4

Poly-Dimension of Antimatroids

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1. Introduction

A partial cube is a graph that can be isometrically embedded into a hypercube. In other words, a partial cube is a subgraph of a hypercube that preserves distances - the distance between any two vertices in the subgraph is the same as the distance between those vertices in the hypercube. Partial cubes were first introduced by Graham and Pollak (Graham & Pollak, 1971) as a model for communication networks and were extensively studied afterwards.

Many important families of combinatorial structures can be represented as subgraphs of partial cubes, for instance, antimatroids. An antimatroid is an accessible set system closed under union. There are two equivalent definitions of antimatroids, one as set systems and the other as languages (Björner & Ziegler, 1992; Korte et al., 1991). An algorithmic characterization of antimatroids based on the language definition was introduced in (Boyd & Faigle, 1990). Later, another algorithmic characterization of antimatroids, which depicted them as set systems was developed in (Kempner & Levit, 2003). Dilworth (Dilworth, 1940) was the first to study antimatroids using another axiomatization based on lattice theory, and they have been frequently rediscovered in other contexts. An antimatroid can be viewed as a special case of either greedoids or semimodular lattices, and as a generalization of partial orders and distributive lattices. While classical examples of antimatroids connect them with posets, chordal graphs, convex geometries etc., a game theory gives a framework, in which antimatroids are considered as permission structures for coalitions (Algaba et al., 2004; 2010). There are also rich connections between antimatroids and cluster analysis (Kempner & Muchnik, 2003). Glasserman and Yao (Glasserman & Yao, 1994) used antimatroids to model the ordering of events in discrete event simulation systems. In mathematical psychology, antimatroids are used to describe feasible states of knowledge of a human learner (Cosyn & Uzun, 2009; Eppstein et al., 2008; Falmagne & Doignon, 2011).

The notion of "antimatroid with repetition" was conceived by Bjorner, Lovasz and Shor (Björner et al., 1991) as an extension of the notion of antimatroid in the framework of non-simple languages. Further they were investigated by the name of "poly-antimatroids" (Kempner & Levit, 2007; Nakamura, 2005), where the set system approach was used.

An antimatroid with the ground set of size n may be isometrically embedded into the **hypercube** $\{0,1\}^n$. A poly-antimatroid with the same ground set is isometrically embedded into n-dimensional integer **lattice** $\mathbb{Z}^n$. In this research we concentrate on interrelations between antimatroids and poly-antimatroids and prove that these two structures are isometrically isomorphic.

The *poly-dimension* of an antimatroid is the minimum dimension d such that the antimatroid is isometrically isomorphic to some d-dimensional poly-antimatroid. This definition is a direct analog of the lattice dimension of graphs (Eppstein, 2005). In this paper the exact characterization of antimatroids of poly-dimension 1 and 2 is given and a conjecture concerning antimatroids of any poly-dimension d is suggested.

This chapter is organized as follows.

Section 1 contains an extended introduction to the theory of antimatroids.

Section 2 gives basic information about distances on graphs and isometric isomorphisms. We concentrate on interrelations between antimatroids and poly-antimatroids. In particular, we construct an isometric isomorphism between these structures.

In Section 3 we introduce the *poly-dimension* of an antimatroid and prove our main theorem characterizing antimatroids of poly-dimension 2. In addition we present a linear labeling algorithm that isometrically embeds such an antimatroid into the integer lattice $\mathbb{Z}^2$.

Section 4 discusses an open problem.

2. Preliminaries

Let E be a finite set. A *set system* over E is a pair $(E, \mathcal{S})$, where $\mathcal{S}$ is a family of sets over E, called *feasible* sets. We will use $X \cup x$ for $X \cup \{x\}$, and $X - x$ for $X - \{x\}$.

Definition 2.1. *(Korte et al., 1991) A finite non-empty set system $(E, \mathcal{S})$ is an antimatroid if*

$(A1)$ *for each non-empty $X \in \mathcal{S}$, there exists $x \in X$ such that $X - x \in \mathcal{S}$*

$(A2)$ *for all $X, Y \in \mathcal{S}$, and $X \nsubseteq Y$, there exists $x \in X - Y$ such that $Y \cup x \in \mathcal{S}$.*

Any set system satisfying $(A1)$ is called *accessible.*

In addition, we use the following characterization of antimatroids.

Proposition 2.2. *(Korte et al., 1991) For an accessible set system $(E, \mathcal{S})$ the following statements are equivalent:*

(i) *$(E, \mathcal{S})$ is an antimatroid*

(ii) *$\mathcal{S}$ is closed under union $(X, Y \in \mathcal{S} \Rightarrow X \cup Y \in \mathcal{S})$*

An "antimatroid with repetition" was invented by Bjorner, Lovasz and Shor (Björner et al., 1991) by studying the set of configurations of the Chip Firing Game (CFG). Further it was investigated by the name of "poly-antimatroid" as a generalization of the notion of the antimatroid for multisets. A *multiset* A over E is a function $f_A : E \to N$, where $f_A(e)$ is a number of repetitions of an element e in A. *A poly-antimatroid* is a finite non-empty multiset system $(E, \mathcal{F})$ that satisfies the antimatroid properties $(A1)$ and $(A2)$. So antimatroids may be considered as a particular case of poly-antimatroids. An example of an antimatroid $(\{x, y, z\}, \mathcal{S})$ and a poly-antimatroid $(\{x, y\}, \mathcal{F})$ is illustrated in Figure 1.

Definition 2.3. *A multiset system $(E, \mathcal{F})$ satisfies the chain property if for all $X, Y \in \mathcal{F}$, and $X \subset Y$, there exists a chain $X = X_0 \subset X_1 \subset ... \subset X_k = Y$ such that $X_i = X_{i-1} \cup x_i$ and $X_i \in \mathcal{F}$ for $0 \leq i \leq k$. We call the system a chain system.*

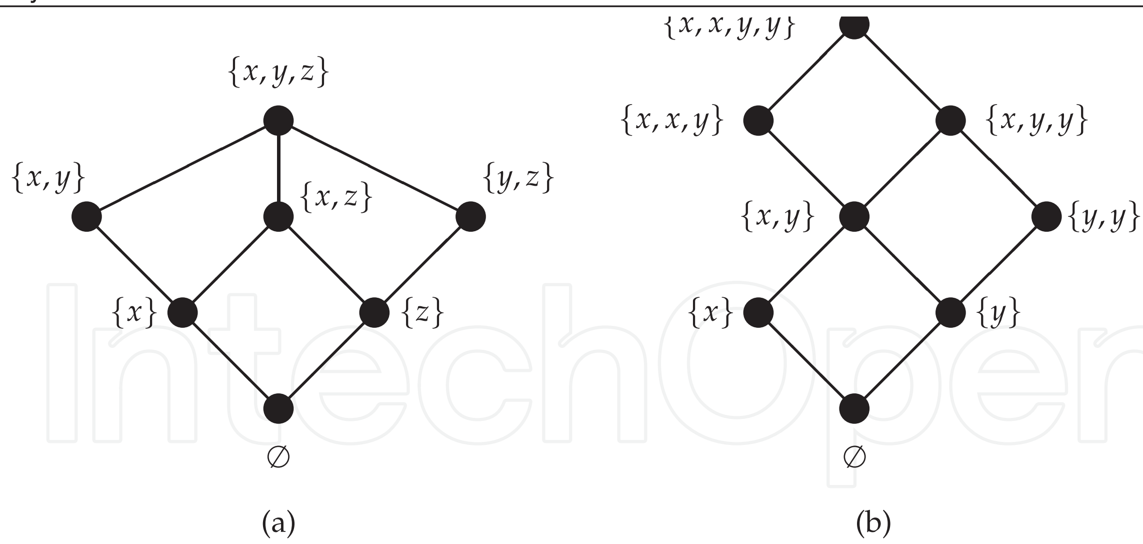


Fig. 1. (a) Antimatroid. (b) Poly-antimatroid.

It is easy to see that this chain property follows from $(A2)$, but these properties are not equivalent. If $\emptyset \in \mathcal{F}$, then accessibility follows from the chain property. In general case, there are accessible set systems that do not satisfy the chain property. Indeed, consider the following example illustrated in Figure 2 (a). Let $E = \{1,2,3\}$ and $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{2,3\}, \{1,2,3\}\})$. It is easy to check that the set system is accessible, but there are no chain from $\{1\}$ to $\{1,2,3\}$. Vice versa, it is possible to construct a system, that satisfies the chain property and it is not accessible. For example, $\mathcal{F} = \{\{1\}, \{3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$ in Figure 2(b).

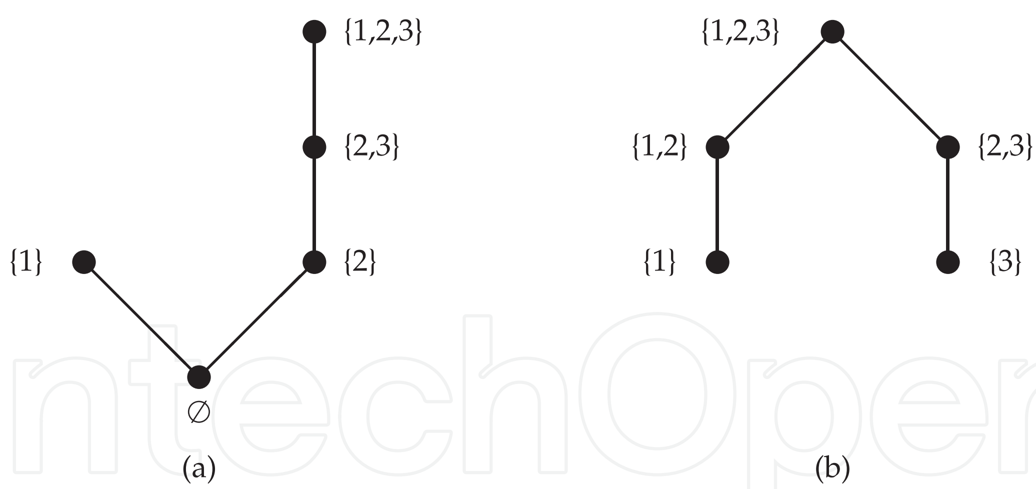


Fig. 2. (a) Accessible system but not chain system. (b) Chain system but not accessible.

In fact, if we have an accessible set system satisfying the chain property, then the same system but without the empty set (or without all subsets of cardinality less then some k) is not accessible, but still satisfies the chain property.

Examples of chain systems include poly-antimatroids, convex geometries, matroids and other hereditary systems (matchings, cliques, independent sets of a graph).

Regular set systems have been defined by Honda and Grabisch (Honda & Grabisch, 2008).

Definition 2.4. *A set system $(E, \mathcal{S})$ is a regular set system if $\emptyset, E \in \mathcal{S}$ and all maximal chains have length n, where $n = |E|$.*

Equivalently, under the condition $\emptyset, E \in \mathcal{S}$, $(E, \mathcal{S})$ is a regular set system if and only if $|A - B| = 1$ for any $A, B \in \mathcal{S}$ such that A is a cover of B.

Actually, a regular set system is a chain system with an additional constraint demanding from both the empty set $\emptyset$ and the ground set E to belong to the set system $\mathcal{S}$.

An algorithmic characterization of chain systems with empty set $\emptyset$, called *strongly accessible set system*, was presented in (Boley et al., 2010).

Augmenting systems were introduced in game theory as a restricted cooperation model. The model is a weakening of the antimatroid structure and strengthening of the chain system.

Definition 2.5. *(Bilbao, 2003) A set system $(E, \mathcal{S})$ is a augmenting system if*

1. $\emptyset \in \mathcal{S}$

2. *it satisfies the chain property*

3. *for all $X, Y \in \mathcal{S}$ with $X \cap Y \neq \emptyset$, we have $X \cup Y \in \mathcal{S}$.*

An augmenting system is an antimatroid if and only if it is closed under union (Bilbao, 2003).

A comparative study of various families of set systems are given in (Grabisch, 2009).

Antimatroids have already been investigated within the framework of lattice theory by Dilworth (Dilworth, 1940). The feasible sets of an antimatroid ordered by inclusion form a lattice, with lattice operations: $X \vee Y = X \cup Y$, and $X \wedge Y$ is the maximal feasible subset of the set $X \cap Y$ called a *basis*. Since an antimatroid is closed under union, it has only one basis.

A finite lattice L is *semimodular* if whenever $x, y \in L$ both cover $x \wedge y$, $x \vee y$ covers x and y.

A finite lattice L is called *join-distributive* (Björner & Ziegler, 1992) if for any $x \in L$ the interval $[x, y]$ is Boolean, where y is the join of all elements covering x. Such lattices are appeared under several different names, e.g., locally free lattices (Korte et al., 1991) and upper locally distributive lattices (ULD) (Felsner & Knauer, 2009; Magnien et al., 2001; Monjardet, 2003).

Proposition 2.6. *(Björner & Ziegler, 1992) For an accessible set system $(E, \mathcal{S})$ the following statements are equivalent:*

(i) $(E, \mathcal{S})$ is an antimatroid.

(ii) $(\mathcal{S}, \subseteq)$ is a join-distributive lattice.

(iii) $(\mathcal{S}, \subseteq)$ is a semimodular lattice.

In fact, the two concepts - antimatroids and join-distributive lattices - are essentially equivalent.

Theorem 2.7. *((Björner & Ziegler, 1992; Korte et al., 1991) A finite lattice is join-distributive if and only if it is isomorphic to the lattice of feasible sets of some antimatroid.*

It is easy to see that feasible sets of a poly-antimatroid ordered by inclusion form a join-distributive lattice as well. This kind of findings was discussed in (Magnien et al., 2001), where it was proved that any CFG is equivalent to a simple CFG (antimatroid), i.e., the lattices of their configuration spaces are isomorphic.

3. Isometry

For each graph $G = (V, E)$ the distance $d_G(u, v)$ between two vertices $u, v \in V$ is defined as the length of a shortest path joining them.

If G and H are arbitrary graphs, then a mapping $f : V(G) \to V(H)$ is an *isometric embedding* if $d_H(f(u), f(v)) = d_G(u, v)$ for any $u, v \in V(G)$.

Let $E = \{x_1, x_2, ...x_d\}$. Define a graph $H(E)$ as follows: the vertices are the finite subsets of E, two vertices A and B are adjacent if and only if the symmetric difference $A \triangle B$ is a singleton set. Then $H(E)$ is the *hypercube* Q_n *on* E (Djokovic, 1973). The hypercube can be equivalently defined as the graph on $\{0, 1\}^d$ in which two vertices form an edge if and only if they differ in exactly one position.

The shortest path distance $d_H(A, B)$ on the hypercube $H(E)$ is the Hamming distance between A and B that coincides with the symmetric difference distance: $d_H(A, B) = |A \triangle B|$.

A graph G is called a *partial cube* if it can be isometrically embedded into a hypercube $H(E)$ for some set E.

A family of sets $\mathcal{S}$ is *well-graded* (Doignon & Falmagne, 1997) if any two sets $P, Q \in \mathcal{S}$ can be connected by a sequence of sets $P = R_0, R_1, ..., R_n = Q$ formed by single-element insertions and deletions ($|R_i \triangle R_{i+1}| = 1$), such that all intermediate sets in the sequence belong to $\mathcal{S}$ and $|P \triangle Q| = n$. This sequence of sets is called a *tight path*.

Any set system $(E, \mathcal{S})$ defines an undirected graph $G_{\mathcal{S}} = (\mathcal{S}, E_{\mathcal{S}})$, where $E_{\mathcal{S}} = \{\{P, Q\} \in \mathcal{S} : |P \triangle Q| = 1\}$.

Theorem 3.1. *(Ovchinnikov, 2008) The graph $G_{\mathcal{S}}$ defined by a set system $(E, \mathcal{S})$ is a partial cube on E if and only if the family $\mathcal{S}$ is well-graded.*

Proposition 3.2. *A family $\mathcal{S}$ of every antimatroid $(E, \mathcal{S})$ is well-graded.*

Proof. We prove that there is a tight path between each $P, Q \in \mathcal{S}$. If $P \subset Q$, then the existence of such path follows immediately from the chain property. If $P \nsubseteq Q$, then there exist chains from both P and Q to $P \cup Q$ that forms the tight path. □

Thus each antimatroid is a partial cube and may be represented as a graph $G_{\mathcal{S}}$ that is a subgraph of the hypercube $H(E)$ or as a set of points of the hypercube $\{0, 1\}^d$. For example, see an antimatroid in Figure 3. The distance in an antimatroid $(E, \mathcal{S})$ considered as a graph coincides with the Hamming distance between sets, i.e., $d_{\mathcal{S}}(A, B) = |A \triangle B|$ for any $A, B \in \mathcal{S}$.

A poly-antimatroid $(E, \mathcal{F})$ may be represented as a set of points in the digital space $\mathbb{Z}^d$, since each $A \in \mathcal{F}$ is defined by the sequence $(f_A(x_1), f_A(x_2), ..., f_A(x_d))$ that may be denoted as a point $(a_1, a_2, ..., a_d)$ in $\mathbb{Z}^d$. For example, see a two-dimensional poly-antimatroid in Figure 4.

The symmetric difference distance can be generalized to multisets by summing the absolute difference of the multiplicities of corresponding elements

$$|A \triangle B| = \sum |f_A(x_i) - f_B(x_i)|$$

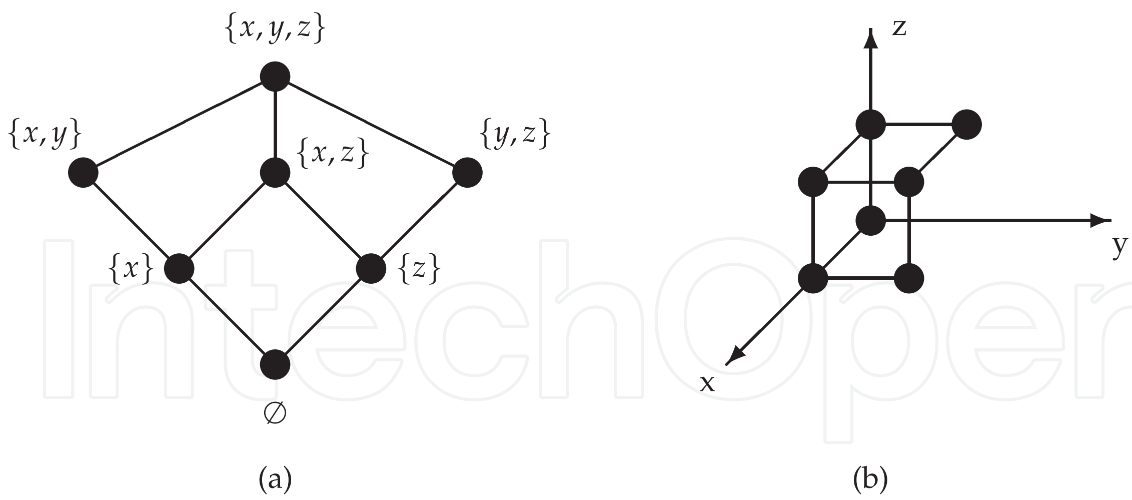


Fig. 3. Representation of an antimatroid: (a) as a graph of a family of subsets and (b) as a set of points in $\mathbb{Z}^3$.

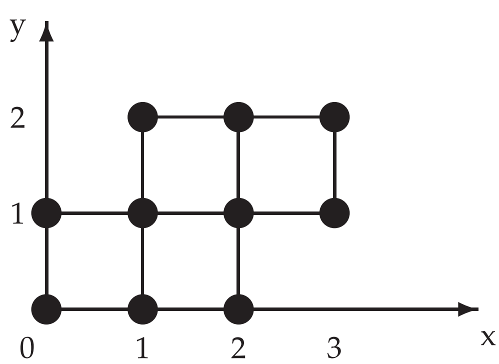


Fig. 4. A two-dimensional poly-antimatroid.

that coincides with the L_1 distance $(||A - B||_1)$ on the digital space $\mathbb{Z}^d$.

A multiset system $(E, \mathcal{F})$ defines an undirected graph $G_{\mathcal{F}}$ in the same way as a set system. So each poly-antimatroid $(E, \mathcal{F})$ may be represented as a graph $G_{\mathcal{F}}$.

These representations of poly-antimatroids are isometrically isomorphic.

Proposition 3.3. *The distance in a poly-antimatroid $(E, \mathcal{F})$ considered as a graph $G_{\mathcal{F}}$ coincides with the symmetric difference distance, i.e., $d_{\mathcal{F}}(A, B) = |A \triangle B|$ for any $A, B \in \mathcal{F}$.*

Proof. For any $A, B \in \mathcal{F}$ there is a multiset $A \cup B \in \mathcal{F}$. Then there is a path in the poly-antimatroid from A to $A \cup B$ (by adding successively all the elements from $B - A$) and from $A \cup B$ to B (by removing successively all the elements from $A - B$). Therefore, $d_{\mathcal{F}}(A, B) \leq |A \triangle B|$. On the other hand, in any path from A to B every two adjacent vertices differ by only one element, and, consequently, $d_{\mathcal{F}}(A, B) \geq |A \triangle B|$. □

Thus a poly-antimatroid with the ground set of size d may be isometrically embedded into the d-dimensional lattice $\mathbb{Z}^d$.

Since each antimatroid is a poly-antimatroid, one may prove that every poly-antimatroid is isometrically isomorphic to some antimatroid.

Consider a poly-antimatroid $(E, \mathcal{F})$ given as a set of points in the digital space $\mathbb{Z}^d$. We use the Djokovic technique (Djokovic, 1973) elaborated by Eppstein (Eppstein, 2005) to embed a finite subset of the lattice $\mathbb{Z}^d$, and, therefore, a poly-antimatroid $(E, \mathcal{F})$, into a hypercube.

Let $\lambda_i = \max\{a_i | A \in \mathcal{F}\}$ and $\tau = \sum_{i=1}^{d} \lambda_i$.

Consider the following mapping $\mu : \mathbb{Z}^d \to \{0,1\}$.

Define $\mu(A) = (\mu_1(A), \mu_2(A), ..., \mu_\tau(A))$, where for each k such that $\sum_{i=1}^{j-1} \lambda_i < k \leq \sum_{i=1}^{j} \lambda_i$

$$\mu_k(A) = \begin{cases} 1 \ a_j \geq k - \sum_{i=1}^{j-1} \lambda_i \\ 0 \quad otherwise \end{cases}.$$

It is clear that $\mu(0, 0, ..., 0) = (0, 0, ..., 0)$.

For example, consider the poly-antimatroid depicted in Figure 4. Mapping μ embeds the poly-antimatroid into a hypercube $\{0,1\}^5$ such that $\mu(1,0) = (1,0,0,0,0)$, $\mu(0,1) = (0,0,0,1,0)$, $\mu(2,1) = (1,1,0,1,0)$ and so on. See an isometrically isomorphic antimatroid in Figure 5.

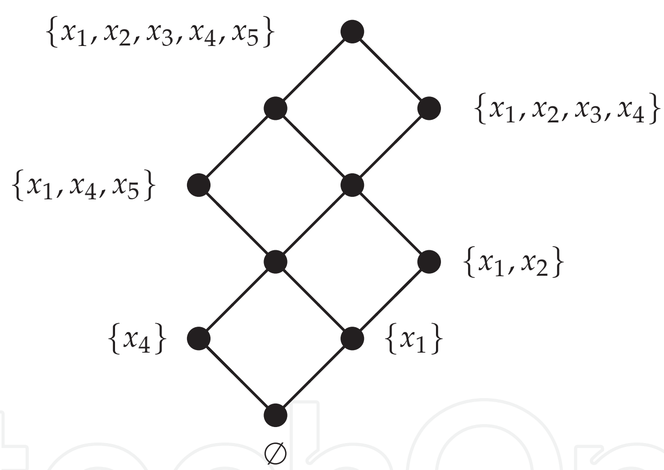


Fig. 5. An isometrically isomorphic antimatroid.

Lemma 3.4. *(Eppstein, 2005) The map μ is an isometry from a poly-antimatroid $(E, \mathcal{F})$ into a hypercube.*

Note that isometry μ is full-dimensional, i.e., each coordinate μ_i takes on both value 0 and 1 for at least one point.

It remains to show that $\mu(\mathcal{F})$ forms an antimatroid. We consider now a hypercube on some set $U = \{u_1, u_2, ..., u_\tau\}$. Then $\widehat{\mu}(A) = \{u_i | \mu_i(A) = 1\}$. Let $S = \{X \subseteq U : \exists A \in \mathcal{F}, X = \widehat{\mu}(A)\}$.

Proposition 3.5. *For every poly-antimatroid $(E, \mathcal{F})$, the set system $(U, S = \widehat{\mu}(\mathcal{F}))$ is an antimatroid.*

Proof. At first, $\emptyset \in S$. Indeed, $\emptyset \in \mathcal{F}$, i.e., $(0,0,...,0) \in \mathcal{F}$ and $\widehat{\mu}(0,0,...,0) = \emptyset$. Then S is accessible, since the map μ is distance-preserving. To see it, consider a non-empty $X \in S$. There is a set $A \in \mathcal{F}$ such that $X = \widehat{\mu}(A)$. Let $A = (a_1, a_2, ..., a_d)$. Since $(E, \mathcal{F})$ is a poly-antimatroid, there is $j \in \{1,2,...,d\}$ such that $B = (a_1, a_2, ..., a_j - 1, ..., a_d) \in \mathcal{F}$. Then $\mu(A)$ and $\mu(B)$ differs only in the position $k = \sum_{i=1}^{j-1} \lambda_i + a_j$, i.e., $\widehat{\mu}(A) - \widehat{\mu}(B) = \{u_k\}$.

Now let us prove, that the family S is closed under union, i.e., $X, Y \in S \Rightarrow X \cup Y \in S$. Since $X, Y \in S$ there are two points $A, B \in \mathcal{F}$ such that $X_1 = \widehat{\mu}(A)$ and $X_2 = \widehat{\mu}(B)$. Let $A = (a_1, a_2, ..., a_d)$ and $B = (b_1, b_2, ..., b_d)$. Then, $A \cup B = (c_1, c_2, ..., c_d)$, where $c_i = \max(a_i, b_i)$. Hence $\mu(A \cup B) = (\nu_1, \nu_2, ..., \nu_\tau)$, where $\nu_i = \max(\mu_i(A), \mu_i(B))$. So, $\widehat{\mu}(A \cup B) = \widehat{\mu}(A) \cup \widehat{\mu}(B)$, which implies $X \cup Y = \widehat{\mu}(A \cup B) \in S$. □

So we can say that the two structures - a poly-antimatroid $(E, \mathcal{F})$ and an antimatroid $(U, \widehat{\mu}(\mathcal{F}))$ are essentially identical.

4. Poly-dimension

Clearly, an antimatroid is a poly-antimatroid as well. The question is what is a minimal dimension d such that an antimatroid $(U, \mathcal{S})$ may be represented by an isomorphic poly-antimatroid $(\widehat{U}, \mathcal{F})$ in the space $\mathbb{Z}^d$.

We will call this minimal dimension $\pi(\mathcal{S})$ the *poly-dimension* of the antimatroid $(U, \mathcal{S})$.

Each antimatroid $(U, \mathcal{S})$ may be considered also as a directed graph $G = (V, E)$ with $V = \mathcal{S}$ and $(A, B) \in E \Leftrightarrow \exists c \in B$ such that $A = B - c$.

Denote *in-degree* of the vertex A as $deg_{in}(A)$, and *out-degree* as $deg_{out}(A)$, where

$$deg_{in}(A) = |\{c : A - c \in \mathcal{S}\}|, deg_{out}(A) = |\{c : A \cup c \in \mathcal{S}\}|$$

Consider antimatroids for which their maximum in-degree and maximum out-degree is at most p, and there is at least one feasible set for which in-degree or out-degree equals p. We will call such antimatroids *p-antimatroids*.

Proposition 4.1. *The poly-dimension of an 1-antimatroid is one.*

Proof. It is easy to see that 1-antimatroid is a chain $\emptyset \subseteq \{a_1\} \subseteq ... \subseteq \{a_1, a_2, ..., a_n\}$, that may be represented as a poly-antimatroid $\emptyset \subseteq \{x\} \subseteq ... \subseteq \{x, x, ..., x\}$ belonging to the axe x. The "only if" part of the proof is clear. □

To find the poly-dimension of a 2-antimatroid we show that 2-antimatroids have the special structure of "sub-grid".

A *poset antimatroid* is a particular case of antimatroid, which is formed by the lower sets of a poset (partially ordered set). The poset antimatroids can be characterized as the unique antimatroids which are closed under intersection (Korte et al., 1991).

Let us prove that any 2-antimatroid is a poset antimatroid.

An *endpoint* of a feasible set X is an element $e \in X$ such that $X - e$ is a feasible set too. A feasible set X is an *e-path* if it has a single endpoint e.

It is easy to see that an e-path is a minimal feasible set containing e.

The following lemma gives the characterization of poset antimatroids in terms of e-paths. We present a proof for the sake of completeness.

Lemma 4.2. *(Algaba et al., 2004) An antimatroid $(U, \mathcal{S})$ is a poset antimatroid if and only if every $i \in U$ has a unique i-path in $\mathcal{S}$.*

Proof. Let $(U, \mathcal{S})$ be a poset antimatroid and let $A, B, A \neq B$ be two distinct i-paths for $i \in U$. Then $i \in A \cap B \in \mathcal{S}$. Since each i-path is a minimal feasible set containing i, then $A \nsubseteq B$ and $B \nsubseteq A$. Hence the chain property implies the existence of an element $j \in A - (A \cap B)$ such that $A - j \in \mathcal{S}$. This is a contradiction with A being an i-path.

Now suppose that every $i \in U$ has a unique i-path in $\mathcal{S}$. Let $X, Y \in \mathcal{S}$. If $X \cap Y = \emptyset$ then $X \cap Y \in \mathcal{S}$ by definition of an antimatroid.

If $X \cap Y \neq \emptyset$ then for every $i \in U$ there exists an i-path $H_1^i \subseteq X$ and $H_2^i \subseteq Y$, because each i-path is a minimal feasible set containing i. By assumption $H_1^i = H_2^i = H^i$. Thus, for all $i \in X \cap Y$, an i-path $H^i \subseteq X \cap Y$, and so $X \cap Y = \bigcup_{i \in X \cap Y} H^i$. Therefore, $X \cap Y \in \mathcal{S}$, since every antimatroid is closed under union. Hence, $(U, \mathcal{S})$ is a poset antimatroid. □

Lemma 4.3. *Any 2-antimatroid $(U, \mathcal{S})$ has a unique i-path for each $i \in U$.*

Proof. Suppose the opposite. Let $A, B \in \mathcal{S}$ be two different i-paths. Note that $A \nsubseteq B$ and $B \nsubseteq A$, since each i-path is a minimal set containing i.

Consider the set $Z = A \cup B$. From the chain property it follows that there is $b \in B - A$, such that $Z - b \in \mathcal{S}$. On the other hand, there is $a \in A - B$ with $Z - a \in \mathcal{S}$. Since $(A - i) \cup (B - i) = Z - i \in S$, we have that $deg_{in}(Z) \geq 3$. □

Lemmas 4.2 and 4.3 imply the following.

Proposition 4.4. *Any 2-antimatroid is a poset antimatroid.*

Denote $C_k = \{X \in \mathcal{S} : |X| = k\}$ a family of feasible sets of cardinality k.

Definition 4.5. *A lower zigzag is a sequence of feasible sets $P_0, P_1, ..., P_{2m}$ such that any two consecutive sets in the sequence differ by a single element and $P_{2i} \in C_k$, and $P_{2i-1} \in C_{k-1}$ for all $0 \leq i \leq m$.*

In the same way we define an *upper zigzag* in which $P_{2i-1} \in C_{k+1}$.

Each zigzag $P_0, P_1, ..., P_{2m}$ is a path connecting P_0 and P_{2m}, and so the distance on the zigzag $d(P_0, P_{2m}) = 2m$ is always no less than the distance $d_S(P_0, P_{2m})$ on an antimatroid $(U, \mathcal{S})$.

In Figure 6 we can see two sets ($A = \{1,2,3,5\}$ and $B = \{1,3,4,5\}$) that are connected by an unique lower zigzag, such that the distance on the zigzag is 4, while $|A \triangle B| = 2$. Note, that the distance on the upper zigzag is indeed 2. For two sets $X = \{1,2,5\}$ and $Y = \{3,4,5\}$ both distances on the lower zigzag and on the upper zigzag are equal to 6, while $|X \triangle Y| = 4$.

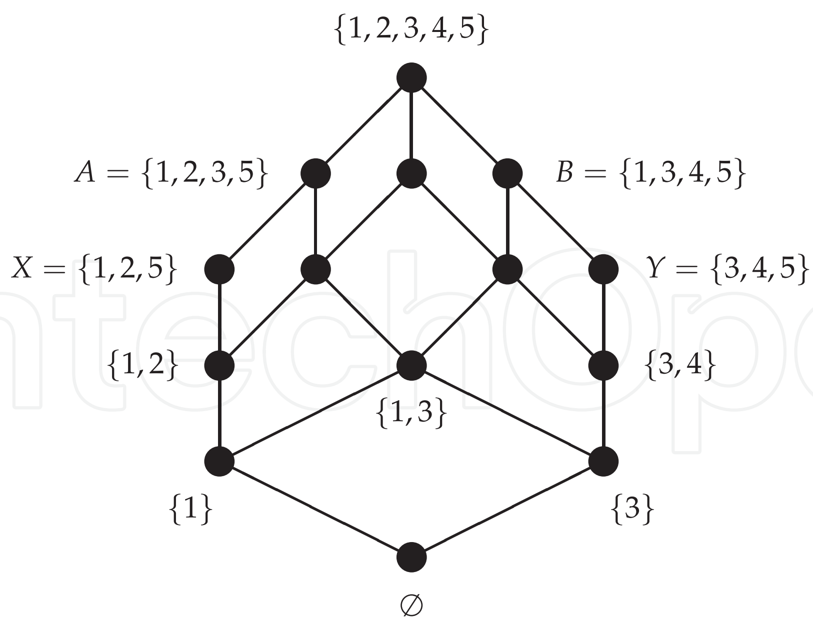


Fig. 6. An antimatroid without distance preserving zigzags.

In order to get the distance on a zigzag equal to the set distance, an antimatroid has to be a poset antimatroid.

Theorem 4.6. *In each poset antimatroid $(U, \mathcal{S})$ every two feasible sets A, B of the same cardinality k can be connected both by a lower and an upper zigzag in such a way that the distance between these sets $d_{\mathcal{S}}(A, B)$ coincides with the distance on the zigzags.*

Proof. We use induction on k. For every pair of one-element feasible sets $\{x\}$ and $\{y\}$ there is a lower zigzag (via $\emptyset$) and an upper zigzag (via $\{x, y\}$) such that the distance between these sets $d_{\mathcal{S}}(\{x\}, \{y\}) = 2$ on the antimatroid coincides with distances on the zigzags.

Assume that the hypothesis of the theorem is correct for all sets $X \in \mathcal{S}$ with $|X| < k$.

Consider two sets $A, B \in C_k$ and let $d_{\mathcal{S}}(A, B) = |A \triangle B| = 2m$. Since $(U, \mathcal{S})$ is a poset antimatroid, the set $A \cap B \in \mathcal{S}$. Hence, by the chain property, there is $a \in A - B$ such that $A - a \in \mathcal{S}$, and $b \in B - A$ with $B - b \in \mathcal{S}$. These two sets belong to C_{k-1}, so there are an upper zigzag $A - a = P_1, P_2, ..., P_{2l-1} = B - b$ connecting $A - a$ with $B - b$ (see Figure 7). Since $d_{\mathcal{S}}(A - a, B - b) = |A \triangle B| - 2 = 2m - 2$, the distance on the upper zigzag between $A - a$ and $B - b$ is $2m - 2$ by the induction hypothesis, i.e., $2l - 1 = 2m - 1$. It is easy to see that neither A nor B belongs to the upper zigzag $A - a = P_1, P_2, ..., P_{2m-1} = B - b$, because otherwise it implies that $d_{\mathcal{S}}(A, B) \leq 2m - 2$. Hence exists a lower zigzag $A = P_0, A - a = P_1, P_2, ..., P_{2m-1} = B - b, P_{2m} = B$ connecting A and B such that the distance on the zigzag between A and B equals $2m$.

Let $P^i = P_{i-1} \cup P_{i+1}$ for each $i = 1, 3, ..., 2m - 1$. All obtained sets are different, because otherwise $d_{\mathcal{S}}(A, B) < 2m$. So there is the upper zigzag by length $2m$ connecting A and B. $\square$

Note that if for a zigzag $P_0, P_1, ..., P_{2m}$ the distance on the zigzag $d(P_0, P_{2m}) = 2m$ is equal to the distance $d_S(P_0, P_{2m}) = |P_0 \triangle P_{2m}|$ then the zigzag preserves the distance for each pair P_i, P_j, i.e., $d_S(P_i, P_j) = d(P_i, P_j) = |j - i|$.

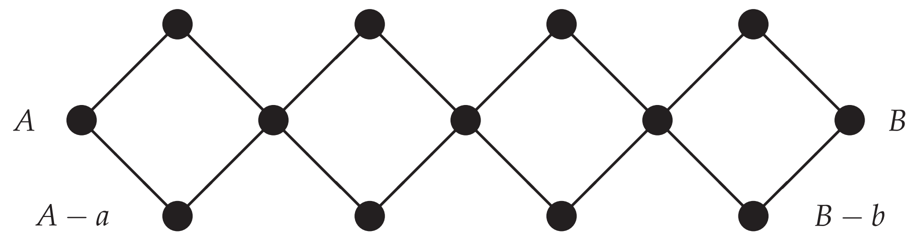


Fig. 7. Zigzags of an antimatroid.

For poset antimatroids there are distance preserving zigzags connecting two given sets, but these zigzags are not obliged to connect all feasible sets of the same cardinality. In Figure 8 we can see that there is a poset antimatroids, for which it is not possible to build two distance preserving zigzags connecting all feasible sets of the same cardinality.

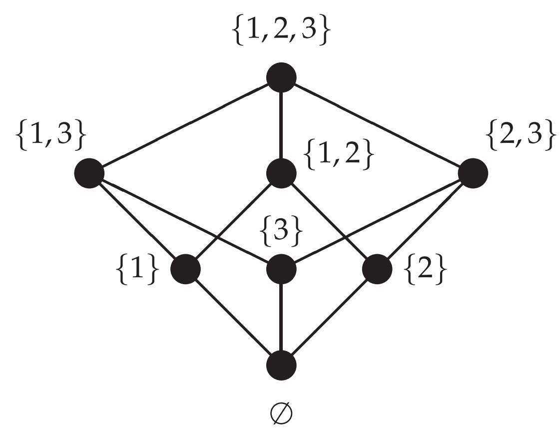


Fig. 8. A poset antimatroid without total zigzags.

It is worth mentioning that in 2-antimatroids all feasible sets of the same cardinality may be connected by one distance preserving zigzag.

Proposition 4.7. *In 2-antimatroid $(U, \mathcal{S})$ all feasible sets C_k of the same cardinality k can be connected both by a lower and an upper zigzag in such a way that the distance between any two sets $d_{\mathcal{S}}(P_i, P_j)$ belonging to the same zigzag coincides with distance on the zigzag $d(P_i, P_j) = |j - i|$.*

Proof. We proceed by induction on cardinality k. For one-elements sets it is obvious. Assume that it is correct for all sets $A \in \mathcal{S}$ with $|A| < k$.

Each set from C_k is obtained from some set of cardinality $k-1$. Since an antimatroid is closed under union and its maximum in-degree and out-degree is 2, the $n+1$ sets of cardinality $k-1$ connected by an upper zigzag form n sets from C_k. In addition to these n sets, each of two extreme sets (P_0 and P_{2n}) can form one more set of cardinality k that does not belong to the upper zigzag. Since each out-degree is less than 3 there are no other sets of cardinality k. So, the set C_k is connected by a lower zigzag. Since an antimatroid is closed under union and its maximum in-degree is 2, all feasible sets of the same cardinality k are also connected by an upper zigzag.

Now prove that the distance on the zigzags coincides with the distance on the antimatroid. Since maximum in-degree and out-degree are 2, the obtained lower zigzag is only lower zigzag connecting P_0 and P_{2n}. Then from Theorem 4.6 it follows that it is a distance preserving zigzag. Similarly, we obtain that the upper zigzag preserves the distance as well. □

For each edge $(A, B) \in \mathcal{S}$ there exists a single element $c \in B$ such that $A = B - c$, and so we can label each edge by such element. Then, each 2-antimatroid consists of quadrilaterals with equal labels on opposite pairs of edges, since if it contains the vertices X, $X \cup a$, $X \cup b$ for some $X \in \mathcal{S}$, and $a, b \in U$, it also contains $X \cup \{a, b\}$. This property, called by $\cup$-coloring, was invented in (Felsner & Knauer, 2009) as the characterization of join-distributive lattices.

Proposition 4.8. *There exists a labeling of edges of a 2-antimatroid by two labels x or y such that opposite pairs of edges in each quadrilateral have equal labels.*

Proof. To obtain such labeling we consequently scan a 2-antimatroid by layers C_k beginning from the empty set ($k = 0$). The first zigzag of edges we mark by two labels x and y in turn, beginning from x or from y arbitrary. From the Proposition 4.7 follows that the next zigzag can be marked alternate by two labels (x, y) in such a way that the opposite pairs of edges in each quadrilateral have equal labels (see Figure 9).

If we reach the layer C_k with one element, the next zigzag is labeled alternate by two labels without dependence on the previous layers. □

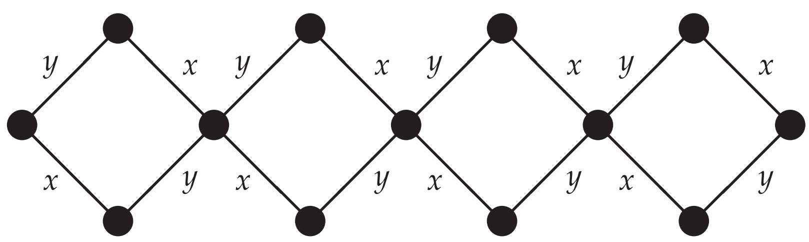


Fig. 9. A zigzag labeling.

It follows from accessibility that for each $A \in \mathcal{S}$ there is a shortest path connecting A with $\emptyset$. The length of such path is equal to the cardinality of A. Let $M(A)$ be a multiset containing all labels on this shortest path. Obviously, $M(A) \in \mathbb{Z}^2$. It is possible that there are some shortest paths between A and $\emptyset$. Prove that all such paths result in the same multiset.

Lemma 4.9. *In every 2-antimatroid $(U, \mathcal{S})$ for each $A \in \mathcal{S}$ all shortest paths connecting A with $\emptyset$ forms the same multiset $M(A)$.*

Proof. We use induction on cardinality k. For one-elements sets it is obviously. Assume it is correct for all sets $A \in \mathcal{S}$ with $|A| < k$. Consider $Y \in C_k$. If Y is obtained from only one set of cardinality $k - 1$, then $M(Y)$ is unique. If $Y = A \cup a = B \cup b$, then there is $X = A \cap B \in \mathcal{S}$ such that $A = X \cup b$ and $B = X \cup a$, since every 2-antimatroid is a poset antimatroid. Let a label of $(X, X \cup b)$ be x, and a label of $(X, X \cup a)$ be y. Thus $M(A) = M(X) \cup x$ and $M(B) = M(X) \cup y$, and so $M(Y) = M(X) \cup \{x, y\}$. Therefore, by induction hypothesis, all shortest path connecting Y with $\emptyset$ forms the same $M(Y)$. □

Proposition 4.10. *In every 2-antimatroid $(U, \mathcal{S})$ the mapping $M : (U, \mathcal{S}) \to \mathbb{Z}^2$, where each $A \in \mathcal{S}$ corresponds to $M(A)$, is an isometry.*

Proof. We have to prove that for all $A, B \in \mathcal{S}$ the distance $d_{\mathcal{S}}(A, B)$ on the antimatroid $(U, \mathcal{S})$ is equal to the distance $||M(A) - M(B)||_1$ on $\mathbb{Z}^2$, i.e., it holds

$$|A \triangle B| = |M(A) \triangle M(B)|.$$

If $A \subseteq B$, then there is a shortest path connecting B with $\emptyset$ via A, and so $|A \triangle B| = |B - A| = |M(B) - M(A)| = |M(A) \triangle M(B)|$.

Consider two incomparable sets $A, B \in \mathcal{S}$. Since a 2-antimatroid is a poset antimatroid, $A \cap B \in \mathcal{S}$. Then there are a shortest path P_A connecting A with $\emptyset$ via $A \cap B$, and a shortest path P_B connecting B with $\emptyset$ via $A \cap B$. Since $|A \triangle B| = |A - A \cap B| + |B - A \cap B|$, remain to prove that the part of P_A from A till $A \cap B$ is labeled by only one label (for example x) and the part of P_B from B till $A \cap B$ is labeled by only one but another label (y).

Indeed, from $A \cap B$ go out two different edges marked by two different labels, where one belongs to P_A and the second belongs to P_B. Let the label of the first edge was x and the label of the second edge was y. Denote obtained sets as A_1 and B_1. Note that $A_1 = (A \cap B) \cup a$ and $B_1 = (A \cap B) \cup b$, where $a \in A - B$ and $b \in B - A$. If $A_1 \neq A$ then from A_1 go out two edges. The first edge $(A_1, A_1 \cup b)$ is labeled by y and does not belong to P_A since $b \in B - A$. The second edge (A_1, A_2) is labeled by x and belongs to P_A. The same is correct for each set on path P_A from A till $A \cap B$. So the part of P_A from A till $A \cap B$ is labeled by label x only. By the same way we obtain that the part of P_B from B till $A \cap B$ is labeled by y only. $\square$

Thus we have the following.

Theorem 4.11. *The poly-dimension of a* 2*-antimatroid is two.*

5. Open problems

It is clear that the poly-dimension of a d-antimatroid is at least d.

Conjecture 5.1. *The poly-dimension of an antimatroid equals d if and only if it is a d-antimatroid.*

6. References

Algaba, E.; Bilbao, J.M.; van den Brink, R. & Jimenez-Losada A. (2004). Cooperative Games on Antimatroids. *Discrete Mathematics*, Vol.282, 1-15.

Algaba, E.; Bilbao, J.M. & Slikker, M. (2010). A Value for Games Restricted by Augmenting Systems. *SIAM J. Discrete Mathematics*, Vol.24, No.3, 992-1010

Bilbao, J.M. (2003). Cooperative Games under Augmenting Systems. *SIAM J. Discrete Mathematics*, Vol.17, No.1, 122-133.

Björner, A.; Lovász, L. & Shor, P.R. (1991). Chip-firing Games on Graphs. *European Journal of Combinatorics*, Vol.12, 283-291.

Björner, A. & Ziegler, G.M. (1992). Introduction to Greedoids, In: *Matroid applications*, N. White, (Ed.), Cambridge Univ.Press, Cambridge, UK.

Boley, M; Horváth,T; Poigné,A & Wrobel,S. (2010). Listing Closed Sets of Strongly Accessible Set Systems with Applications to Data Mining. *Theoretical Computer Science - TCS* , Vol. 411, No. 3, 691-700.

Boyd, E.A. & Faigle, U. (1990). An Algorithmic Characterization of Antimatroids. *Discrete Applied Mathematics*, Vol.28, 197-205.

Cosyn, E. & Uzun, H.B. (2009). Note on Two Necessary and Sufficient Axioms for a Well-graded Knowledge Space. *Journal of Mathematical Psychology*, Vol.53, No.1, 40-42.

Dilworth, R.P. (1940). Lattices with Unique Irreducible Decomposition. *Ann. of Mathematics*, Vol.41, 771-777.

Djokovic, D.Z. (1973). Distance Preserving Subgraphs of Hypercubes. *J. Combin. Theory Ser.B*, Vol.14, 263-267.

Doignon, J.-P. & Falmagne, J.-Cl. (1997). Well-graded Families of Relations. *Discrete Mathematics*, Vol.173, 35–44.

Eppstein, D. (2005). The Lattice Dimension of a Graph. *European Journal of Combinatorics*, Vol.26, 585-592.

Eppstein, D. (2008). Upright-Quad Drawing of st-Planar Learning Spaces. *Journal of Graph Algorithms and Applications*, Vol.12, No.1, 51-72.

Eppstein, D.; Falmagne, J.-Cl. & Ovchinnikov, S. (2008). *Media Theory: Interdisciplinary Applied Mathematics*, Springer-Verlag, Berlin.

Falmagne, J.-Cl. & Doignon, J.-P., (2011). *Learning Spaces: Interdisciplinary Applied Mathematics*, Springer-Verlag, Berlin.

Felsner, S. & Knauer, K. (2009). ULD-Lattices and Δ-Bondes. *Combinatorics, Probability and Computing*, Vol.18, No.5, 707-724.

Glasserman, P. & Yao, D.D. (1994). *Monotone Structure in Discrete Event Systems*, Wiley Inter-Science, Series in Probability and Mathematical Statistics.

Grabisch, M. (2009). The Core of Games on Ordered Structures and Graphs. *4OR: A Quarterly Journal of Operations Research*, Vol.7, No.3, 207-238.

Graham, R.L. & Pollak, H.O. (1971). On the Addressing Problem for Loop Switching. *Bell System Technical Journal*, Vol.50, 2495-2519.

Honda, A. & Grabisch, M. (2008). An Axiomatization of Entropy of Capacities on Set Systems, *European Journal of Operational Research*, Vol.190 , 526-538.

Kempner, Y. & Levit, V.E. (2003). Correspondence between Two Antimatroid Algorithmic Characterizations. *The Electronic Journal of Combinatorics*, Vol.10.

Kempner, Y. & Levit, V.E. (2007). A Geometric Characterization of Poly-antimatroids, *ENDM (Electronic Notes in Discrete Mathematics)*, Vol.28, 357-364.

Kempner, Y. & Muchnik, I. (2003). Clustering on Antimatroids and Convex Geometries, *WSEAS Transactions on Mathematics*, Vol.2, No.1, 54-59.

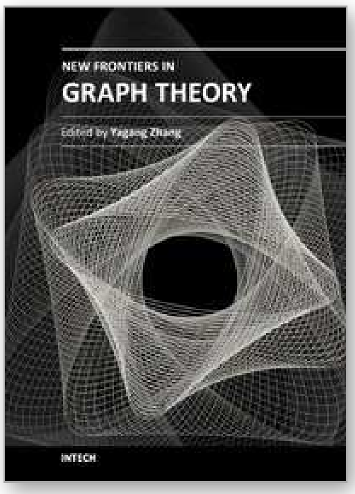
Korte, B.; Lovász, L. & R. Schrader, R. (1991). *Greedoids*, Springer-Verlag, New York/Berlin.

Magnien, C.; Phan, H.D. & Vuillon, L. (2001). Characterization of Lattices Induced by (extended) Chip Firing Games, *Discrete Mathematics and Theoretical Computer Science Proceedings AA (DM-CCG)*, 229-244.

Monjardet, B. (2003). The Presence of Lattice Theory in Discrete Problems of Mathematical Social Sciences. Why, *Math. Social Science*, Vol.46, 103-144.

Nakamura, M. (2005). Characterization of Polygreedoids and Poly-antimatroids by Greedy Algorithms, *Oper.Res. Lett.*, Vol.33, No. 4, 389-394.

Ovchinnikov, S. (2008). Partial cubes: Structures, Characterizations, and Constructions, *Discrete Mathmatics*, Vol.308, 5597–5621.



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Nowadays, graph theory is an important analysis tool in mathematics and computer science. Because of the inherent simplicity of graph theory, it can be used to model many different physical and abstract systems such as transportation and communication networks, models for business administration, political science, and psychology and so on. The purpose of this book is not only to present the latest state and development tendencies of graph theory, but to bring the reader far enough along the way to enable him to embark on the research problems of his own. Taking into account the large amount of knowledge about graph theory and practice presented in the book, it has two major parts: theoretical researches and applications. The book is also intended for both graduate and postgraduate students in fields such as mathematics, computer science, system sciences, biology, engineering, cybernetics, and social sciences, and as a reference for software professionals and practitioners.