka 1969, Montague 1974, Palomäki 1997, Duzi et al. 2010), intensional notions are defined as (possibly partial, but indeed set-theoretical) functions from the possible worlds to extensions in those worlds.

Also Nicola Guarino, in his key article on "ontology" in (1998), where he emphasized the intensional aspect of modelling, started to formalize his account of "ontology"[8] by the possible world semantics in spite of being aware that the possible world approach has some disadvantages, for instance, the two concepts "trilateral" and "triangle" turn out to be the same, as they have the same extension in all possible worlds.

[8]From Guarino's (1998) formalization of his view of "ontology", we will learn that the "ontology" for him is a set of axioms (language) such that its intended models approximate as well as possible the conceptualization of the world. He also emphasize that "it is important to stress that an ontology is *language-dependent*, while a conceptualization is *language-independent*." Here the word "conceptualization" means "a set of conceptual relations defined on a domain space", whereas by "the ontological commitments" he means the relation between the language and the conceptualization. This kind of language dependent view of "ontology" as well as other non-traditional use of the word "ontology" is analyzed and critized in Palomäki (2009).

In all these possible worlds approaches intensional notions are once more either reduced to extensional set-theoretic constructs in diversity of worlds or as being non-logical notions left unexplained. So, when developing an adequate presentation of a concept theory it has to take into account both formal (logic) and contentual (epistemic) aspects of concepts and their relationships.

5.2 Nominalism, conceptualism, and conceptual realism (Platonism)

In philosophy ontology is a part of metaphysics,[9] which aims to answer at least the following three questions:

1. What is there?
2. What is it, that there is?
3. How is that, that there is?

The first is (1) is perhaps the most difficult one, as it asks what elements the world is made up of, or rather, what are the building blocks from which the world is composed. A Traditional answer to this question is that the world consists of things and properties (and relations). An alternative answer can be found in Wittgenstein's Tractatus 1.1: "The world is the totality of facts, not of things", that is to say, the world consists of facts.

The second question (2) concerns the basic stuff from which the world is made. The world could be made out of one kind of stuff only, for example, water, as Thales suggests, or the world may be made out of two or more different kinds of stuff, for example, mind and matter.

The third question (3) concerns the mode of existence. Answers to this question could be the following ones, according to which something exists in the sense that:

a. it has some kind of concrete space-time existence,
b. it has some kind of abstract (mental) existence,
c. it has some kind of transcendental existence, in the sense that it extends beyond the space-time existence.

The most crucial ontological question concerning concepts and intensionality is: "What modes of existence may concepts have?" The traditional answers to it are that

i. concepts are merely predicate expressions of some language, i.e. they exist concretely, (nominalism);
ii. concepts exist in the sense that we have the socio-biological cognitive capacity to identify, classify, and characterize or perceive relationships between things in various ways, i.e. they exist abstractly, (conceptualism);
iii. concepts exist independently of both language and human cognition, i.e. transcendentally, (conceptual realism, Platonism).

If the concepts exist only concretely as linguistic terms, then there are only extensional relationships between them. If the concepts exist abstractly as a cognitive capacity, then

[9]Nowadays there are two sense of the word "ontology": the traditional one, which we may call a philosophical view, and the more modern one used in the area of information systems, which we may call a knowledge representational view, (see Palomäki 2009).

conceptualization is a private activity done by human mind. If the concepts exist transcendentally independently of both language and human cognition, then we have a problem of knowledge acquisition of them. Thus, the ontological question of the mode of existence of concepts is a deep philosophical issue. However, if we take an ontological commitment to a certain view of the mode of the existence of concepts, consequently we are making other ontological commitments as well. For example, realism on concepts is usually connected with realism of the world as well. In conceptualism we are more or less creating our world by conceptualization, and in nominalism there are neither intensionality nor abstract (or transcendental) entities like numbers.

6. Conclusion

In the above analysis of the different senses of IS-A relation in the Section 2 we took our starting point Brachman's analysis of it in (Brachman 1983), and to which we gave a further analysis in order to show that most of those analysis IS-A relation is interpreted as an extensional relation, which we are able to give set theoretical interpretation. However, for some of Brachman's instances we were not able to give an appropriate set theoretical interpretation, and those were the instances concerning concepts. Accordingly, in the Section 3 we turned our analysis of IS-IN relation following Aristotelian-Leibnizian approach to it, and to which we were giving an intensional interpretation; that is, IS-IN relation is an intensional relation between concepts. A formal presentation of the basic relations between terms, concepts, classes (or sets), and things was given in the Section 4 as well as the basic axioms of the intensional concept theory *KC*. In the last Section 5 some of the basic differences between the IS-IN relation and the IS-A relation was drawn.

So, in this Chapter we maintain that an IS-IN relation is not equal to an IS-A relation; more specifically, that Brachman's analysis of an *extensional* IS-A relation in his basic article: "What IS-A Is and Isn't: An Analysis of Taxonomic Links in Semantic Networks", (1983), did not include an *intensional* IS-IN relation. However, we are not maintain that Brachman's analysis of IS-A relation is wrong, or that there are some flaw in it, but that the IS-IN relation is different than the IS-A relation. Accordingly, we are proposing that the IS-IN relation is a conceptual relation between concepts and it is basically intensional relation, whereas the IS-A relation is to be reserved for extensional use only.

Provided that there are differences between intensional and extensional view when constructing hierarchical semantic networks, we are not allowed to identify concepts with their extensions. Moreover, in that case we are to distinguish the intensional IS-IN relation between concepts from the extensional IS-A relation between the extensions of concepts. However, only a thoroughgoing nominalist would identify concepts with their extensions, whereas for all the others this distinction is necessarily present.

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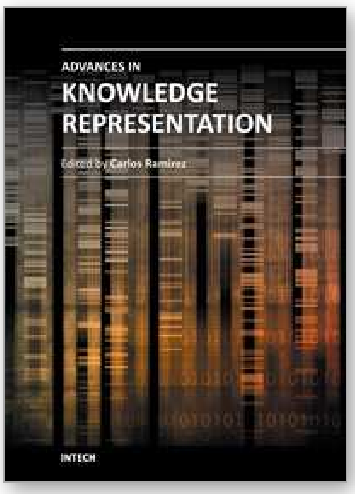

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